Consider a finite deformation of a material from its reference position at \( X_A \), to its deformed position at \( x_i \). The deformation of the material is given by

\[
x_i = g_i(X_A)
\]  

(1)

The deformation gradient has components in a rectangular Cartesian coordinate system given by

\[
F_{iA} = \frac{\partial x_i}{\partial X_A}
\]  

(2)

The components of the inverse deformation gradient are given by

\[
f_{Ai} = \frac{\partial X_A}{\partial x_i}
\]  

(3)

which is the inverse of the deformation gradient, that is,

\[
F_{iA} f_{Aj} = \delta_{ij} \quad \text{and} \quad f_{Ai} F_{iB} = \delta_{AB}
\]  

(4)

The inverse deformation gradient may be thought of as the deformation gradient for the inverse deformation

\[
X_A = G_A(x_i)
\]  

(5)

where the roles of \( x_i \) and \( X_A \) are reversed.

The Jacobian of the deformation is given by

\[
J = \det(F_{iA})
\]  

(6)

If we assume that there is a strain energy per unit reference volume, \( W_0 \), which only depends on the Lagrangian strains (and hence can only represent a homogeneous material in the reference state)

\[
E_{AB} = \frac{1}{2}(F_{iA} F_{iB} - \delta_{AB})
\]  

(7)

that is

\[
W_0 = W_0(E_{AB})
\]  

(8)
then the Cauchy stress is given by

\[ \tau_{ij} = \frac{1}{J} \frac{\partial W_0}{\partial F_{jA}} F_{iA} . \] (9)

(a) Using (4) show that

\[ \frac{\partial f_{Ai}}{\partial F_{jB}} = f_{Aj} f_{Bi} \] (10)

and use this to rewrite (9) as

\[ \tau_{ij} = -\frac{1}{J} \frac{\partial W_0}{\partial f_{Ai}} f_{Aj} . \] (11)

(b) Verify the identity

\[ \frac{\partial}{\partial X_A} (J f_{Ai} \tau_{ij}) = 0 \] (12)

and use it to show that

\[ \frac{\partial}{\partial X_A} (J f_{Ai} \tau_{ij}) = 0 \quad \text{is equivalent to} \quad \frac{\partial \tau_{ij}}{\partial x_i} = 0 , \] (13)

which is the equation of static stress equilibrium in the absence of body forces.

(c) Use (9) and (13) to express equilibrium as

\[ \frac{\partial}{\partial X_A} \left( \frac{\partial W_0}{\partial F_{iA}} \right) = 0 \] (14)

and show that

\[ \frac{\partial}{\partial X_A} \left( \frac{\partial W_0}{\partial F_{iA}} F_{iB} \right) = \frac{\partial W_0}{\partial F_{iA}} F_{iA,B} = \frac{\partial W_0}{\partial X_B} . \] (15)

We will now shift our attention to the strain energy \( W \) of the body per unit volume in the deformed configuration. It is related to \( W_0 \) by

\[ W_0 = JW . \] (16)
First note that (11) and (13) give

\[ \frac{\partial}{\partial x_i} \left( \frac{1}{f_{Ai}} \frac{\partial W_0}{\partial f_{Aj}} f_{Aj} \right) = 0 . \]  \hspace{1cm} (17)

(d) Verify the identity

\[ \frac{\partial J}{\partial f_{Ai}} f_{Aj} = -J \delta_{ij} \]  \hspace{1cm} (18)

and use it and (16) to simplify (17) to

\[ \frac{\partial}{\partial x_i} \left( \frac{\partial W}{\partial f_{Ai}} \right) f_{Aj} = 0 . \]  \hspace{1cm} (19)

(c) Explain why (19) can be reduced to

\[ \frac{\partial}{\partial x_i} \left( \frac{\partial W}{\partial f_{Ai}} \right) = 0 . \]  \hspace{1cm} (20)

(f) Compare (20) and (14). Discuss what this comparison shows about the problem that the inverse deformation (5) solves in comparison to the deformation (1).

For question (h) below you may limit your discussion to isotropic materials, so that

\[ W_0(E_{AB}) = \bar{W}(I_1, I_2, J) , \]  \hspace{1cm} (21)

where \( I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \ I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \) and \( J = \lambda_1 \lambda_2 \lambda_3 \), where the \( \lambda_k \) are the principal stretches of the deformation (1).

(g) State some conditions under which the deformation (1) (with strain energy \( W_0 \)) and the inverse deformation (5) (with strain energy \( W \)) are possible in the same material. In other words, under what conditions does \( W_0 = W = \bar{W} \) satisfy (16):

\[ \bar{W}(I_1, I_2, J) = JW(I_2, I_1, 1/J) \]  \hspace{1cm} (22)