1 System Model and Setup

Figure 1 shows the helicopter state variables involved in the longitudinal helicopter dynamics. The helicopter pitch attitude $\theta$ can be controlled by the pilot through intermediary of the rotor system which generates a pitching moment on the fuselage proportional to the rotor incidence $a$.

![Diagram](image)

Figure 1: Main variables for the helicopter pitch attitude dynamics.

The rotor can be approximated by a disc that tilts relative to the helicopter body about the rotor hub. The rotor thrust is approximately perpendicular to the rotor disc. We assume that the thrust magnitude is constant.

This rotor motion is controlled by the longitudinal cyclic input ($\delta_{lon}$). Its dynamics can be approximated by a first-order system:

$$\dot{a} = -\frac{1}{\tau} a - q + \frac{A_{lon}}{\tau} \delta_{lon}$$  \hspace{1cm} (1)

where $\tau$ is the rotor time constant, $q = \dot{\theta}$, and $A_{lon}$ is the input sensitivity.
The rotor angular displacement will impart a pitching moment on the helicopter body due to two contributions: the tilting of the thrust vector (change in its line of action), and the stiffness of the rotor attachment to the helicopter.

a) Assuming that the helicopter center of gravity (c.g.) is situated along the rotor shaft at a distance $h$ below the rotor head, and that the rotor hub stiffness is modeled by a linear torsional spring with coefficient $k_h$: determine the total pitching moment as a function of the rotor motion $\alpha$.

If you assume that the rotor angle $\alpha$ is small, you can obtain the following linear differential equation relating the rotor motion $\alpha$ and the pitching rate $\dot{\alpha}$:

$$ \dot{\alpha} = M_\alpha \alpha $$  \hspace{1cm} (2)

where $M_\alpha$ is the rotor pitch moment derivative.

b) Write out the expression for $M_\alpha$ in terms of the helicopter moment of inertia about the pitching direction $I_\theta$, the helicopter thrust $T$ and the distance $h$. 

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c) Write the state-space model for the helicopter pitching dynamics using the state vector $x = [\theta, q, a]^T$ and the input $u = \delta_{lon}$.

d) Show that the transfer function from the longitudinal input $\delta_{lon}$ and the pitching rate $q$ is

$$\frac{q(s)}{\delta_{lon}(s)} = \frac{M_a A_{lon}/\tau}{s^2 + s/\tau + M_a}$$

(3)
e) Draw a detailed block diagram for the helicopter pitching dynamics. Make sure to clearly represent the rotor and fuselage subsystems.
2 System Analysis

In this section you will analyze the pitch response characteristics of the helicopter (as described by the transfer function in Eq. 3). Use the following values for the parameters: $M_n = 240; A_{lon} = 11\text{deg}/\%$ and $\tau = 0.1\text{sec}$.

a) Determine the expressions for the damping ratio and the natural frequency $w_n$ and $\xi$ in terms of the physical parameters of the transfer function and then give their numerical values.

b) Draw the bode diagram of the transfer function for the provided parameter values (use the figures provided at the end).

c) Knowing that the control input is constrained ($|\delta_{lon}| \leq 1$), what is the maximum steady-state pitching rate that the helicopter can attain and what amount of rotor pitching angle $\alpha$ results?
d) Draw the time response of the pitch rate to a step input $\delta_{\text{on}}=1$ (represents 100% of the admissible range). Make sure to indicate the response rise time $t_r$ and the peak overshoot $M_p$ based on calculated values (use: $t_r = 1.8/\omega_n$; $M_p = 5\%$ for $\xi = 0.7$; $M_p = 15\%$ for $\xi = 0.5$; $M_p = 35\%$ for $\xi = 0.3$).

e) What system parameter would you change to increase the maximum pitching rate without affecting the transient characteristics of the response?
3 Control System Design

You would like to design a simple pitch attitude control system. For this task a quasi-steady model is used instead of the full-state model.

a) The quasi-steady model is obtained by using steady-state rotor dynamics. Show that the transfer function for the pitch angle for the simplified model is:

\[
\frac{\theta(s)}{\delta_{\text{on}}(s)} = \frac{M_a A_{\text{on}}}{s^2 + M_a \tau s}
\]  

(4)

b) Add the drawing of step response for the quasi-steady model.

c) For a first design a simple proportional feedback is used. The nominal gain value is \( k=0.1 \) (\%/degrees). Draw the Bode plot and determine the stability margins and the crossover frequency.

d) What is the resulting response rise time?
e) You want to increase the speed of response. The specification is a settling time $t_s$ under 1 sec (use $t_s = \frac{4\delta}{\sigma}$, where $\sigma = \xi \omega_n$). What should you be concerned about knowing that the model used for the control design disregards the rotor dynamics?

f) Backup your argument with a quantitative analysis.
g) To complete the design, the effect of disturbances need to be accounted for. We can assume that wind gusts will produce a pitching moment on the helicopter. Draw the closed-loop block diagram showing where the disturbance enters the system. Make sure to label all the signals.

h) How would you quantify the effect of the disturbance on the response of the close-loop system?
i) If you wanted to account for the rotor dynamics in the control design but the rotor time constant $\tau$ was uncertain how would you model this uncertainty?
Figure 2: Bode plot for pitching dynamics
Figure 3: Pitch rate step response
Figure 4: Stability margins for pitch angle control system