1. A hyperelastic material is one for which a strain energy density (units of energy per unit of reference volume) function exists. Suppose this function is given by $W(E)$, where $E$ is the Lagrangian strain tensor. Then the material’s constitutive relation is given in terms of the Second Piola-Kirchhoff stress tensor $S$ by

$$S = \frac{\partial W}{\partial E} \leftrightarrow S_{IJ} = \frac{\partial W}{\partial E_{IJ}}.$$  

(1)

It is often convenient to linearize the constitutive relation and express it in terms of increments of $S$ and $E$ from a given configuration, represented by a particular value of the Lagrangian strain $E^A$. Thus,

$$\Delta S = K^A : \Delta E \leftrightarrow \Delta S_{IJ} = K^A_{IJKL} \Delta E_{KL},$$  

(2)

where $K^A \equiv \frac{\partial^2 W}{\partial E \partial E} \bigg|_{E = E^A}$.

A common numerical method, called the updated Lagrangian approach, uses an incremental technique to solve boundary value problems. In this method, after the stresses and deformations for an increment of loading are determined, the current configuration is taken as a new reference configuration and the next increment of loading is considered with quantities measured from this new reference. Thus, it is important to be able to find the incremental stress-strain relationships expressed with respect to the new reference configuration.

Let the current configuration be $x = \phi^A(X)$ and define this as the new reference configuration: $X^* \equiv \phi^A(X)$. Then, additional deformations away from this new reference configuration can be described by a new deformation mapping such that the new current configuration is given by

$$x = \phi(X^*) = \phi(\phi^A(X)).$$  

(3)
Answer the following:

(a) (10 pts) The deformation gradient from the new reference is
\[ \mathbf{F}^* = \nabla^* \mathbf{x} \equiv \frac{\partial \mathbf{x}^*}{\partial \mathbf{X}^*}. \]
Find an expression for the deformation gradient from the original reference configuration, \( \mathbf{F} \), in terms of \( \mathbf{F}^* \) and the deformation gradient to the new reference configuration \( \mathbf{F}^A = \nabla_0 \mathbf{X}^* \equiv \frac{\partial \mathbf{X}^*}{\partial \mathbf{X}} = \frac{\partial \phi^A}{\partial \mathbf{X}}. \)

(b) (15 pts) Find an expression for the Lagrangian strain tensor from the original reference configuration, \( \mathbf{E} \), in terms of the Lagrangian strain tensor from the original to the new reference configuration, \( \mathbf{E}^A \), the Lagrangian strain tensor from the new reference configuration \( \mathbf{E}^* \), and \( \mathbf{F}^A \).

(c) (10 pts) Give a definition (in terms of \( W(\mathbf{E}) \)) for a new strain energy density function \( W^*(\mathbf{E}^*) \) that provides the strain energy, per unit volume in the new reference configuration, \( \mathbf{E}^* \), in the material as a function of the Lagrangian strain from the new reference, \( \mathbf{E}^* \).

(d) (15 pts) Linearize this function about the new reference configuration \( (\mathbf{E}^* = 0) \) to obtain the incremental stress-strain relation
\[ \Delta \mathbf{S}^* = \mathbf{K}^* : \Delta \mathbf{E}^* \leftrightarrow \Delta \mathbf{S}_{ij}^* = \mathbf{K}_{ijkl}^* \Delta \mathbf{E}_{KL}^*, \tag{4} \]
and show that
\[ \mathbf{K}_{ijkl}^* = \frac{1}{J^A} F^A_{iP} F^A_{jQ} F^A_{kR} F^A_{lS} \mathbf{K}_{PQRS}^*, \tag{5} \]
where \( J^A \equiv \det(\mathbf{F}^A) \).

Note:
The relations given in Eqs. (2) and (4) provide convenient prescriptions for the increments of the symmetric second Piola-Kirchhoff stress in terms of increments of the symmetric Lagrangian strain. The former is appropriate for use with a Lagrangian solution method, such as a standard finite element method, where all quantities are computed with respect to a single fixed reference configuration. The latter is appropriate for use with an incremental updated Lagrangian solution method, where at each step the increment of second Piola-Kirchhoff stress \( \Delta \mathbf{S}^* \) is computed, then the associated increment of Cauchy stress is determined from \( \Delta \mathbf{S}^* \) and added to the value of Cauchy stress computed from the previous increment.

In practice, the strain energy density function is usually the available quantity that specifies the body’s constitutive relation. In this case Eq. (5) is required in order to obtain \( \mathbf{K}^* \) for use in the updated Lagrangian method.
2. The tensor $\mathcal{K}^A$ from the previous problem is called the *tangent stiffness* tensor. When evaluated at $\mathbf{E} = \mathbf{0}$ the tangent stiffness is denoted $\mathcal{K}^0$. It is often useful to express the increment of Lagrangian strain in terms of the stress increments:

$$\Delta \mathbf{E} = \mathcal{D}^0 : \Delta \mathbf{S} \quad \leftrightarrow \quad \Delta E_{IJ} = \mathcal{D}^0_{IJKL} \Delta S_{KL}, \quad (6)$$

where $\mathcal{D}^0$ is called the *tangent compliance* tensor.

When the incremental stress-strain and strain-stress relations are put in Voigt notation we have the matrix relations

$$\Delta \mathbf{S} = \mathbf{K}^0 \Delta \mathbf{E} \quad \leftrightarrow \quad \Delta S_i = K^0_{ij} \Delta E_j, \quad (7)$$

and

$$\Delta \mathbf{E} = \mathbf{D}^0 \Delta \mathbf{S} \quad \leftrightarrow \quad \Delta E_j = D^0_{ij} \Delta S_j, \quad (8)$$

where $\mathbf{K}^0$ and $\mathbf{D}^0$ are $6 \times 6$ matrices and $\mathbf{D}^0 = (\mathbf{K}^0)^{-1}$.

Consider a material that is isotropic in the original reference configuration. For such a material one has:

$$\mathcal{K}^0_{IJKL} = \lambda \delta_{IJ} \delta_{KL} + \mu (\delta_{IK} \delta_{JL} + \delta_{JK} \delta_{IL}) \quad (9)$$

and

$$\mathcal{D}^0_{IJKL} = \frac{1}{4\mu} (\delta_{IK} \delta_{JL} + \delta_{JK} \delta_{IL}) - \frac{\lambda}{2\mu(3\lambda + 2\mu)\delta_{IJ} \delta_{KL}}. \quad (10)$$

(a) (10 pts) Find an example of a second-order tensor $\mathbf{Z} \neq \mathbf{0}$ such that for the above isotropic material

$$\mathbf{K}^0 : \mathbf{Z} = \mathbf{0}. \quad (11)$$

(b) (10 pts) Explain why this shows that $\mathbf{K}^0$ does not have an inverse, and thus, $\mathbf{D}^0 \neq (\mathbf{K}^0)^{-1}$.

(c) (15 pts) Finally, explain the apparent contradiction between the two equations:

$$\mathbf{D}^0 = (\mathbf{K}^0)^{-1} \quad \text{and} \quad \mathbf{D}^0 \neq (\mathbf{K}^0)^{-1}, \quad (12)$$

That is, the inverse exists in Voigt form, but not for the general $4^{th}$ order tensor expression.

(d) (15 pts) Without evaluating the expression, do you think that $\mathbf{K}^A$ evaluated at an arbitrary finite deformation would have the same isotropic form as $\mathbf{K}^0$ in Eq. (9)? Explain.