2012 AEM Preliminary Exam — Continuum Mechanics

1. (25 points) The balance of linear momentum of continuum mechanics, expressed in the deformed (or current) configuration of a body is

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\text{grad} \mathbf{v}) \right) = \text{div} \mathbf{\sigma} + \rho \mathbf{b}, \quad x \in \Omega_t \subset \mathbb{R}^3, \quad t > 0, \]

(1)

where \( \mathbf{v}(x,t) \) is the velocity field, \( \mathbf{b} = \mathbf{b}(x,t) \) is the body force per unit mass, and \( \mathbf{\sigma} = \mathbf{\sigma}(x,t) \) is the Cauchy stress tensor. The symbols grad and div above denote gradient and divergence with respect to the spatial position vector \( x \). In rectangular Cartesian components \( ((\text{grad} \mathbf{v}) v)_i = u_{i,j}v_j \).

Recall the fundamental relation between the spatial velocity \( \mathbf{v}(x,t) \) and the referential (or Lagrangian) motion \( \mathbf{y}(X,t) \) velocity given by the ordinary differential equation,

\[ \dot{\mathbf{y}}(X, t) = \mathbf{v}(\mathbf{y}(X,t), t), \quad \mathbf{y}(X,0) = \mathbf{X}. \]

(2)

Let \( \mathbf{F} = \nabla \mathbf{y} = \partial \mathbf{y}/\partial \mathbf{X} \) be the deformation gradient and \( J = \det \mathbf{F} \). Use Nanson’s formula, \( \mathbf{N}dA = J\mathbf{F}^{-T}\mathbf{N}dA_0 \), the divergence theorem, and the principle of conservation of mass to derive the balance of linear momentum equations expressed in the reference configuration of the body (These are partial differential equations defined for \( \mathbf{X} \in \Omega_0, \quad t > 0 \)). In particular, show that this procedure provides the natural definition,

\[ \mathbf{P} = J\mathbf{\sigma}\mathbf{F}^{-T}, \]

(3)

for the first Piola–Kirchhoff stress tensor.

2. (15 points) Give the general form of the constitutive relation (i.e., an expression for the Cauchy stress as a function of the the left Cauchy–Green tensor) for an incompressible, isotropic nonlinearly elastic material. Explain all terms. For reference, recall that the principal invariants of the left Cauchy–Green tensor \( \mathbf{B} \) are

\[ I_1 = \text{tr} \mathbf{B}, \quad I_2 = \frac{1}{2}((\text{tr} \mathbf{B})^2 - \text{tr}(\mathbf{B}^2)), \quad I_3 = \det \mathbf{B} \]

(4)

Be sure to write the constraint of incompressibility and explain the meaning of the pressure.

3. (20 points) A Neo-Hookean material is a special incompressible isotropic elastic material with the simple stored energy function \( W(\mathbf{F}) = c_1(I_1 - 3) \), where \( I_1 \) is defined above and of course \( \mathbf{B} = \mathbf{F}\mathbf{F}^T \). Write the form of the Cauchy stress of the Neo-Hookean material. For reference, recall that the Piola–Kirchhoff stress \( \mathbf{P} = \partial W/\partial \mathbf{F} \) and the formula relating Piola–Kirchhoff stress to Cauchy stress \( \mathbf{\sigma} \) is given above.

4. (40 points) A rubber seal (or gasket) for a vacuum system is designed to fit in a groove in the wall of a stainless steel cylinder. The seal is in the shape of a ring with rectangular cross-section in its deformed configuration as shown in Figure 1. The main principles behind the design are that, in the deformed configuration, the seal should exert a uniform radial traction on a rod that is to be inserted into the cylinder. There should also be no traction on the top and bottom faces of the ring, as shown, so that it is able to seat properly in the groove (not shown). The ring is to be made of (incompressible) Neo-Hookean material with material constant \( c_1 \). The overall goal of this problem is to determine the reference configuration, which is taken to be the stress-free, zero-traction configuration.
Figure 1: A seal for a vacuum system. The deformed configuration is shown in red. It has uniform radial traction on the curved surfaces and no traction on the top and bottom surfaces. The lower picture shows a cross-sectional view with the dashed line being the centerline.

(a) Using the usual formal linearization $F = I + \varepsilon \mathbf{H} + \ldots$, find the linearized elastic stress-strain relation for the Neo-Hookean material. Interpret $c_1 > 0$ in terms of the shear modulus of linearized isotropic elasticity.

(b) Suppose the radial traction is $-\tau \mathbf{n^\pm}$ where $\tau > 0$ is given, $\mathbf{n^+}$ is the outward normal at any point on the outer surface and $\mathbf{n^-}$ is the outward normal at any point on the inner surface of the deformed ring ($\tau$ is the same constant for points on both the inner and outer surfaces). Make an assumption about the deformation in this case, appropriate for an incompressible Neo-Hookean material. Your deformation should satisfy the equations of equilibrium with no body forces. Verify that your assumption is consistent with the boundary conditions given here. Calculate the deformation in terms of $\tau$ and the material constant $c_1 > 0$.

(c) Using your deformation, calculate the detailed shape of the reference configuration in terms of $\tau, c_1, r_0, r_i, h$.

(d) Is the inner radius of the ring in the reference configuration larger or smaller than the deformed (current) value of $r_i$?

(e) Assuming you successfully solved the above, consider the equilibrium solution you obtained. In polar coordinates consider a small angular section of the ring $d\theta$. On both the inside and outside surfaces we have tractions $-\tau \mathbf{n^\pm}$ and therefore we have approximately the resultant force $-r_o d\theta h \tau \mathbf{n^+}$ on the outside and $-r_i d\theta h \tau \mathbf{n^-}$ on the inside. But this seems to imply that there is a resultant force on this angular section given by $(r_i - r_o) d\theta h \tau \mathbf{n^+}$. But this cannot be true because the ring is supposed to be in equilibrium. What is wrong with this argument?