1. The pitching equations of motion for a Boeing 777 aircraft are given as:

\[
\begin{align*}
\dot{\alpha} &= -0.313\alpha + 56.7q + 0.23\delta_e \\
\dot{q} &= -0.0139\alpha - 0.426q + 0.0203\delta_e \\
\dot{\theta} &= 57.3q
\end{align*}
\]  

(1) \hspace{1cm} (2) \hspace{1cm} (3)

$\alpha$ is the angle of attack, $q$ the pitch rate, $\theta$ the angle and $\delta_e$ is the elevator deflection (in radians).

(a) (5 points) Determine the transfer function between $\delta_e$ and pitch angle $\theta$.

(b) (5 points) Write the state-space equations including the output equation.
(c) (5 points) Draw a block diagram for the entire system (line up all integrators on the same line).

(d) (5 points) What kind of physical insight can you get from the block diagram.
2. Stability Analysis

(a) (5 points) What are the poles and zeros of the plant.

(b) (5 points) Draw the root locus
(c) (5 points) The control requirements are:

- Overshoot: less than 10%
- Rise time: less than 2sec
- Settling time: less than 10sec
- Steady state error: less than 2%

Transform the control requirements into a requirement on damping ratio $\xi$ and natural frequency $\omega_n$.

(d) (5 points) What can you say about the control performance for a simple gain from the root locus, i.e., can you satisfy the design requirements?
(e) (10 points) Sketch the bode plot and determine the phase and gain margins.

(f) (5 points) Do these phase and gain margin satisfy the design requirements?
3. (a) (5 points) Based on the transfer function, what type of transient response do you expect for the open-loop?

(b) (5 points) What closed-loop performance do you expect with a simple gain? (Tip: use your insight from the root locus).

(c) (5 points) Determine the steady-state error from the bode plot?
4. Observability/Controllability.
   (a) (5 points) Is the plant observable and controllable given the output is the pitch angle $\theta$?

(b) (5 points) Is the plant observable given the output is the angle of attack $\alpha$?
(c) (5 points) Is the plant observable given the output is the pitch rate $q$?

(d) (5 points) What are the implications for control design for these different scenarios?
5. **Control system design** To improve the closed-loop pitch response someone suggests using a lead compensator with transfer function:

\[ G(s) = K_c \frac{s + 0.9}{s + 20} \]  

(4)

(a) (5 points) Sketch the root locus for the compensated plant and comment on what is achieved thanks to the proposed control element.
(b) (10 points) Draw the bode plot for the compensated plant. Use the bode plot to determine a gain $K_c$ which satisfies the phase and gain margin requirements.
6. Satellite Orbit Transfer

The equations of motion for a satellite of mass \( M \) orbiting at an altitude \( r \) are given by:

\[
\ddot{r} = r\dot{\theta}^2 - k \frac{1}{r^2} + u_r
\]

\[
\ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r} + \frac{1}{r} u_t
\]

\( \omega = \dot{\theta} \) is the angular velocity, and \( k = GM \), where \( G \) is the gravitational constant. The control inputs are the tangential thruster \( u_t \) and the radial thruster \( u_r \).

(a) (5 points) Determine the equation for the steady-state orbit defined by \( r_0 = 1 \) and \( \omega_0 = 1 \).

(b) (5 points) Linearize the equations of motion around the steady state orbit and determine the state-space equations. Use the following state vector \( \delta x_0 = [\delta r, \delta \dot{r}, \delta \theta, \delta \omega]' \).
7. Computation of Control Trajectory

A control trajectory \([\delta u_t, \delta u_r]') to steer the satellite from \(\omega_0\) and \(r_0\) at \(t_0 = 0\) to \(\omega_0 + \delta \omega\) and \(r_0 + \delta r\) at \(t_1 = 1\) can be found using

\[
u(t) = B^* e^{A(t-\tau)} M^{-1}(t, t_0) x_1 \tag{7}\]

where \(M(t, t_0)\) is the controllability gramian, given by:

\[
M(t_1, t_0) = \int_{t_0}^{t_1} e^{A(t-\tau)} B B^* e^{A^*(t-\tau)} d\tau \tag{8}
\]

\(B^*\) stands for the complex conjugate transpose of \(B\). Since we don’t care about \(\delta \theta\), for the following questions use the reduced system with state vector \(\delta x_0 = [\delta r, \delta \dot{r}, \delta \omega]'\).

(a) (10 points) Determine the matrix exponential \(e^{A(t-\tau)}\)
(b) (10 points) The steering problem can be formulated for different combinations of the two thrusters. Consider scenario 1: only thruster $u_t$ is used. Scenario 2: only thruster $u_r$ is used. Explain what you have to verify to determine if the orbit transfer can be achieved for each of the two scenarios?