Robust Control: Past Successes and Future Directions

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Outline

• Brief Overview of Robust Control

• Robustness of Time-Varying Systems

• Future Directions

• Conclusions
Pillars of Robust Control

1. Multivariable Optimal Control
   - $H_2$, $H_\infty$, DK-synthesis

2. Fundamental Limitations of Dynamics & Control
   - Bode sensitivity integral, complementary sensitivity integrals, constraints due to right-half plane poles and zeros.

3. Uncertainty Modeling and Robustness Analysis
   - Linear Fractional Transformations (LFTs), Structured Singular Value ($\mu$), Integral Quadratic Constraints (IQCs)
Pillars of Robust Control

1. Multivariable Optimal Control
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2. Fundamental Limitations of Dynamics & Control
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3. Uncertainty Modeling and Robustness Analysis
   - Linear Fractional Transformations (LFTs), Structured Singular Value ($\mu$), Integral Quadratic Constraints (IQCs)
Many design objectives: Stability, disturbance rejection, reference tracking, noise rejection, moderate actuator commands, adequate robustness margins.

Basic Limitation: \( S + T = 1 \)

Typically require \(|S| \ll 1\) at low frequencies for reference tracking and disturbance rejection.
Conservation of Sensitivity

Suppose $P$ is stable so that $C = 0$ is a stabilizing controller.

\[
S(s) := \frac{1}{1 + P(s)C(s)} \equiv 1
\]

AND

\[
\int_{0}^{\infty} \ln |S(j\omega)| \, d\omega = 0
\]
Conservation of Sensitivity

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$$S(s) := \frac{1}{1 + P(s)C(s)} \equiv 1$$

**AND**

$$\int_{0}^{\infty} \ln |S(j\omega)| \, d\omega = 0$$

Improving sensitivity at some frequencies leads to degradations at others.
Bode Integral Theorem [1]

If $PC$ is stable, relative degree 2 and $S(s)$ is stable. Then:

$$\int_0^\infty \ln |S(j\omega)| \, d\omega = 0$$

This a key conserved quantity in feedback design. Improving performance (e.g. increased bandwidth) comes at the expense of reduced robustness (peak in $|S|$) [2].

Trade-off degrades further if open loop is unstable [3].

Plant Uncertainty

A simplified model $P$ is used for control design.

Experimental frequency responses (blue) and simplified model (black).
Plant Uncertainty

A simplified model $P$ is used for control design.

- Actual dynamics are complex and have part-to-part variation.
- We lose model fidelity as we go to higher frequencies.

Experimental frequency responses (blue) and simplified model (black).
Stability Margins: Safety Factors for Control

An Approach:
1. Build an analysis model (possibly of high fidelity)
2. Assess the impact of parametric model errors, e.g. statistical sampling methods or classical gain/phase margins

Issue: Even high fidelity models fail to capture certain aspects of the dynamics, i.e. there are “unknown unknowns.”
Non-parametric (Dynamic) Uncertainty

Model *nominal* behavior with LTI system $G_0$.

Uncertainty modeled by LTI systems $\tilde{G}$ close to $G_0$ in frequency response, e.g. small additive error.

\[ |\tilde{G}(\omega) - G_0(\omega)| \leq \alpha(\omega) \]

Error Bound

\[ |E(\omega)| \leq \alpha(\omega) \]
Advanced Robustness Analysis

Move beyond classical SISO stability (gain/phase) margins

1. Multi-loop (MIMO) systems with multiple uncertainties

2. More detailed uncertainty descriptions including
   - Parametric,
   - Non-parametric (dynamic)
   - Nonlinearities, e.g. saturation

3. Consider both robust stability and robust performance

Developments go back to the Lur’e problem (40’s) with key contributions in the 80’s and 90’s:

- $\mu$: Safonov, Stein, Doyle, Packard, ...
- IQCs: Yakubovich, Megretski, Rantzer, ...
Numerical Algorithms and Software

Reliable software to create uncertainty models & perform analyses.

- Matlab’s Robust Control Toolbox (Safonov & Chiang), (Balas, Doyle, Glover, Packard, & Smith), (Gahinet, Nemirovski, Laub, & Chilali)
- ONERA’s Systems Modeling, Analysis and Control Toolbox (Biannic, Burlion, Demourant, Ferreres, Hardier, Loquen, & Roos)

Example Matlab code to assess robustness of simple feedback loop.

```matlab
% Unstable plant with parametric uncertainty
a = ureal('a',1, 'Range', [0.8 1.1]);
b = ureal('b',2, 'Range', [1.7 2.6]);
P = tf(b, [1 -a]);

% Actuator with non-parametric (dynamic) unc.
nomAct = tf(10, [1 10]);
DeltaE = ultidyn('DeltaE',[1 1]);
A = nomAct + 0.1*DeltaE;

% Uncertain closed-loop (d->e) with PI control
C = tf([3 4.5],[1 0]);
R = feedback(-P, A*C);

% Robust stability and worst-case gain
[StabMargin, DestabilizingUncert] = robstab(R);
[wcGain, OffendingUncertainty] = wcgain(R);
```
Numerical Algorithms and Software

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Numerical algorithms continue to be developed, e.g. in Matlab:

- Structured $H_\infty$ (R2010b) and systune (R2014a): Based on work by (Gahinet, Apkarian, Noll)
- $\mu$ without frequency gridding (R2016b): (Gahinet, Balas, Packard, Seiler) and (Biannic, Ferreres, Roos)
- Automatic regularization for $H_2$ (R2017b) and $H_\infty$ synthesis (R2018b): (Gahinet, Packard, Seiler)
- Multi-loop disk margins (R2018b): (Gahinet, Packard, Seiler)
(My) Theoretical Contributions to Robust Control

Fundamental limits in vehicle platoons
(Seiler, Pant, Hedrick)

Networked Control
(Seiler & Sengupta)

Robustness of Linear Parameter Varying Systems using IQCs
(Seiler, Pfifer, Wang, & Venkataraman)
(My) Applications of Robust Control

- 787 Flight Control Electronics
- Wind farm modeling and control (Annoni ‘16, Singh, Hoyt)
- Individual turbine control (Wang ‘16, Ossmann, Theis)
- UAV control with a single aerodynamic surface (Venkataraman ‘18)
- Flexible aircraft (Kotikalpudi ’17, Theis ’18, Gupta, Pfifer)
- Dual stage hard disk drives with Seagate (Honda ‘16)

(Years refer to Ph.D. theses.)
Outline

• Brief Overview of Robust Control

• Robustness of Time-Varying Systems
  • Joint work with M. Arcak, A. Packard, M. Moore, and C. Meissen at UC, Berkeley + Jyot Buch at Minnesota.
  • Funded by ONR BRC with B. Holm-Hansen at Tech. Monitor

• Future Directions

• Conclusions
Time-Varying Systems

Wind Turbine
Periodic / Parameter-Varying

Flexible Aircraft
Parameter-Varying

Vega Launcher
Time-Varying
(Source: ESA)

Robotics
Time-Varying
(Source: ReWalk)

Few numerically reliable methods to assess the robustness of time-varying systems.
(Robust) Finite-Horizon Analysis

Uncertain LTV System

\[
\begin{bmatrix}
\dot{x}(t) \\
v(t) \\
e(t)
\end{bmatrix} =
\begin{bmatrix}
A(t) & B_1(t) & B_2(t) \\
C_1(t) & D_1(t) & D_2(t) \\
C_2(t) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
w(t) \\
d(t)
\end{bmatrix}
\]

\[x(0) = 0\]

Uncertainty set \(\Delta\) can be block-structured with parametric / non-parametric uncertainties and nonlinearities.

Analysis Objective

Derive bound on \(\|e(T)\|_2\) that holds for all disturbances \(\|d\|_2,[0,T] \leq 1\) and uncertainties \(\Delta \in \Delta\) on the horizon \([0,T]\).
Integral Quadratic Constraints (IQC) [1,2]

The robustness analysis uses constraints on the I/O behavior of $\Delta$ expressed as (time-domain) IQCs.

Definition

$\Delta$ satisfies the IQC on $[0, T]$ defined by a stable filter $\Psi$ and matrix $M$ if:

$$\int_0^T z(t)^T M z(t) \, dt \geq 0 \quad \forall v \in L_2[0, T] \text{ and } w = \Delta(v)$$

Example: Sector-bounded Nonlinearity

$\Delta$ is a sector-bounded nonlinearity, $f$.

$$(w(t) - \alpha v(t)) \cdot (\beta v(t) - w(t)) \geq 0 \quad \forall t$$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0 \quad \forall t$$
Example: Sector-bounded Nonlinearity

\[ \Delta \text{ is a sector-bounded nonlinearity, } f. \]
\[ (w(t) - \alpha v(t)) \cdot (\beta v(t) - w(t)) \geq 0 \quad \forall t \]

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\[ := M \quad := z(t) \]

\[ \int_0^T z(t)^T M z(t) \, dt \geq 0 \]

\[ \Delta \text{ satisfies the IQC on } [0, T] \]
\[ \text{defined by } \Psi := I_2 \text{ and } M. \]
Example: Non-parametric (Dynamic) Uncertainty

\[ \Delta \text{ is stable, LTI with } \| \Delta \|_\infty := \sup_\omega |\Delta(\omega)| \leq 1 \]
Example: Non-parametric (Dynamic) Uncertainty [1]

\[ \Delta \text{ is stable, LTI with} \]
\[ \| \Delta \|_{\infty} := \sup_{\omega} |\Delta(\omega)| \leq 1 \]

For any \( D(s) \)

\[ \int_{0}^{T} z(t)^{T} M z(t) \, dt \geq 0 \]

\( \Delta \) satisfies the IQC on \([0, T]\) defined by
\( \Psi := \text{diag}(D, D) \) and \( M := \text{diag}(1, -1) \)

Additional IQC Details

• A dictionary of additional IQC for various uncertainties / nonlinearities is given in [1].
  • IQCs for passive operators, static memoryless nonlinearities (Popov, Zames-Falb), time-delays, real parameters, etc.
  • Many IQCs are specified in the frequency domain
• Most IQCs are related to previous robust stability results
  • IQC for sector nonlinearities related to the circle criterion
  • IQC for LTI uncertainties related to D-scales in $\mu$ analysis
• A technical J-spectral factorization result can be used to convert freq. domain IQCs into time-domain IQCs [2,3].

Robustness Analysis

The robustness analysis is performed on the extended (LTV) system of \((M, \Psi)\) using the constraint on \(z\).

\[
\begin{bmatrix}
\dot{x}_e(t) \\
z(t) \\
e(t)
\end{bmatrix}
= 
\begin{bmatrix}
A(t) & B_1(t) & B_2(t) \\
C_1(t) & D_1(t) & D_2(t) \\
C_2(t) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_e(t) \\
w(t) \\
d(t)
\end{bmatrix}
\]

with
\[
\int_0^T z(t)^T M(t) z(t) \, dt \geq 0
\]
Robust Finite Horizon Analysis

**Theorem [1,2]**
Assume $\Delta$ satisfies the IQC defined by $(\Psi, M)$.
If there exists $P(\cdot) = P(\cdot)^T$ such that
(i) $P(T) = C_2(T)^T C_2(T)$, and
(ii) $V(x, t) := x^T P(t)x$ satisfies
\[
\frac{d}{dt} V(x, t) - \gamma^2 d(t)^T d(t) + z(t)^T M z(t) \leq 0 \quad \forall t \in [0, T]
\]
then $\|e(T)\|_2 \leq \gamma \|d\|_2, [0,T]$

**Proof**
Integrate dissipation inequality from $t = 0$ to $t = T$:
\[
\begin{align*}
V(x(T), T) - V(x(0), 0) &= e(T)^T e(T) - 0 - \gamma^2 \int_0^T d(t)^T d(t) dt + \int_0^T z(t)^T M z(t) dt \\
&\geq 0
\end{align*}
\]

**Robust Finite Horizon Analysis**

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$$

then $\|e(T)\|_2 \leq \gamma \|d\|_2, [0,T]$

Dissipation inequality can be recast as a differential LMI:

$$
\begin{bmatrix}
\dot{P} + A^T P + PA & PB_1 & B_2 \\
B_1^T P & 0 & 0 \\
B_2^T P & 0 & -\gamma^2 I
\end{bmatrix} + (\cdot)^T M \begin{bmatrix} C_1 & D_1 & D_2 \end{bmatrix} \leq 0
$$

$\forall t \in [0, T]$

Numerical Algorithms and Software

• Robustness Algorithms
  • Differential LMI can be “solved” via convex optimization using basis functions for $P(\cdot)$ and gridding on time [1].
  • A more efficient algorithm mixes the differential LMI and a related Riccati Differential Equation condition [2].
  • Similar methods developed for LPV [4,5] and periodic systems [6].

• LTVTools Software [3]
  • Time-varying state space system objects, e.g. obtained from Simulink snapshot linearizations.
  • Includes functions for nominal and robustness analyses.

[3] https://z.umn.edu/LTVTools
Nonlinear dynamics [MZS]:
\[ \dot{\eta} = f(\eta, \tau, d) \]
where
\[ \eta = [\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2]^T \]
\[ \tau = [\tau_1, \tau_2]^T \]
\[ d = [d_1, d_2]^T \]
\( \tau \) and \( d \) are control torques and disturbances at the link joints.

Nominal Trajectory in Cartesian Coordinates
Analysis

Nonlinear dynamics:

\[ \dot{\eta} = f(\eta, \tau, d) \]

Linearize along the finite-horizon trajectory \((\bar{\eta}, \bar{\tau}, d = 0)\)

\[ \dot{x} = A(t)x + B(t)u + B(t)d \]

Design finite-horizon state-feedback LQR gain.

Goal: Compute bound on the final position accounting for disturbances and LTI uncertainty \(\Delta\) at 2nd joint.
Monte-Carlo Simulations

LTV simulations with randomly sampled disturbances and uncertainties (overlaid on nominal trajectory).
Robustness Bound

Cyan disk is bound computed in 102 sec using IQC/DI method
Bound accounts for disturbances $\|d\|\leq 5$ and $\|\Delta\|\leq 0.8$
Randomly sample $\Delta$ to find “bad” perturbation and compute corresponding worst-case disturbance using method in [1].

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  • Robustness in Reinforcement Learning
  • Optimization as Robust Control

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“Model-Free” Reinforcement Learning

- **Goal:** Train a control policy from data to maximize a cumulative reward
  - Training data obtained from a simulator or the real system
  - Often assume state feedback
  - Many algorithms (Q-learning, value iteration, policy iteration, policy search) [1,2,3]
  - Algorithms have close connections to dynamic programming and optimal control.

Is Robustness an Issue in RL?

Training via simulation

Training on real system
Is Robustness an Issue in RL?

Training via simulation

• Training can exploit flaws in the simulator [1].
• Loss of performance transitioning from simulator to real system.

Training on real system

Is Robustness an Issue in RL?

Training via simulation
• Training can exploit flaws in the simulator [1].
• Loss of performance transitioning from simulator to real system.

Training on real system
• Part to part variation (train on one system and implement on many)
• Changes in system dynamics over time (temperature dependence, environmental effects, etc....)

Initial Investigations [1]

• Use linear optimal control problems to understand performance of RL techniques
  • RL provides most benefit for problems that can’t be addressed by standard system ID + linear optimal control
  • However, LTI problems can be used as “test” cases

• Develop (model-free) methods to recover robustness
  • Model uncertainty is different from process noise
  • What is the appropriate regularizer?

Linear Quadratic Gaussian (LQG)

Minimize

\[ J_{LQG}(u) := \lim_{N \to \infty} \frac{1}{N} E \left[ \sum_{t=0}^{N} x_t^T Q x_t + u_t^T R u_t \right] \]

Subject To:

\[ x_{t+1} = A x_t + B u_t + B_w w_t \]
\[ y_t = C x_t + v_t \]

The optimal controller has an observer/state-feedback form

\[ \hat{x}_{t+1} = A \hat{x}_t + B u_t + L (y_t - C \hat{x}_t) \]
\[ u_t = -K \hat{x}_t \]

Gains \((K,L)\) computed by solving two Riccati equations. This solution is model-based, i.e. it uses data \(A,B,C\), etc
Reinforcement Learning

- Partially Observable Markov Decision Processes (POMDPs)
  - Set of states, $S$
  - Set of actions, $A$
  - Reward function, $r : S \times A \rightarrow \mathbb{R}$
  - State transition probability, $T$
  - Set of observations and observation probability, $O$

- Many methods to synthesize a control policy from input/output data to maximize the cumulative reward

$$J_{RL}(a) := E \left[ \sum_{t=0}^{N} r(s_t, a_t) \right]$$

- The LQG problem is a special case of this RL formulation
Doyle’s Example (‘78 TAC)

- LQR state-feedback regulators have provably good margins.
- Doyle’s example shows that LQG (output-feedback) regulators can have arbitrarily small input margins.
Doyle’s Example (‘78 TAC)

- LQR state-feedback regulators have provably good margins.
- Doyle’s example shows that LQG regulators can have arbitrarily small input margins.
- Doyle’s example can also be solved within RL framework using direct policy search:

\[
\begin{align*}
    z_{t+1} &= A_K(\theta)z_t + B_K(\theta)y_t \\
    u_t &= C_K(\theta)z_t
\end{align*}
\]

where

\[
A_K(\theta) := \begin{bmatrix} 0 & \theta_1 \\ 1 & \theta_2 \end{bmatrix}, \quad B_K(\theta) := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_K^T(\theta) := \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}
\]

- RL will converge to the optimal LQG control with infinite data collection. Thus RL can also have poor margins.
What is an Appropriate Regularizer for Robustness?

- Increase process noise during training?
  - This causes margins to decrease on Doyle’s example
  - **Process noise is not model uncertainty**

- Modify reward to increase state penalty or decrease control penalty?
  - Again, this causes margins to decrease on Doyle’s example
  - Trading performance vs. robustness via the reward function can be difficult or counter-intuitive
Proposed Method to Recover Robustness

Inject synthetic gain/phase variations at the plant input (and output?) during the training phase.

\[ \Delta = 1 + \delta \] where \( \delta \) is \( U[-b,b] \).
Results On Doyle’s Example
Results on Simplified Flexible System

- Model has 4-states (Rigid body and lightly damped modes)
- RL applied to 3-state controller parameterization
  - LQG controller is not in the control policy parameterization
  - Still converges to policy with small margins
  - Robustness recovered with synthetic perturbations during training
Longer Term Goals/Questions

• Develop (model-free) methods to recover robustness. What is the appropriate regularizer?

• Understand how to merge lower level (model-based) control with higher level (model-free) methods.
  • What is an appropriate merging point?
  • What is a useful model abstraction for higher level (model-free) methods?

• Can we make any rigorous claims about the proposed method? Performance certification?

• What are fundamental performance limits on RL policies?
  • What will an RL-trained algorithm do for a fundamentally difficult problem, e.g. $G(s) = (s-1)/(s-2)$?
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First-order Optimization Algorithms

Assumptions on $f$

- Strongly convex ($m$)
- Lipschitz gradients ($L$)

$$\min_{x \in \mathbb{R}^n} f(x)$$

First-Order Algorithm

- Input: Gradient at iterate
- Output: Next iterate

**Gradient Descent**

$$x_{k+1} = x_k - \alpha \nabla f(x_k)$$

**Heavy-Ball**

$$x_{k+1} = x_k - \alpha \nabla f(x_k) + \beta (x_k - x_{k-1})$$

**Nesterov’s Method**

$$x_{k+1} = y_k - \alpha \nabla f(y_k)$$
$$y_k = (1 + \beta)x_k - \beta x_{k-1}$$
First-order Optimization as Robust Control [1]

Robust Control Perspective
- Uncertain plant, $\nabla f$
- Controller $G$ (algorithm) is finite-dim, strictly proper, LTI system

Automated Analysis with IQC/SDP
- Characterize $\nabla f$ with IQCs
- “Small LMIs” to certify convergence rate
- Analytical proofs guided by SDP solns.
- Extensions including algorithm design

Extension to Stochastic Optimization

Finite Sum Minimization

- Certain convexity/Lipschitz assumptions
- Application to empirical risk minimization, e.g. in supervised learning.

\[
\min_{x \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^{n} f^i(x)
\]
Extension to Stochastic Optimization

Finite Sum Minimization

- Certain convexity/Lipschitz assumptions
- Application to empirical risk minimization

\[
\min_{x \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^{n} f^i(x)
\]

Stochastic Gradient is widely used

- Fixed stepsize: Convergence to tolerance of optimal
- Decreasing stepsize: Sublinear convergence

Many recent methods (SAGA, Finito, SDCA) with linear convergence and similar iteration cost as SG.

**SAGA**

Randomly sample \( i_k \) at each step

\[
x_{k+1} = x_k - \alpha \left( \nabla f^i_k(x_k) - y^i_k + \frac{1}{n} \sum_{i=1}^{n} y^i_k \right)
\]

\[
y_{k+1}^i := \begin{cases} 
\nabla f^i(x_k) & \text{if } i = i_k \\
y^i_k & \text{else}
\end{cases}
\]
Extension to Stochastic Optimization [1,2]

Express stochastic optimization techniques with:

- Uncertain plant, $\nabla f$
- Markov Jump System representation for optimization algorithm

Automated Analysis with IQC/SDP

- Characterize $\nabla f$ with IQCs
- “Small” SDPs to certify convergence-rate
- Analytical proofs guided by SDP solns.

\[ \eta_{k+1} = A^i_k \eta_k + B^i_k y_k \]
\[ u_k = C \eta_k \]

\[ y_k = \begin{bmatrix} \nabla f^1(u_k) \\ \vdots \\ \nabla f^n(u_k) \end{bmatrix} \]


Longer Term Goals/Questions

• Determine if finite horizon analysis analysis tools can be used to assess convergence rates.
  • Related work on finite horizon Performance Estimation Problem (PEP) [1]
• Are IQC rate bounds tight for strictly convex, Lipschitz bounded functions? If no, then for what class are they tight?
  • Initial results prove tightness for stability boundary but likely not true in general [2].
• Can methods from robust synthesis be used to design algorithms with faster convergence?
  • IQC synthesis is non-convex so this would require some heuristics
  • Would require new robust synthesis methods for jump systems.

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Conclusions

- Robust control has a long history with many successes
  1. Multivariable Optimal Control
  2. Fundamental Limitations of Dynamics & Control
  3. Uncertainty Modeling and Robustness Analysis
- Robust control techniques can solve emerging problems
  1. Robustness in controls designed via data-driven (RL) methods
  2. Optimization as Robust Control
- Acknowledgements:
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https://www.aem.umn.edu/~SeilerControl/
Backup Slides
Typical $S+T=1$ Tradeoff

Large loop gain $|PC|$:  
Good reference tracking  
Poor noise rejection  

Small loop gain $|PC|$:  
Poor reference tracking  
Good noise rejection  

\[ P = \frac{1}{s+0.1} \]  
\[ C = \frac{s+1.5}{s} \]  
\[ S = \frac{1}{1+PC} \]  
\[ T = \frac{PC}{1+PC} \]
Typical $S+T=1$ Tradeoff

Crossover Region:
Poor reference tracking AND Poor noise rejection

Large loop gain $|PC|:$
Good reference tracking
Poor noise rejection

Small loop gain $|PC|:$
Poor reference tracking
Good noise rejection

$P = \frac{1}{s+0.1}$
$C = \frac{s+1.5}{s}$
$S = \frac{1}{1+PC}$
$T = \frac{PC}{1+PC}$
Typical Sensitivity Objectives

- **Performance:** “Small” $|S|$ up to 0 dB bandwidth $\Omega_s$
- **Robustness:** $|S| \leq 2$ (=6dB) at all frequencies (No Peaks)

Typical sensitivity response (red) and design objectives (black)
Bode Integral Theorem [1,2]

Assume $PC$ has relative degree 2 and $S(s)$ is stable. Then:

$$ \int_{0}^{\infty} \ln |S(j\omega)| \, d\omega = \pi \sum_{k=1}^{N_u} \text{Re}(p_k) \geq 0 $$

where $p_k$ are the unstable (RHP) poles of $PC$.

(Note: $|S|$ (dB) $\approx$ 8.7 ln $|S|$)

Bode Integral Theorem [1,2]

Assume $PC$ has relative degree 2 and $S(s)$ is stable. Then:

$$\int_0^\infty \ln |S(j\omega)| \, d\omega = \pi \sum_{k=1}^{N_u} \Re(p_k) \geq 0$$

where $p_k$ are the unstable (RHP) poles of $PC$.

(Note: $|S|$ (dB) $\approx 8.7 \ln |S|$)

This a key conserved quantity in feedback design. Improving performance (e.g. increased bandwidth) comes at the expense of reduced robustness (peak in $|S|$) [3].

Bode Integral Theorem and “Peaking”

A procedure to avoid peaking could be:

• Obtain significant Sensitivity reduction over \([0, \Omega_s]\).
  
  This incurs a large negative integral which must be balanced.

• Maintain \(|S(j\omega)|\) slightly larger than 1 over a wide interval.
  
  This incurs a positive integral balancing the negative integral.

• Make \(|PC|\) approach 0 quickly at higher frequencies so that \(|S|\) quickly approaches 1.

Accumulate a large area by having \(|S|\) just exceed 1 over a large frequency range.
Available Bandwidth

The Bode Integral theorem may appear to be a minor constraint, e.g. spreading area over a large frequency band.

Stein (‘89 Bode Lecture, ’03 CSM):

*a key fact about physical systems is that they do not exhibit good frequency response fidelity beyond a certain bandwidth. ... Let us call that bandwidth the “available bandwidth,” $\Omega_a$*

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Available Bandwidth

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*a key fact about physical systems is that they do not exhibit good frequency response fidelity beyond a certain bandwidth. ... Let us call that bandwidth the “available bandwidth,” \( \Omega_a \)*

The available bandwidth due to physical (hardware) constraints requires positive area be accumulated over a finite frequency band. **Consequence:** Improving performance (e.g. increased bandwidth) comes at the expense of reduced robustness (peak in \(|S|\)).
Consequence of Available Bandwidth

$|PC|$ must roll-off quickly above $\Omega_a$

\[ \int_0^\infty \ln |S(j\omega)| \, d\omega = \pi p \quad \text{roughly} \quad \Omega_S \leq \frac{\Omega_a \ln M - \pi p}{M} \]

*Performance is constrained by the Bode integral and robustness requirements.*
Stability Margins: Safety Factors for Control

Classical Margins: Largest gain/phase variations that can be tolerated before closed-loop instability occurs.

- Gain: $\alpha P$ where $\alpha$ varies from its nominal $\alpha_{nom}=1$
- Phase: $e^{j\theta}P$ where $\theta$ varies from its nominal $\theta_{nom}=0$
Uncertainty Modeling

Consider SISO feedback system:

- Unstable plant with uncertain pole and input gain: 
  \[ P(s) = \frac{b}{s-a} \text{ where } a \in [0.8, 1.1] \text{ and } b \in [1.7, 2.6] \]

- First-order actuator with additive dynamic uncertainty 
  \[ A(s) = A_0(s) + E(s) \text{ where } A_0(s) = \frac{10}{s+10} \& |E(\omega)| \leq 0.1, E \text{ stable} \]

- Proportional-Integral control 
  \[ C(s) = \frac{3s+4.5}{s} \]
Uncertainty Modeling

Separate known from the uncertain

Uncertainty is typically very structured
Uncertainty Modeling

Re-center and re-scale to normalize the uncertainties

\[ e \rightarrow C \rightarrow q_c \rightarrow A \rightarrow q \rightarrow u \rightarrow P \rightarrow y \]

\[ \begin{pmatrix} \delta_a \\ \delta_b \\ \Delta_E \end{pmatrix} \]

Uncertainty set is structured:

\[ \Delta := \{ \text{diag}(\delta_a, \delta_b, \Delta_E) : \delta_a, \delta_b \in \mathbb{R} \text{ and } \Delta_E \text{ LTI, stable} \} \]

where:

1. \( \Delta=0 \) gives nominal behavior
2. Range of modeled uncertainty is

\[ \| \Delta \|_{\infty} := \sup_{\omega} \vartheta(\Delta) \leq 1 \]
Robustness Metrics

**Stability Margin:** $\kappa_m := \inf_{\Delta \in \Delta} \| \Delta \|_\infty$

s.t. $\Delta$ causes instability

**Worst-case Gain:** $\sup_{\Delta \in \Delta, \| \Delta \|_\infty \leq 1} \| T_{d \rightarrow e}(M, \Delta) \|_\infty$

**Comments:**

- System is robustly stable if and only if $\kappa_m > 1$.
- Both metrics can be converted to a (freq. domain) $\mu$ test.
- Algorithms compute bounds that provide guarantees on performance and bad instances of uncertainties.
- IQCs extend the framework to include nonlinearities.
(Nominal) Finite Horizon Analysis

Nominal LTV System

\[
\begin{bmatrix}
\dot{x}(t) \\
e(t)
\end{bmatrix}
= \begin{bmatrix}
A(t) & B(t) \\
C(t) & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
d(t)
\end{bmatrix}
\]

\[x(0) = 0\]

Analysis Objective

Derive bound on \(\|e(T)\|_2\) that holds for all disturbances \(\|d\|_2, [0, T] \leq 1\) on the horizon \([0, T]\).
Nominal Analysis with Dissipation Inequalities

**Theorem [1,2]**
If there exists $P(\cdot) = P(\cdot)^T$ such that

(i) $P(T) = C(T)^T C(T)$, and

(ii) $V(x, t) := x^T P(t) x$ satisfies

$$\frac{d}{dt} V(x, t) - \gamma^2 d(t)^T d(t) \leq 0 \quad \forall t \in [0, T]$$

then $\|e(T)\|_2 \leq \gamma \|d\|_2, [0, T]$

**Proof**
Integrate dissipation inequality from $t = 0$ to $t = T$:

$$\underbrace{V(x(T), T) - V(x(0), 0)}_{= e(T)^T e(T)} - \gamma^2 \int_0^T d(t)^T d(t) dt \leq 0$$

Nominal Analysis with Dissipation Inequalities

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Dissipation inequality can be recast as a differential LMI:

$$
\begin{bmatrix}
\dot{P} + A^T P + PA & PB \\
BT P & -\gamma^2 I
\end{bmatrix} \leq 0 \quad \forall t \in [0, T]
$$

Nominal Analysis with Dissipation Inequalities

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Comments
• The dissipation inequality is equivalent to Riccati conditions [3] but enables extensions to robustness analysis.
• **Numerically reliable algorithm to construct worst-case disturbance [4].**

Example: Non-parametric (Dynamic) Uncertainty

\[ v \xrightarrow{\Delta} w \]

\( \Delta \) is stable, LTI with
\[ \| \Delta \|_\infty := \sup_\omega |\Delta(\omega)| \leq 1 \]

\[ |\hat{w}(\omega)| \leq |\hat{v}(\omega)| \quad \forall \omega \]

\[ \int_{-\infty}^{\infty} X(\omega) \left[ |\hat{v}(\omega)|^2 - |\hat{w}(\omega)|^2 \right] d\omega \geq 0 \]
for any \( X(\omega) \geq 0 \)

**Frequency-Domain IQC**

\[ \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -X(\omega) \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} d\omega \geq 0 \]
for any \( X(\omega) \geq 0 \)

**Spectral Factorization**

\[ X(\omega) = D(\omega)^* D(\omega) \]

where
\[ \Psi := \text{diag}(D, D) \text{ and } M := \text{diag}(1, -1) \]
Example: Non-parametric (Dynamic) Uncertainty

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\[ \int_{-\infty}^{\infty} X(\omega) \left[ |\hat{v}(\omega)|^2 - |\hat{w}(\omega)|^2 \right] d\omega \geq 0 \]

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for any \( X(\omega) \geq 0 \)

(Invoke Causality)

\[ \int_{0}^{T} z(t)^T M z(t) \, dt \geq 0 \]

\[ \Delta \text{ satisfies the IQC on } [0, T] \text{ defined by} \]

\[ \Psi := \text{diag}(D, D) \text{ and } M := \text{diag}(1, -1) \]

Closed-Loop Robust L2-to-Euclidean Gain

Two Controllers:

- Finite-Horizon LQR with state feedback
- Output Feedback using high pass filter $\pi s / (\pi s + 1)$ to estimate angular rates

Finite horizon robustness is degraded by output feedback with rate estimates.
Closed-Loop Robust L2-to-Euclidean Gain

Two Controllers:

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Closed-Loop Robust L2-to-Euclidean Gain

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Impact of Using High Pass Rate Estimator

Worst-Case Robust Gain Plot

Closed-Loop with $||d|| \leq 5$
Partial Dictionary of IQCs [1]

<table>
<thead>
<tr>
<th>Uncertainty</th>
<th>IQC Multiplier</th>
</tr>
</thead>
</table>
| 1. Passive                                       | \[
\begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix}
\] |
| 2. Norm-bounded LTI                               | \[
\begin{bmatrix}
X(j\omega) & 0 \\
0 & -X(j\omega)
\end{bmatrix}
\] where $X(j\omega) \geq 0$ |
| 3. Constant Real Parameter                       | \[
\begin{bmatrix}
X(j\omega) & Y(j\omega) \\
Y(j\omega)^* & -X(j\omega)
\end{bmatrix}
\] where $X(j\omega) \geq 0$ and $Y(j\omega) = -Y(j\omega)^*$ |
| 4. Varying Real Parameter                        | \[
\begin{bmatrix}
X & Y \\
Y^T & -X
\end{bmatrix}
\] where $X \geq 0$ and $Y = -Y^T$. |
| 5. Unit Saturation                               | \[
\begin{bmatrix}
0 & 1 + H(j\omega) \\
1 + H(j\omega)^* & -2(1 + ReH(j\omega))
\end{bmatrix}
\] where $\|h\|_1 \leq 1$. |

LTV Toolchain

% Matlab snapshot linearizations
% along nominal trajectory
io(1) = linio('TwoLinkRobotOL/Input Torque',1,'input');
io(2) = linio('TwoLinkRobotOL/Two Link Robot Arm',1,'output');
sys = linearize('TwoLinkRobotOL',io,Tgrid);

% Construction of LTV Model
G = tvss(sys,Tgrid);
Summary: Recovering Robustness in RL

- Robustness issues can arise in output-feedback controllers trained by RL [2]
  - Linear Quadratic Gaussian (LQG) Control can be solved via RL
  - A well-known counterexample by Doyle [1] demonstrates that LQG controllers can have arbitrarily small margins.

Summary: Recovering Robustness in RL

• Robustness issues can arise in output-feedback controllers trained by RL [2]
  • Linear Quadratic Gaussian (LQG) Control can be solved via RL
  • A well-known counterexample by Doyle [1] demonstrates that LQG controllers can have arbitrarily small margins.

• Robustness can be recovered by introducing (synthetic) input perturbations during the RL training [2].