1. Problem 1: MIMO Control Design

The state-space model for a multi-input multi-output plant is given as:

\[
A = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix} ; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; C = \begin{bmatrix} 1 & 10 \\ -10 & 1 \end{bmatrix} ; D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .
\]  

(a) (5 points) Determine the transfer function matrix \( G(s) \) for the system.
(b) (5 points) What are the poles and zeros of the system. Is the system stable?
(c) (5 points) Draw a block diagram of the plant.
(d) (5 points) Determine the poles of the closed loop system for the feedback gain $K=I$. Does this feedback gain stabilize the system?
(e) (5 points) Assume there is a perturbation in each control channels $\epsilon_1$ and $\epsilon_2$ respectively, i.e., the changes in the control variables are

$$u_1' = (1 + \epsilon_1)u_1 \quad (2)$$
$$u_2' = (1 + \epsilon_2)u_2 \quad (3)$$

Write out the input matrix $B'$ that accounts for these perturbations and determine the corresponding closed-loop system matrix.
(f) (10 points) What is the characteristic polynomial of the perturbed system?

(g) (5 points) Based on the characteristic polynomial, determine the non-trivial conditions ($\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$) for which the system will be stable.
2. Problem 2: System Identification

(a) (5 points) An experiment is performed with a linear time invariant system $G(s)$. A sine-wave input $u(t)$ is applied to the system and after some time the system output $y(t)$ reaches a steady-state. The left figure below shows both the input and output of the system. The input has been applied since $t = 0$ but the figure only shows $u(t)$ and $y(t)$ for $7 \leq t \leq 11$ sec. Use this experimental to compute the magnitude and phase of $G(j\omega)$ at a single frequency $\omega$. 

![Input and Output Graph](image-url)
(b) (5 points) Assume that the input/output data from Part 1a was generated by a first order system of the form \( G(s) = \frac{c_1}{s+c_2}. \) Use the results of Part 1a to determine the coefficients \( c_1 \) and \( c_2. \)

(c) (5 points) Sketch the Bode plot for the first order system \( G(s) \) computed in Part 1b on the axes below. Also mark the location on the magnitude and phase plots corresponding to the input/output data from Part 1a.
(d) (5 points) Explain how you would generalize this procedure to identify a model for a higher order system using experimental input/output data.
3. **Problem 3: Strong Stabilization** Consider the classical feedback diagram shown in Figure 1 below. The plant \( G(s) \) is called *strongly stabilizable* if it can be stabilized by a controller \( K(s) \) that is itself stable. In other words, \( G(s) \) is strongly stabilizable if there is a stable \( K(s) \) such that the closed-loop system is stable.

(a) (10 points) Suppose that the system dynamics \( G(s) \) are modeled by:

\[
G(s) = \frac{s - 1}{s(s - 2)}
\]  
(4)

Is the system \( G(s) \) strongly stabilizable? If yes, then provide a controller \( K(s) \) such that both \( K(s) \) and the closed-loop system are stable. If no, then sketch a proof using classical arguments (Bode, Nyquist, and/or root locus) for why no stable \( K(s) \) can stabilize \( G(s) \).

![Figure 1: Feedback Loop](image-url)
(b) (5 points) A flight control system is safety critical. The control law is typically implemented on multiple processors to ensure safe operation even if a single processor fails. Consider a simple dual redundant implementation of the control law. The same control law $K(s)$ is implemented on two computers. Initially computer 1 controls the system, i.e. $u_1$ is the control command, while computer 2 is operating in a stand-by mode. Computer 2 takes over if a failure is detected in Computer 1, i.e. $u_2$ is the control command after a failure is detected in Computer 1. Assume that the computers operate on different clocks and hence they sample the data at slightly different times.

If the plant $G(s)$ is not strongly stabilizable then the controller $K(s)$ will be an unstable system. What issues will arise in a dual redundant implementation if $K(s)$ is unstable? How might you resolve these issues?

Figure 2: Dual Redundant Computing Architecture
4. Problem 4: Short Answer

(a) (5 points) The singular value decomposition (SVD) provides useful insight for multi-input multi-output analysis. A matrix $G$ can be decomposed into $G = U \Sigma V^H$, where $\Sigma$ contains the singular values $\sigma_i(G) = \sqrt{\lambda_i(G^H G)}$, and $U$ and $V$ are unitary matrices composed, respectively, of the input and output singular vectors. The steady-state model of a system gives the following results:

$$ G = \begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix} \begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix} \begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^T \quad (5) $$

What physical interpretation can you draw from the SVD decomposition (you can draw a diagram if you want)?

(b) (5 points) You are responsible for designing a control law that will be implemented on a processor that runs at 100Hz. What is the approximate delay associated with the control law computations? What limitation does this place on the bandwidth of the closed-loop system?
(c) (5 points) In what situations might open loop control be preferable to closed loop control? Why?

(d) (5 points) Suppose you’ve designed a PD control law of the form:

\[ u(t) = k_p e(t) + k_d \dot{e}(t) \]  \hspace{1cm} (6)

\( e(t) = r(t) - y(t) \) is the tracking error between the reference signal \( r(t) \) and plant output \( y(t) \). What issues might arise if you implement this PD control law? What are some possible solutions to deal with these implementation issues?
(e) (5 points) You are responsible for designing a controller for a system whose dynamics are modeled by the transfer function \( G(s) = \frac{s^2+4}{s^2+5s+6} \). Your boss asks you to design a controller that tracks reference signals \( r(t) = \sin(\omega t) \) with less than 1\% error for all \( 0 \leq \omega \leq 10 \) rad/sec. What recommendation would you make? Why?