A horizontal beam is free to rotate about a vertical axis passing through its center of mass at O. The beam has a moment of inertia $I$ about its center of mass. In addition, an external moment $u(t)$ may be applied about the axis of rotation. At a point A located a distance $r$ from the center of mass, a light rigid rod of length $\ell$ is attached by a hinge to the beam. The rod has a mass $m$ attached to the tip. The rod moves in a vertical plane. A torsional spring with stiffness $k$ acts at the hinge; the spring is undeformed when the rod is vertical. A general configuration is shown in Figure 1.

At time $t$ the beam has rotated through angle $\alpha(t)$ with respect to an original reference position. At the same time the rod will make an angle $\theta(t)$ with respect to the vertical. If we assume that the acceleration due to gravity is $g$, then the equations of motion for the configuration are

$$\{I + m[\ell \sin(\theta) - r]^2\} \dddot{\alpha} + 2m[\ell \sin(\theta) - r] \cos(\theta) \ddot{\alpha} = u(t),$$

$$m \ell^2 \dddot{\theta} + m[\ell \sin(\theta) - r] \ell \cos(\theta) \ddot{\alpha}^2 + k \theta - mg \ell \sin(\theta) = 0.$$
1. Consider first the case $u(t)=0$ and $\dot{\alpha}(t)=\omega_0$, with $\omega_0=\text{constant} \neq 0$ corresponding to a steady uniform rotation of the beam about the vertical axis. Using (1) and (2), find an equation satisfied by $\theta^*$, where $\theta^*$ is the value of $\theta(t)$ when the rod is at a relative equilibrium position with respect to the rotating beam. Draw a picture showing the forces which act on the mass when it is in this position. Discuss.

2. Let the configuration be in steady motion as in part 1 with $\dot{\alpha}(t)=\omega_0$ and $\theta^*=\text{constant}$. Assume that $\theta^*>0$. Linearize the equations of motion (1) and (2) about this steady motion. Is the relative equilibrium for this $\theta^*$ stable? Unstable? Explain.

3. Assume $\dot{\alpha}(t)$ is no longer constant and we would like to apply an external moment $u(t)$ which places the small mass directly over the center of the beam and keeps it there. Is it possible to do this? If so, what should $u(t)$ be; if not, why?

4. Suppose it is decided that a force on the small mass proportional to the airspeed is significant, i.e. $\vec{F}_{\text{drag}} = -c\vec{V}_{\text{mass}}$. How will this change the equations of motion and what will be the effect on the equilibria?

5. For the configuration shown in the figure, write down the kinetic and potential energy. If $u(t)=0$, show that energy is conserved along solutions of (1) and (2). What kinetic and potential energy correspond to the linearized equations of motion?