Problem 1

Consider the negative feedback system shown below. The plant models the roll-rate dynamics of an aircraft with aileron deflection $\delta_a$ as input and roll rate $p$ as output. The rigid body dynamics are given by $G_{\text{rig}}(s) = \frac{1}{s+1}$.

![Figure 1: Roll Rate Feedback Loop](image)

A. Design a controller $K(s)$ so that the closed-loop has poles in the left half plane with natural frequency $\omega_{\text{des}} = 6$ rad/sec and damping ratio $\zeta_{\text{des}} = 0.9$. Moreover, your controller should achieve zero steady-state error for step reference commands. Perform your design using the rigid body dynamics $G = G_{\text{rig}}$. 
B. The rigid body assumption is only an approximation. Assume the aircraft flexibility is modeled with a lightly damped mode:

\[ G_{flex}(s) = \frac{100^2}{s^2 + 10s + 100^2} \]  

(1)

The aircraft model with rigid and flexible dynamics is given by \( G(s) = G_{rig}(s)G_{flex}(s) \). Sketch the Bode plot for the loop transfer function \( L(s) := G(s)K(s) \) using the controller designed in part A. You may use straight-line approximations in your sketch but the peak of the flexible mode should be drawn accurately. (Hint: The peak of the flex mode is approximately given by \( |G_{flex}(j\omega_n)| \approx \frac{1}{\sqrt{2}} |G_{flex}(0)| \) where \( \omega_n \) and \( \zeta \) are the natural frequencies of the flexible mode).
C. Another design requirement is that the controller should not excite the flexible mode. Quantitatively, the controller must ensure the loop transfer function satisfies $|L(j\omega_n)| \leq 0.1$ at the natural frequency of the flexible mode. Does your controller designed in part A satisfy this design requirement? If not, then briefly explain how you would modify your control design to meet this requirement.
Problem 2
Consider a linear, time-invariant system $G$ with input $u$ and output $y$. An experiment is performed with a unit step input applied at time $t = 0$ from zero initial conditions. The results are shown below.

Figure 2: Unit Step Response

A. What is the transfer function for the system $G$? Your answer will depend on the parameters $L$ and $R$. 
B. The Ziegler-Nichols method was a popular technique for PID control design dating back to the 1940’s. One version consists of the following steps:

(i) Connect the plant $G$ in negative feedback with a proportional controller.

(ii) Increase the gain of the controller until the closed-loop system becomes unstable and begins to oscillate. Let $K_c$ denote this critical proportional gain and $T_c$ the period of the oscillations.

(iii) Construct the PID controller as

$$C(s) = K_P \left[ 1 + \frac{1}{T_I s} + T_D s \right]$$

where the gains are selected as $K_P = 0.6 K_c$, $T_I = \frac{T_c}{2}$ and $T_D = \frac{T_c}{8}$.

What are the critical values $K_c$ and $T_c$ for the plant identified in part A? What is the corresponding PID controller $C(s)$ designed using Ziegler-Nichols? Your answers will depend on the parameters $L$ and $R$. 
C. Show that if $L = R = 1$ then the loop transfer function can be written as

$$G(s)C(s) = 0.15\pi \frac{(s + 1)^2 e^{-s}}{s^2}$$
D. Continue to assume $L = R = 1$. It can be shown that the closed-loop system is stable. Use the Bode plot for $G(s)C(s)$ shown below to calculate the: phase crossover frequencies, (upper and lower) gain margins, gain crossover frequencies, and phase margin. **Note:** This part can be completed even if you could not complete the previous parts.

![Bode Plot](image)

**Figure 3: Unit Step Response**
E. How would your answers in part D change if the system parameters $L$ and $R$ are not necessarily equal to 1?
Problem 3
Consider the standard negative feedback system shown below.

A. Suppose the plant transfer function is:

\[ G(s) = \frac{(s - 1)(s - 4)}{(s + 1)(s + 3)(s - 2)} \]

Can this system be stabilized using proportional control, \( K(s) = K_p \)? If yes, then provide the range of gains \( K_p \) for which the closed-loop is stable.
B. Continue to assume that $G$ is the third-order system given in part A. Suppose the controller $K(s)$ is stable. What conclusions can you draw about the stability of the closed-loop system? Justify your answer.
Problem 4

Consider the standard negative feedback loop shown below with a simple integral controller $K(s) = \frac{K_p}{s}$. Assume the plant has the form $G(s) = \frac{1}{s}H(s)$ where $H$ is a proper system. Furthermore, assume the closed-loop is stable.

Let $y(t)$ be the output for a unit step input ($r(t) = 0$ for $t < 0$ and $r(t) = 1$ for $t \geq 0$) with zero initial conditions. Show that the system response must have overshoot, i.e. there is some time $t$ for which $y(t) > 1$. 

![Feedback Loop Diagram]