Problem 1

In stochastic dynamic programming, Bellman’s equation quantifies how a representative household’s optimal consumption varies with its capital stock, $k$. This relationship can be written as a non-linear partial differential equation in terms of the value function, $V = V(k,t)$, as:

$$\frac{\partial V}{\partial t} = \frac{1}{\omega} \left( \frac{\partial V}{\partial k} \right)^{\frac{\omega-1}{\omega}} + (\lambda k^\theta - \delta k) \frac{\partial V}{\partial k} + \frac{1}{2} \sigma^2 k^2 \frac{\partial^2 V}{\partial k^2} - \rho V$$

(1)

where $t$ is time, and $\omega, \lambda, \theta, \delta, \sigma,$ and $\rho$ are constants between zero and unity.

The capital stock, $k$, varies from zero to a large value, at which $\frac{\partial V}{\partial k} \rightarrow 0$. The value of $V$ at $k = 0$ is fixed at $V(0,t) = V_0$.

Solutions for the steady-state, $\frac{\partial V}{\partial t} \rightarrow 0$, are sought.

1. Using fluid dynamics language, describe the expected behavior of each term in Eq. (1).
2. Propose an explicit finite-difference method for this problem and explain your choice of differencing approach.
3. What would you expect the stability criterion to be for your method?
4. Propose an implicit finite-difference method and describe how to solve the resulting system of equations including the influence of the boundary conditions. Provide justification for the design of your method.

Problem 2

In von Neumann stability analysis, a scheme’s stability is assessed by considering the magnitude of the complex amplification factor, $G$. For the linear wave equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$

the diffusive and dispersive errors are given by deviations from unity of:

$$\epsilon_D = |G|, \quad \epsilon_\phi = \frac{\Phi}{\sigma \phi}$$
where $\phi$ is the phase angle of the test function,

$$u_i^n = \sum_{j=-N}^{N} U_j^n e^{i\phi}$$

$\sigma = \frac{a \Delta t}{\Delta x}$ is the Courant number, $I$ is the imaginary number, $\phi = k \Delta x$, and $k$ the wavenumber. $\Delta x = N/L$, where $L$ is the length of the solution domain. $\Phi$ is the numerical phase speed,

$$\tan \Phi = -\Im(G)/\Re(G)$$

1. Consider the leapfrog method and compute the diffusive and dispersive error as a function of the Courant number and the phase, $\phi$. Explain the implications of this analysis, particularly for high wavenumber data.

2. The leapfrog method produces a numerical solution for a wavepacket of phase $\phi = \pi/6.25$ as shown in the figure below (computed with $\sigma = 0.8$ for 80 time steps); explain this behavior based on the error properties.

3. What are the implications for the viability of the leapfrog scheme for the linear wave equation?

The leapfrog scheme is given by:

$$u_i^{n+1} = u_i^{n-1} + \sigma(u_{i+1}^n - u_{i-1}^n)$$