1. Consider the following 4 methods for solving the convection equation:

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad \text{where} \quad c = \frac{a \Delta t}{\Delta x} \]

\[ u_{j}^{n+1} = u_{j}^{n} - c(u_{j}^{n} - u_{j-1}^{n}) \]

\[ u_{j}^{n+1} = \frac{1}{2}(u_{j+1}^{n} + u_{j-1}^{n}) - \frac{c}{2}(u_{j+1}^{n} - u_{j-1}^{n}) \]

\[ u_{j}^{n+1} = u_{j}^{n} - \frac{c}{2}(u_{j+1}^{n} - u_{j-1}^{n}) + \frac{c^2}{2}(u_{j+1}^{n} - 2u_{j}^{n} + u_{j-1}^{n}) \]

\[ u_{j}^{n+1} = u_{j}^{n} - \frac{c}{2}(3u_{j}^{n} - 4u_{j-1}^{n} + u_{j-2}^{n}) + \frac{c^2}{2}(u_{j-2}^{n} - 2u_{j-1}^{n} + u_{j-3}^{n}) \]

Below is one computation by each method. The initial data was a step function at \( x = 0 \); the parameters are \( a = 1, \Delta x = 0.1 \), and \( \Delta t = 0.05 \).
(a) Clearly explain which figure (a)-(d) was produced by which numerical method [1]-[4]. Determine the Modified Equation for each method and use it to support your answer. Be sure to convert all time derivatives of \( u \) to space derivatives.

(b) Use the Modified Equations to determine the order of spatial accuracy for each method.

(c) Based on the Modified Equations, what can you determine about the stability criteria (restrictions on \( c \)) for methods [1] and [2]?

(d) Based on the Modified Equation analysis, explain the purpose of the second and third terms (on the right-hand-side) of methods [3] and [4].

(e) Formally determine if method [2] is Total-Variation-Diminishing (TVD) and also formally determine if method [3] is TVD.

2. Next, consider the nonlinear Burgers' equation:

\[
\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = 0 \quad \text{where} \quad E = \frac{1}{2} u^2 \quad \text{(conservative form)}
\]

applied to a discontinuous initial waveform: (where \( k \geq 0 \))

\[
u_0(x) = \begin{cases} 
     u_L = 1 + k, & x < 0, \\
     u_R = k, & x > 0,
\end{cases}
\]

where \( u_L \) and \( u_R \) are values on either side of the discontinuity. For example, one may have \( u_L = u_j \) and \( u_R = u_{j+1} \).

(a) A general method to solve this equation is written in flux-function formulation as:

\[
u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} (S_{j+1/2}^n - S_{j-1/2}^n),
\]

where the flux function is given by:

\[
S_{j+1/2}^n = \frac{\Delta x}{2\Delta t} (u_j^n - u_{j+1}^n) + \left(E_j^n + E_{j+1}^n\right)
\]

Which method (for the convection equation) is this analogous to? [1], [2], [3], or [4]?

(b) The “shock speed” is defined as the wave speed at \( x = 0 \) (or equivalently, the characteristic speed at \( j + 1/2 \)). Simply using the original Burgers’ equation,

\[
\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = 0,
\]

how is the characteristic speed defined? Use a basic discretization across the discontinuity to determine an expression for the shock speed resulting from the discontinuous waveform given above.
Furthermore, if \( u \) is a smooth function, Burgers’ equation may be written equivalently as:
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad \text{(non-conservative form)}
\]
which may be discretized using an upwind scheme (for \( u \geq 0 \)):
\[
u_j^{n+1} = u_j^n - u_j^n \frac{\Delta t}{\Delta x} (u_j^n - u_{j-1}^n)
\]
(c) If this non-conservative, upwind scheme is used for the discontinuous waveform (take \( k = 0 \) for simplicity), what is the shock speed? Is this correct compared to your answer to (b)? What happens to the waveform in time using this method?

(d) If the conservative, flux-function formulation method from (a) is used, what is the shock speed? Is it correct compared to your answer to (b)?

3. Finally, consider the Cauchy-Riemann Equations (non-dimensionalized):
\[
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \quad \text{and} \quad \omega = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \text{where} \quad \rho = \left[1 - \frac{1}{2} (u^2 + v^2 - 1)\right]^{1/2}
\]
Which can be written:
\[
\frac{\partial F(Q)}{\partial x} + \frac{\partial G(Q)}{\partial y} = 0 \quad \text{with} \quad Q = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F = \begin{pmatrix} -\rho u \\ v \end{pmatrix}, \quad \text{and} \quad G = \begin{pmatrix} -\rho v \\ -u \end{pmatrix}.
\]
Furthermore, a fictitious time-derivative is added, such that the solution to the Cauchy-Riemann Equations are obtained from the steady-state numerical solution to:
\[
\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0
\]
(a) Explain the advantages/disadvantages of adding a fictitious time-derivative.

(b) Formally characterize this system of equations (hyperbolic, elliptic, parabolic). What are the wave speeds (if any) in both the \( x - t \) and \( y - t \) planes?

(c) Write out the flux jacobians \( A = \frac{\partial F}{\partial Q} \) and \( B = \frac{\partial G}{\partial Q} \).

(d) Describe the steps involved in formulating an implicit method to solve this system of equations with the general form:
\[
\frac{Q_{i,j}^{n+1} - Q_{i,j}^n}{\Delta t} + \left. \frac{\partial F}{\partial x} \right|_{i,j}^{n+1} + \left. \frac{\partial G}{\partial y} \right|_{i,j}^{n+1} = 0
\]
Specifically, first linearize $F^{n+1}$ and $G^{n+1}$ using Taylor Series expansion, and ultimately form an "implicit operator matrix" $[N]$ that operates on $\Delta Q = Q^{n+1} - Q^n$:

$$[N] \Delta Q = \text{RHS}$$

What are the specific terms that comprise $[N]$ and the RHS?

What would an appropriate discretization stencil be for the spatial gradients? Why?

What is the general form of matrix $[N]$ (ex. is it tridiagonal)?

Discuss a suitable method to solve this system of equations and comment on its advantages/disadvantages.