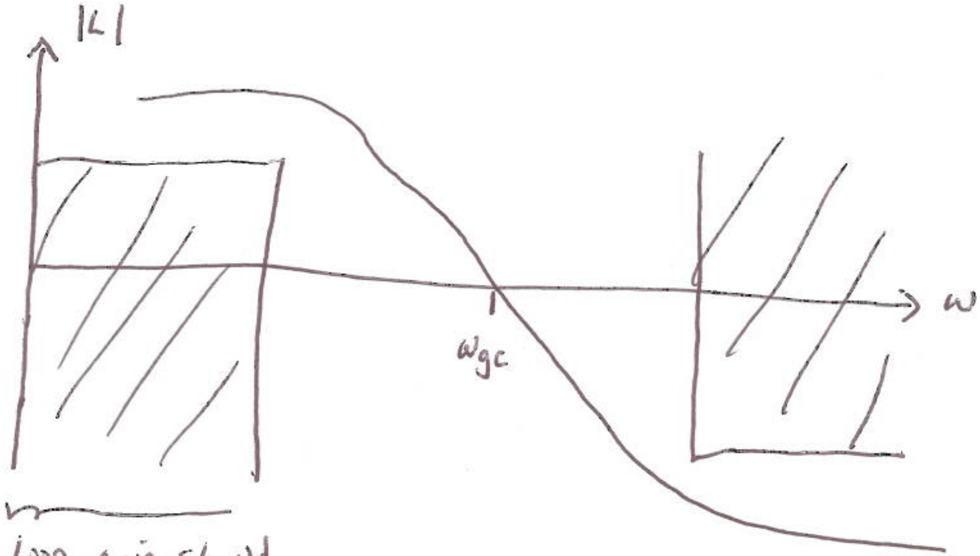
Lead Control

Up to now we've mainly focused on low/high frequency performance constraints within the loop-shaping framework:



Loop gain should be large at low frequencies for good tracking ($|S| \ll 1$)

Loop gain should be small at high frequencies for good noise rejection. ($|T| \ll 1$)

We mentioned that, as a rule of thumb, the slope of the loop gain should be > -30 dB/dec for approximately 1 decade of freq. surrounding the gain crossover freq ω_{cg} (i.e. for $\omega \in [\omega_{cg}/\sqrt{10}, \sqrt{10}\,\omega_{cg}]$)

This rule of thumb prevents us from creating a loop-gain that is too steep at cross-over.

We can now make the following connection between this rule-of-thumb and the robustness of the feedback system.

By the Bode gain/phase formula, if the slope of $|L(j\omega)| > -30\,\mathrm{dB/dec}$ then (assuming the poles and zeros of L are in the LHP) the accumulated phase of L satisfies

$$\measuredangle L(j\omega_{gc}) - \measuredangle L(0) > -135^\circ$$

If $\measuredangle L(0) = 0$ then we have $\measuredangle L(j\omega_c) > -135^\circ$

Thus the phase margin is at least 45°.

As we mentioned before, $\pm 45^\circ$ of phase margin is typically required for good robustness to time delays and other model variations.

In making this argument we made several assumptions:

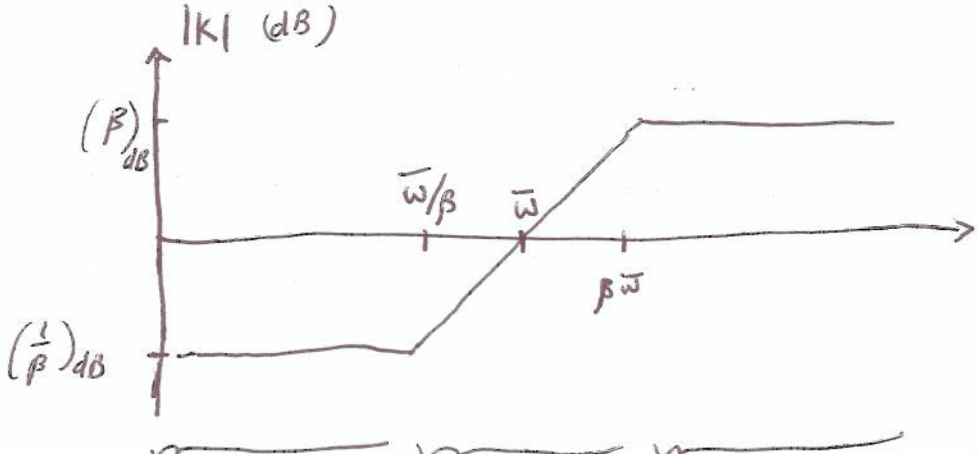
- $\measuredangle L(0) = 0$
- poles and zeros of L in the LHP.

The assumptions can be relaxed with a more detailed stability analysis. In general the same conclusion holds: <u>The slope of $|L|$ must be (approximately) $> -30\,\mathrm{dB/dec}$ near the cross-over frequency for good robustness.</u>

We can use a <u>lead controller</u> to increase the slope near cross-over and hence increase the phase margin.

Recall the form of a lead controller is:

$$K(s) = \frac{\beta s + \bar{\omega}}{s + \beta \bar{\omega}} \qquad \text{where } \beta > 1$$



At low frequencies ~~K(jω)~~ $K(j\omega) \approx 1/\beta$

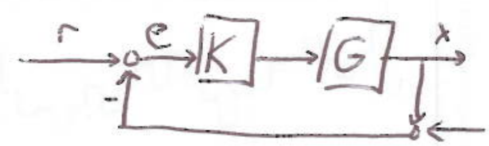
At high frequencies $K(j\omega) \approx \beta$

Approximate (straight-line) slope is $+20$ dB/dec

If β is large ($\beta > 10$) then $|K|$ will have approx. a slope of $+20$ dB/dec over a wide frequency range around $\bar{\omega}$. By the Bode gain/phase formula, it must have $+90°$ of phase near $\bar{\omega}$. Unfortunately if $\beta > 10$, ~~we~~ $|K|$ is ~~about~~ $< 1/10$ at low frequencies and > 10 at high frequencies. Generally this will degrade our low/high frequency performance as specified by requirements on $|L|$.

Thus we typically choose $\bar{\omega} = \omega_{cg}$ (where ω_{cg} is the cross-over freq. of the existing loop-shape) and β as small as possible to achieve the desired phase margins. There are precise formulas to compute the β to achieve a specific phase margin but it is just as easy to iterate by hand.

Ex) $G(s) = 1/s^2$



Requirements

1) Closed-loop is stable
2) $|S(j\omega)| \leq 0.01$ for $\omega < 0.1$ rad/sec [Good low-freq tracking]
3) $|T(j\omega)| \leq 0.01$ for $\omega > 10$ rad/sec [Good high freq noise rejection]
4) $45°$ of Phase Margin
5) Gain cross-over of $\omega_{gc} = 1$ rad/sec, i.e. $|L(j\omega_{gc})| = 1$

Convert 2) and 3) into requirements on L:

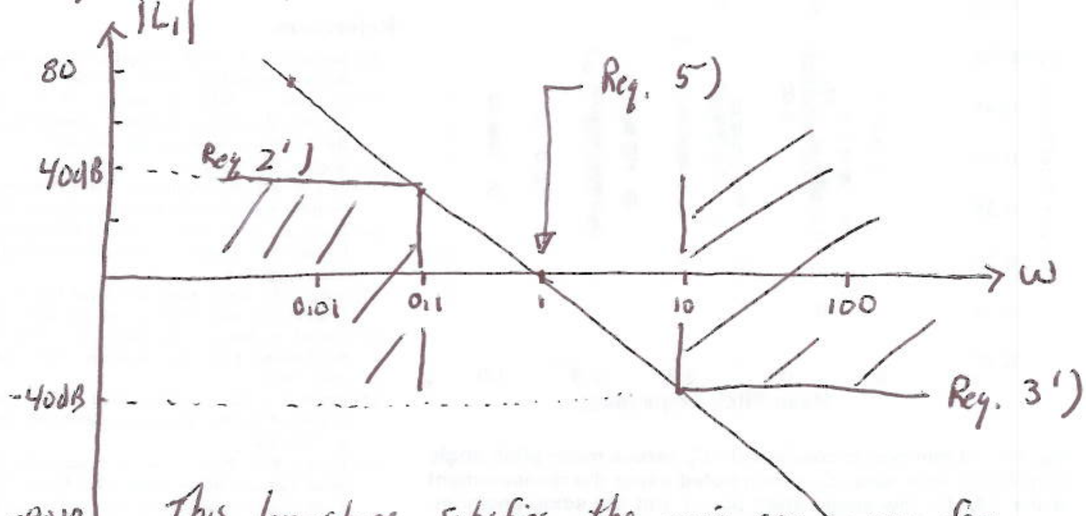
2') $|L(j\omega)| \geq 100$ for $\omega < 0.1$ rad/sec

[Recall that $S = 1/1+L$ so approximately $|S| < b << 1$ if $|L| > 1/b$]

3') $|L(j\omega)| \leq 0.01$ for $\omega > 10$ rad/sec

[Recall that $T = L/1+L$ so approximately $|T| < b << 1$ if $|L| < b$]

Start the design with the simplest possible controller: $K_1(s) = 1$
This is simply proportional control with gain $= 1$. The loop transfer function is $L_1(s) = G(s)K_1(s) = 1/s^2$.



This loop-shape satisfies the gain cross-over freq (Requirement 5) as well as the requirements for tracking and noise rejection (2' and 3')

Unfortunately the slope of $|L_1|$ is -40 dB/dec everywhere. ~~By the Bode~~ Moreover $\measuredangle L_1 = -180^\circ$ for all frequencies because $L_1(j\omega) = \frac{1}{(j\omega)^2} = \frac{-1}{\omega^2}$.

Thus the phase margin is 0° $\left[\begin{aligned} \text{Phase Margin} &= |180^\circ + \measuredangle L(j\omega_{gc})| \\ &= 180^\circ - 180^\circ = 0^\circ \end{aligned} \right]$

In addition the closed-loop is not stable. The closed-loop poles are located at the zeros of:

$$0 = 1 + L_1(s) = 1 + \frac{1}{s^2} = \frac{s^2+1}{s^2}$$

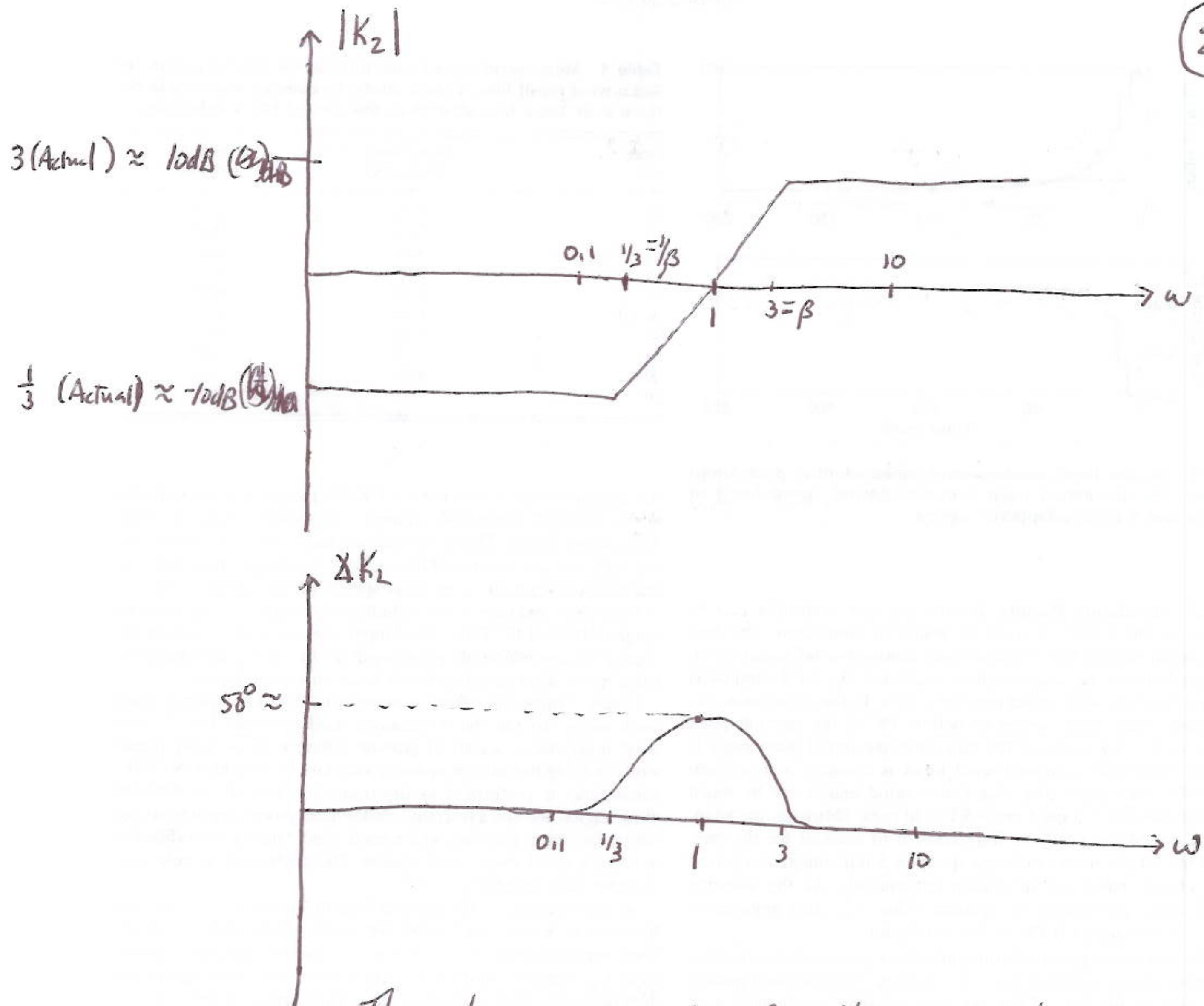
$\Rightarrow$ Closed-loop poles at $s = \pm j$

Notice that the poles are located on the imaginary axis at the frequency of 1 rad/sec. This is the ~~freq~~ critical frequency where the gain is $=1$, i.e. $\omega_{gc} = 1$ rad/sec.

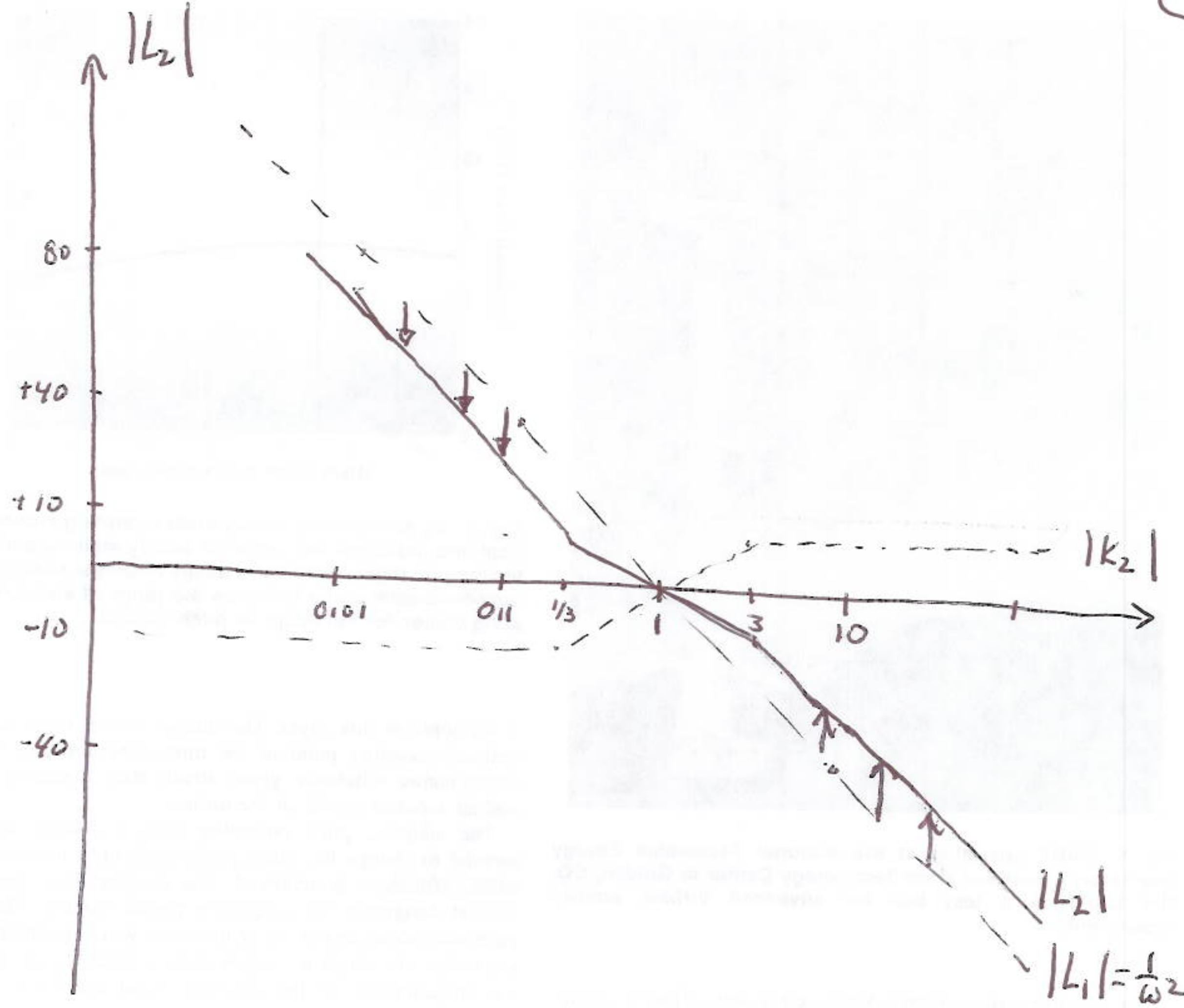
We can use a lead controller to increase the slope near cross-over and hence increase the phase margin. Choose the second stage as:

$$K_2(s) = \frac{\beta s + \bar{\omega}}{s + \beta \bar{\omega}}$$

We'll choose $\bar{\omega} = 1$ rad/sec since that is the gain cross-over frequency and $\beta \approx 3$. The choice of β requires a bit of trial and error but remember that we'd like to choose it as small as possible while still achieving the desired phase margin.



The larger we make β the more phase increase we'll get at $\omega_{cg} = 1$ rad/sec (with a maximum of $+90^\circ$ of phase increase). The drawback is the decrease in gain at low frequencies and increase in gain at high frequencies.



After including the lead control the loop shape is:

$$L_2 = G K_1 K_2 = \left(\frac{1}{s^2}\right)\left(\frac{3s+1}{s+1/3}\right)$$

The lead control:

a) Makes the slope of $|L_2|$ more "shallow" near ω_{gc} thus increasing the phase [and making the closed-loop stable]

but.. b) it causes the low frequency gain to be decreased and it increases the high frequency gain.

Requirements 2') and 3') are no longer satisfied due to the effects of the lead control. We would need one more iteration to increase the low freq gain [low freq. boost] and decrease the high freq. gain [rolloff]