AEM WPE Dynamics 2014

A particle of mass $M$ moves in a plane under the action of a central force centered at $O$. Denoting the position vector to $M$ with respect to $O$ by $r$ and its magnitude by $r = |r|$ the central force has the expression

$$ F = -\frac{\mu M}{r^2} e_r $$

where $\mu$ is a known constant, and $e_r$ is the radial unit vector along $r$. The angle between $r$ and the fixed X-axis is denoted $\theta$. Another particle of mass $m$ is located at a distance $\rho$ from $M$ and at an angle $\phi$, defined as the angle between $r$ and $\rho$–the position vector of $m$ with respect to $O$. There is an attractive force between $M$ and $m$ of magnitude $f = \frac{GMm}{\rho^2}$, where $G$ is a constant.

The particle $m$ is not affected by the central force at $O$ and moves in the same plane as $M$.

a) Using the variables $r, \theta, \rho$, and $\phi$ as generalized coordinates, derive Lagrange's equations for the system.

b) Verify the expressions for the generalized momenta, i.e. $\frac{\partial \mathcal{L}}{\partial r}$, $\frac{\partial \mathcal{L}}{\partial \theta}$, $\frac{\partial \mathcal{L}}{\partial \rho}$, $\frac{\partial \mathcal{L}}{\partial \phi}$ by interpreting them in physical terms (i.e. determining whether they are momenta, angular momenta, etc.).

c) Verify the correctness of your equations of motion by deriving the corresponding equations using a Newton-Euler approach. Note that you will need to write the Newton-Euler equations in terms of the same variables as in (a) and along the same directions.

d) Suppose that there is an additional force $T = T_r e_r + T_\theta e_\theta$ on $m$. How would you modify the equations of motion in (a)? The unit vectors $e_r$ and $e_\theta$ are defined in the figure.

Hint: a force that obeys the inverse square law, i.e. $F = -\frac{k}{r^2} e_r$ is associated with a potential energy of the form $V = -\frac{k}{r}$. 