Model Order Reduction by Parameter-Varying Oblique Projection

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Abstract—A method to reduce the dynamic order of linear parameter-varying (LPV) systems in grid representation is developed in this paper. It approximates balancing and truncation by an oblique projection onto a dominant subspace. The approach is novel in its use of a parameter-varying kernel to define the direction of this projection. Parameter-varying state transformations in general lead to parameter rate dependence in the model. The proposed projection avoids this dependence and maintains a consistent state space basis for the reduced-order system. The method is compared with LPV balancing and truncation for a nonlinear mass-spring-damper system. It is shown to yield similar accuracy, while the required computation time is reduced by a factor of almost 100,000.

I. INTRODUCTION

Linear parameter-varying (LPV) models are particularly useful for the design of gain-scheduled controllers due to the availability of powerful synthesis techniques and computational tools [1], [2]. These techniques produce controllers that are at least of the same dynamic order as the plant model. One limitation is that the synthesis requires the solution of linear matrix inequalities (LMIs). The computation required for this synthesis grows rapidly with increasing state dimension. For many physically motivated models, directly obtaining models with a low number of states is not easy. For instance, structural mechanics models are often obtained from finite element analysis with a dense grid of nodes and hence these models have a large number of states. Similarly, unsteady aerodynamic models can be recovered from the projection. This is a crucial feature for simulation and control if the LPV model is meant to approximate a nonlinear dynamic system. The effectiveness of the approach is compared to LPV balancing and truncation in Section IV on a nonlinear spring-mass-damper system with 100 states. The proposed method is shown to achieve similar accuracy with a major reduction in computational effort. It is further shown to accurately reduce a system with 1,000 states, where balancing becomes computationally intractable.

II. LPV MODEL ORDER REDUCTION

In this section, LPV systems are introduced and the model order reduction problem is formulated. The problems arising from parameter-varying transformations are highlighted, and it is shown how model order reduction can be stated as an oblique projection.

A. LPV Modeling and Reduced Order Models

LPV systems are dynamic systems whose state space matrices are continuous functions of a time-varying parameter
vector \( \rho(t) \in \mathbb{R}^{n_\rho} \). Based on physical considerations, the admissible parameter trajectories are confined to a compact set \( \mathcal{P} \subset \mathbb{R}^{n_\rho} \). This infinite dimensional set is commonly approximated by a finite dimensional subset \( \{ \rho_k \}_{k=1}^\infty \subset \mathcal{P} \), called a grid. The state space equations for an LPV system with state vector \( x(t) \in \mathbb{R}^{n_x} \) and input vector \( u(t) \in \mathbb{R}^{n_u} \) are

\[
\dot{x}(t) = A(\rho(t)) x(t) + B(\rho(t)) u(t) - \frac{d}{dt} \bar{x}(\rho(t)) \quad (1)
\]

\[
y(t) = C(\rho(t)) x(t) + D(\rho(t)) u(t),
\]

where the term \( \frac{d}{dt} \bar{x}(\rho(t)) \) is included to allow a parameter-varying equilibrium point \( \bar{x}(\rho(t)) \). Such a term naturally arises if an LPV is obtained as the linearization of a nonlinear system with respect to a parameter-varying trim condition, see [14] for details. For a constant \( \bar{x} \), the notion commonly encountered in the literature is recovered.

The problem of LPV model order reduction consists of finding an approximation for \( (1) \) as

\[
\dot{z}(t) = A_{\text{red}}(\rho(t)) z(t) + B_{\text{red}}(\rho(t)) u(t) - \frac{d}{dt} \bar{z}(\rho(t)) \quad (2)
\]

\[
y(t) = C_{\text{red}}(\rho(t)) z(t) + D_{\text{red}}(\rho(t)) u(t).
\]

The reduced state \( z(t) \in \mathbb{R}^{n_z} \) should be of much lower dimension than \( x(t) \in \mathbb{R}^{n_x} \), while the input-output behavior from \( u \) to \( y \) should be as similar as possible to that of the original model. Further, stability of the original model should be preserved in the reduced-order model. Finally, the equilibrium \( \bar{x} \) needs to be sufficiently well approximated by \( \bar{z} \) so that the results can be related back to the original nonlinear system.

In the remainder of this paper, time dependence is dropped and parameter dependence is denoted by the subscript \( \rho \), i.e., \( A_\rho := A(\rho(t)) \).

**B. Balancing and Truncation for LTI Model Reduction**

For a fixed parameter \( \rho = \rho_k \), system \( (1) \) simplifies to the standard LTI system

\[
\dot{x} = Ax + Bu \\
y = Cx + Du.
\]

A standard model reduction method for LTI systems is balancing and truncation [5]. The first step is obtaining the controllability Gramian \( X_c \) and the observability Gramian \( X_o \) as solutions to the Lyapunov equations

\[
AX_c + X_c A^T + BB^T = 0, \quad (4)
\]

\[
A^TX_o + X_o A + CCT = 0. \quad (5)
\]

Given a state \( x_0 \), the minimum energy required to steer the system from \( x = 0 \) to \( x = x_0 \) is \( \epsilon_c = \frac{1}{2} X_c^{-1} x_0 \). Further, \( \epsilon_o = \frac{1}{2} X_o x_0 \) is the energy of the free response to the initial condition \( x_0 \) [5]. The ratio \( \epsilon_c/\epsilon_o \) thus measures how much a state is affected by the input and how much it affects the output. A transformation \( [x_1 \ x_2] = T x \) can be calculated so that \( TX_cT^T = (T^{-1})^T X_c T^{-1} = \Sigma_H \), where \( \Sigma_H \) is diagonal and contains the eigenvalues of the product \( X_c X_o \) in descending order of magnitude. These singular values are exactly the ratios \( \epsilon_c/\epsilon_o \) for each state in the new coordinates. System \( (3) \) can hence be partitioned as

\[
\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u \\
\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_2 u \\
y = C_1 x_1 + C_2 x_2 + D u.
\]

The states that are both highly controllable and observable are represented by \( z := x_1 \). The states \( x_2 \) contribute little to the input-output behavior and are removed from the state vector by truncation, leading to a reduced-order model

\[
\dot{z} = A_1 z + B_1 u \\
y = C_1 z + D u. \quad (7)
\]

**C. Balancing for LPV Models**

For parameter-varying systems as defined by \( (1) \), balancing was extended in [3] by introducing parameter-varying generalized Gramians \( X_{c,\rho} \) and \( X_{o,\rho} \) that satisfy the LMI

\[
-\frac{d}{dt} X_{c,\rho} + A_{\rho} X_{c,\rho} + X_{c,\rho} A_{\rho}^T + (B_{\rho} B_{\rho}^T) \prec 0, \quad (8)
\]

\[
\frac{d}{dt} X_{o,\rho} + A_{\rho}^T X_{o,\rho} + X_{o,\rho} A_{\rho} + (C_{\rho}^T C_{\rho}) \prec 0. \quad (9)
\]

Minimization of trace\( (X_{c,\rho} X_{o,\rho}) \) subject to \( (8) \) and \( (9) \) can be used to calculate a parameter-varying balancing transformation so that \( T_{\rho} X_{c,\rho} T_{\rho}^{-1} = (T_{\rho}^{-1})^T X_{o,\rho} T_{\rho}^{-1} = \Sigma_{H,\rho}^{1/2} \), see [4] for an iterative approach to this nonconvex problem. The diagonal matrix \( \Sigma_{H,\rho} \) in this case contains the parameter-varying eigenvalues of \( X_{c,\rho} X_{o,\rho} \) ordered by decreasing magnitude along its diagonal. Such a transformation implies

\[
\frac{d}{dt} \tilde{z}_1 = \frac{d}{dt} T_{\rho} x - \frac{\partial T_{\rho}}{\partial \rho} \dot{\rho} x + T_{\rho} \dot{x} \quad (10)
\]

and consequently the resultant system

\[
\frac{d}{dt} \tilde{z}_2 = T_{\rho} A_{\rho} + \frac{\partial T_{\rho}}{\partial \rho} \dot{\rho} T_{\rho}^{-1} \tilde{z}_2 + T_{\rho} B_{\rho} u - T_{\rho} \frac{d}{dt} \bar{x}_{\rho} \quad (11)
\]

\[
y = C_{\rho} T_{\rho}^{-1} \tilde{z}_2 + D_{\rho} u,
\]

depends on the parameter rate \( \dot{\rho} \) in addition to the original parameter \( \rho \). A parameter-varying transformation thus inevitably increases the complexity of the model since the parameter space is enlarged. A reduced-order LPV model without additional rate dependence can be obtained as described in II-B only if solutions to \( (8) \) and \( (9) \) are restricted to parameter independent matrices. In this case, more conservative solutions are to be expected. Further, even such parameter independent solutions require extensive computational effort to be calculated by numerical methods.

**D. Projection Perspective on Model Order Reduction**

The truncation operation applied to turn \( (6) \) into \( (7) \) can be expressed as replacing \( [x_1 \ x_2] \) with \( [I_{nx} \ 0_{nx \times (n_x-n_z)}]^T x_1 \) and multiplying the state equation from the left by \( [I_{nz} \ 0_{nz \times (n_x-n_z)}] \). An equivalent representation of the reduced-order system \( (7) \) is thus

\[
\begin{align*}
\dot{z} &= W^T A_{\text{red}} z + W^T B_{\text{red}} u \\
y &= CV z + D u
\end{align*}
\]

\[
\begin{align*}
\tilde{z} &= W^T z \\
y &= W^T B u
\end{align*}
\]
with \( V = T^{-1} \begin{bmatrix} t_{x \rho} & 0_{n_x \times (n_z - n_x)} \end{bmatrix}^T \), \( W^T = \begin{bmatrix} t_{x \rho} & 0_{n_x \times (n_z - n_x)} \end{bmatrix} \). It is shown in this section that the reduced-order model (12) obtained from balancing and truncation is a Petrov-Galerkin approximation of the original system, i.e., an approximation obtained by oblique projection. Taking this perspective makes it possible to extend model order reduction by projection in Section III-A to LPV systems and to consequently construct an approximation to LPV balancing and truncation in Section III-B.

An oblique projection is a linear operation defined by a matrix \( \Pi = V (W^T V)^{-1} W^T \) with \( V \in \mathbb{R}^{n_x \times n_x} \), \( W \in \mathbb{R}^{n_z \times n_z} \) and \( \text{rank}(W^T V) = n_z \). Hence, \( \Pi \) is idempotent, i.e., \( \Pi^2 = \Pi \). It is completely characterized by its range space \( \text{span}(\Pi) = \text{span}(V) \) and its nullspace \( \ker(\Pi) = \text{span}(W^T \Pi) = \text{span}(W) \). This fact is easy to prove by replacing \( V \) and \( W \) with their respective thin QR-factorizations. A vector space is said to be projected along \( V \) if the subspace \( \text{span}(V) \subseteq \mathbb{R}^{n_x} \) to a lower dimensional subspace \( \text{span}(V) \subseteq \mathbb{R}^{n_z} \). Reference [15] shows that any projection can be parameterized by \( V \) and a symmetric positive definite matrix \( S \in \mathbb{R}^{n_x \times n_x} \) as \( \Pi = \Pi^T = \Pi^2 = \Pi^T \Pi = \Pi \). The desired approximation is hence given by \( x_{\text{approx}} = V z \), where \( z \) is the solution to (17).

The procedure is known as Petrov-Galerkin approximation, see e.g. [16], [17], [18]. The unique solution to (16) is

\[
\dot{z} = W^T A V z + W^T B u .
\]

There is a rich geometric interpretation for this projection, see [18] for details.

### III. Dominant Subspace Approximation by Parameter-Varying Oblique Projection

Section II-C revealed that a parameter-varying state transformation introduces an additional parameter rate dependence. The same is true if an oblique projection is constructed as in Section II-D from a parameter-varying transformation and the truncation operator. This section shows as the main result of this paper that it is possible to construct a parameter-varying projection that does not introduce rate dependence. It is then shown how a dominant subspace approximation for LPV systems is obtained.

#### A. Main Result: Parameter-Varying Oblique Projections

Constructing a reduced-order model for an LPV system essentially requires the same steps as in Section II-D. Replacing \( x \) in (1) with \( V_{\rho} z \) and left multiplying the resulting equation by \( W_{\rho}^T \) shows that any parameter-varying projection \( \Pi_{\rho} = V_{\rho} W_{\rho}^T \) with \( W_{\rho}^T V_{\rho} = I_{n_z} \) leads to a reduced-order model

\[
\dot{z} = W_{\rho}^T \left( A_{\rho} V_{\rho} - \frac{\partial V_{\rho}}{\partial \rho} \dot{\rho} \right) z + W_{\rho}^T B_{\rho} u - W_{\rho}^T \frac{d}{dt} (V_{\rho} \dot{z}_{\rho}) \quad y = C_{\rho} V_{\rho} z + D_{\rho} u .
\]

System (19) depends on \( V_{\rho} \) but not on the time derivative of \( W_{\rho} \). Hence, rate dependence can be avoided by restricting parameter dependence in the projection to the kernel. Such a parameter-varying oblique projection \( \Pi_{\rho} = V W_{\rho}^T \) is obtained from the parameterization (13) when only the symmetric matrix \( S \) is parameter dependent, i.e.,

\[
W_{\rho}^T = (V^T S_{\rho} V)^{-1} V^T S_{\rho} .
\]
Since $V$ is now constant and $W_p^T V = I_{n_z}$ still holds for all parameters, the projected state space equations (19) simplify to

$$
\begin{align*}
\dot{\bar{x}} &= \frac{A_{\bar{m},p}}{W_p^T} \bar{x}_p V z + \frac{B_{\bar{m},p}}{W_p^T} B_p u - \frac{d}{dt} \bar{x}_p \\
y &= C_p V \bar{x} + D_p u .
\end{align*}
$$

(21)

System (21) has exactly the structure of the desired reduced-order system (2). The key is that $V$ is constant. State consistency for the LPV system is hence preserved with both $x \approx V \bar{x}$ and $\dot{x} \approx V \dot{\bar{x}}$. The direction along which the full-order model is projected onto this constant subspace is, on the other hand, allowed to vary with the parameter.

From the perspective of a Petrov-Galerkin approximation, this is equivalent to enforcing (16) over a varying test space span($W_p$).

If the intent is to simulate or control a nonlinear system using the reduced-order LPV model, the equilibrium reduced-order state can be calculated by the Petrov-Galerkin approximation $W_p^T (V \dot{\bar{x}}_p - \bar{x}_p) = 0$, i.e., as $\bar{x}_p = W_p \bar{x}_p$.

B. Dominant Subspace Approximation

Recall from Section II-D that balancing and truncation can be interpreted as an oblique projection. Given the controllability and observability Gramians, either as solutions to (4) and (5) for LTI models or as parameter independent solutions that satisfy (8) and (9) for LPV models, this projection can be directly constructed from what is known as the square root algorithm [19]. Doing so requires the Cholesky factorizations $L_o L_o^T$ and $X_o = \bar{\Sigma}_o$, i.e., as $\bar{\Sigma}_o = L_o^T L_o$, as well as the singular value decomposition (SVD) of the product $L_o^T L_o$.

$$
L_o^T L_o = [U_1 \ U_2] \begin{bmatrix}
    \Sigma_1 \\
    \Sigma_2
\end{bmatrix} \begin{bmatrix}
    N_1 \\
    N_2
\end{bmatrix}^T .
$$

(22)

The singular values are ordered by descending magnitude, such that the diagonal matrix $\Sigma_1$ contains the largest $n_z$ singular values. The orthogonal matrices $[U_1 \ U_2]$ and $[N_1 \ N_2]$ contain the corresponding left and right singular vectors. The oblique projection for balancing and truncation is

$$
\Pi_{bal} = L_o U_1 \begin{bmatrix}
    \frac{1}{2} \Sigma_1^{-1/2} N_1^T L_o^T \\
    \frac{1}{2} \Sigma_2^{-1/2} N_2^T L_o^T
\end{bmatrix} .
$$

(23)

i.e., a projection onto span($L_o U_1$) along ker($N_1^T L_o$), see [17], [19] for details.

The goal of this section is to extend (23) to a parameter-varying projection as introduced in Section III-A. As a first step, projection (23) is rewritten in terms of the parameterization (13). Since $\bar{\Sigma}_1^{-1} = (\Sigma_1 N_1^T N_1 \Sigma_1)^{-1} \Sigma_1$,

$$
\Pi_{bal} = L_o U_1 (\Sigma_1 N_1^T L_o) \Sigma_1^{-1} \bar{\Sigma}_1^{-1} N_1^T L_o .
$$

It further follows from (22) that $\Sigma_1 N_1^T = U_1^T L_o^T L_o$ and thus

$$
\Pi_{bal} = L_o U_1 \left( U_1^T L_o^T L_o \right)^{-1} U_1^T L_o^T L_o L_o^T .
$$

Replacing finally $L_o U_1 = Q R$ by its thin QR-factorization and $L_o L_o^T$ by $X_o$ yields

$$
\Pi_{bal} = Q \left( Q^T X_o Q \right)^{-1} Q^T X_o .
$$

(24)

Equation (24) has the desired form (13) with $V = Q$ and $S = X_o$.

This suggests the following approximation of LPV balancing and truncation by a parameter-varying projection: Find a constant basis $\bar{Q}$ such that span($\bar{Q}$) approximates the parameter-varying dominant subspace span($L_o U_1$) and replace $X_o$ by the parameter-varying observability Gramian $X_o$, i.e.,

$$
\Pi_{bal} = \bar{Q} \left( \bar{Q}^T X_o, Q \right)^{-1} \bar{Q}^T X_o, Q .
$$

(25)

This method only requires calculation of the Cholesky factors $L_o U_1$, $L_o, X_o$, and the SVD (22) at each grid point $\rho_k$ over the grid $\{\rho_1, \ldots, \rho_g\}$. The matrix $L_o$ is then formed from interpolation of $L_o, L_o^T L_o, L_o^T X_o$. An approximation for span($L_o U_1$) could be obtained by adopting a prevalent approach from parametric model reduction to build a common basis from the SVD of the orthonormal bases calculated at each grid point [12]. That approach only considers the directions but not their individual importance. Thus, a direction that varies little or not at all and hence appears at all grid points would be given priority over a varying direction regardless how difficult it might be to reach. Instead, span($L_o U_1$) is approximated by $\bar{Q} = \bar{U}_1 \in \mathbb{R}^{n_z \times n_z}$ from the SVD

$$
\left[ L_o U_1, \ldots L_o U_1 \right] = \left[ \bar{U}_1 \ \bar{U}_2 \right] \begin{bmatrix}
    \bar{\Sigma}_1 \bar{N}_1^T \\
    \bar{\Sigma}_2 \bar{N}_2^T
\end{bmatrix} .
$$

(26)

The singular values $\bar{\Sigma}_1$ in (26) still provide a measure of how easy the subspace spanned by $\bar{U}_1$ is to reach. Thus, they provide a meaningful threshold to decide on the order of the approximation.

There are several noteworthy properties of projection (25). First, it results in a reduced-order LPV model (21) without additional rate dependence. Second, since $X_o$ is a continuous function of $\rho$, interpolation is accurate for a sufficiently dense grid. Further, $W_p^T V = I_{n_z}$ is exactly enforced by (25) for all parameter values regardless of the method used to interpolate $X_o$. Third, when applied to an LTI system, $\bar{Q}$ is simply an orthonormal basis for span($L_o, U_1$) and hence (25) exactly coincides with balancing and truncation, or more precisely, with the balancing-free square-root algorithm [20]. Finally, a result from Reference [15] is invoked to show that for frozen parameters, all poles of the reduced-order system are in the left half plane. Multiplying the Lyapunov equation (5) for the original system from the left by $\bar{Q}^T$ and from the right by $\bar{Q}$ results in

$$
\bar{Q}^T A^T \bar{Q} + \bar{Q} C^T \bar{Q} = 0 .
$$

(27)

Using $A_{red} = (\bar{Q}^T X_o \bar{Q})^{-1} \bar{Q}^T \bar{Q} A \bar{Q}$ and $C \bar{Q} = C_{red}$, it can be shown that (27) is equivalent to

$$
A_{red}^T (\bar{Q}^T X_o \bar{Q}) + (\bar{Q}^T X_o \bar{Q}) A_{red} = C_{red} C_{red} = 0 .
$$

(28)

Since $X_o$ is symmetric positive definite, so is $\bar{Q}^T X_o \bar{Q}$ and consequently $A_{red}$ has all its eigenvalues in the left half plane. This guarantees stability of the reduced-order model for “slowly” varying parameters, see [13] for details.
IV. Application Example

A nonlinear mass-spring-damper system, taken from [21], is used to demonstrate the approach. It represents the interconnection of \( M \) blocks with mass \( m = 1 \) kg, that are each connected both to their neighboring blocks and the initial system by a linear damper with damping constant \( d = \frac{1}{m} N_s \) and a nonlinear spring with stiffness \( k(q) = k_1 + k_2 q^2 \), \( k_1 = 0.5 \) N, \( k_2 = 1 \) N. An external force \( \rho \) and a controlled force \( u \) are acting on the \( M \)th block. The equations of motion for the \( i \)th block in terms of its displacement \( q_i \) are thus

\[
m \ddot{q}_i = \begin{cases} F_{1,i} - F_{1,2} & i = 1 \\ F_{i} - F_{i,i-1} - F_{i,i+1} & i = 2, \ldots, M - 1 \\ F_{M} - F_{M,M-1} + \rho + u & i = M. \end{cases}
\]

The force \( F_{i,j} = d(q_i - q_j) + k_1(q_i - q_j) + k_2(q_i - q_j)^2 \) is caused by the relative motion of neighboring blocks and \( F_i = d \dot{q}_i + k(q_i) q_i \) is due to the connection with the initial system. With state vector \( \xi := [q_1, \ldots, q_M, \dot{q}_1, \ldots, \dot{q}_M]^T \), the system is written as \( \dot{\xi} = f(\xi, u, \rho) \). The output of the system is the displacement \( q_M \) as \( h(\xi, u, \rho) \) of the \( M \)th block. For each value of \( \rho \in \mathcal{P} \), an equilibrium point \( \bar{x} \) is defined by \( 0 = f(\bar{x}(\rho), 0, \rho) \) with a corresponding equilibrium output \( \bar{q}_M = h(\bar{x}(\rho), 0, \rho) \). Linearization around \( (\bar{x}(\rho(t)), 0, \bar{q}_M(\rho(t))) \) yields an LPV system of the form (1) in the perturbation variables \( x := \xi - \bar{x}(\rho) \) and \( y = q_M - \bar{q}_M(\rho) \), see [14], [21] for details.

A. Comparison to LPV Balanced Truncation

The number of blocks for the mass-spring-damper model is selected as \( M = 50 \). The admissible parameter range is restricted to \( \mathcal{P} = [0, 2] \) and the nonlinear system is linearized on a grid \( \{\rho_k\}_{k=1}^{N} = \{0, 1, 2\} \). For this example, the proposed approach is compared to the standard LPV balancing and truncation method [4]. The function \texttt{lpvbalreal} of the LPVTools toolbox [2] is used to solve for a constant balancing transformation and parameter independent generalized Gramians by minimizing \texttt{trace}(X_c X_o) subject to (8) and (9). An oblique projection is then formed as described in Section II-D. Obtaining a reduced-order model with 4 states requires vastly different computational effort with the two methods. The proposed projection takes 0.1 seconds, whereas solving the LMIs for balancing takes more than 2.5 hours on a standard desktop computer. The resulting reduced-order models, on the other hand, are very similar. An upper bound \( \bar{\gamma} \) for the induced \( \mathcal{L}_2 \)-norm of the error system

\[
\begin{align*}
\dot{\tilde{x}} & = \begin{bmatrix} A_{\rho} & (I_n - \Pi_{\rho}) A_{\rho} V \\ 0_{n_s \times n_z} & W^T_{\rho} A_{\rho} V \end{bmatrix} \tilde{x} + \begin{bmatrix} (I_n - \Pi_{\rho}) B_{\rho} \\ \Pi_{\rho} B_{\rho} \end{bmatrix} u \\
0 & C_{\rho} \tilde{x} \\
\end{align*}
\]

where \( \tilde{x} = x - V z \) is calculated for both methods using the LPVTools function \texttt{lpvnorm}. This guarantees \( \|e\|_2 < \bar{\gamma} \|u\|_2 \) and certifies stability of the reduced-order system for arbitrary fast parameter variations. Balancing and truncation yields a slightly better error bound \( \bar{\gamma} = 3.9e-03 \) compared to \( \bar{\gamma} = 12.1e-03 \) for the proposed method. Additionally, the maximum \( \mathcal{H}_\infty \)-norm error \( \gamma \) for “frozen” parameters at all grid points is calculated as a lower bound for the induced \( \mathcal{L}_2 \)-norm and the frequency responses are shown in Fig. 1. The proposed method results in \( \gamma = 1.5e-03 \) compared to \( \gamma = 2.2e-03 \) for balancing. The time-domain responses for an external force \( \rho(t) = (1 + \cos(0.5 t)) \) N and a step input \( u = 0.5 \) N after 25 s are compared in Fig. 2. Both methods approximate the output of the nonlinear system very well and result in an identical mean square error of \( 1.5e-03 \).

Table I summarizes these results.

<table>
<thead>
<tr>
<th>LPV Balancing</th>
<th>Proposed Projection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computation Time(^\dagger)</td>
<td>9170 s</td>
</tr>
<tr>
<td>( \mathcal{L}_2 )-norm error bound</td>
<td>3.9e-03</td>
</tr>
<tr>
<td>( \mathcal{H}_\infty )-norm error(^\dagger)</td>
<td>2.2e-03</td>
</tr>
<tr>
<td>Output MSE</td>
<td>1.5e-03</td>
</tr>
</tbody>
</table>

\(^\dagger\) on a 64-bit desktop PC with 3.4 GHz 8-core CPU and 8 GB RAM
\(^\dagger\) for frozen parameters at grid points

![Frequency response at frozen parameters \( \rho = 0, 1, 2 \) for original model (---100 states) and reduced-order model from balancing and truncation (– -4 states) and parameter-varying projection (---4 states). The relative error is shown in the magnitude plot for balancing and truncation (-------) and parameter-varying projection (------).](image1)

![Nonlinear simulation of original model (---100 states), equilibrium output (•••), and reduced-order model from balancing and truncation (– -4 states) and parameter-varying projection (---4 states). The relative error is shown in the magnitude plot for balancing and truncation (-------) and parameter-varying projection (------).](image2)
system. Computation with the proposed approach, on the other hand, takes 47 seconds and results in a fifth order model. Since calculation of the $L_2$-norm error bound is also intractable, the “frozen” parameter frequency responses are used as surrogates and shown in Fig. 3. They again match very well for all grid points. The nonlinear simulation for an external force $\rho(t) = (5 + 5 \cos(0.5t))$ N and a step input $u = 0.5$ N after 25 s is shown in Figure 4 and also indicates excellent agreement.

![Frequency response at frozen parameters](image)

Fig. 3. Frequency response at frozen parameters $\rho = 1, \ldots, 10$ for original model (--- 1000 states), reduced-order model (- - - 5 states), and relative error (· · ·).

![Nonlinear simulation](image)

Fig. 4. Nonlinear simulation of original model (--- 1000 states), equilibrium output (- - -), and reduced-order model (—— 5 states).

V. CONCLUSION AND EXTENSIONS

A. Conclusion

A model order reduction method for LPV systems is developed in this paper. It is shown to approximate balancing and truncation within a fraction of its required computation time and is hence also applicable to systems where balancing and truncation becomes intractable.

B. Future Extensions

In case the Lyapunov equations become intractable to solve by standard means, low-rank approximations of Gramians can be used. Such approximations can be calculated even for large-scale systems with tens of thousands of states, e.g., by Krylov subspace methods [22]. The use of frequency-weighted or frequency-limited Gramians also lends itself well to the approach and could allow to specify a frequency region of interest. Finally, the basis space can be calculated by entirely different means. For instance, $V$ can be calculated by Krylov moment matching algorithms, while a parameter-varying $W_p$ is still obtained from the observability Gramian. This provides an extension of the SVD-Krylov algorithm proposed in [23] to LPV systems.

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