Consider the diffusion equation,
\[
\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2},
\]
on \(0 \leq x \leq 1\), subject to \(\phi(0) = 0\) and \(\phi(1) = 1\).

We solve this equation on a uniform spatial mesh, with spatial step \(\Delta x\) and time step \(\Delta t\), using the following two schemes.

(I) \[
\left( \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} \right) = \nu \left( \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2} \right)
\]

(II) \[
\left( \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} \right) = \nu \left( \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2} \right) + \nu \left( \frac{\phi_i^{n+1} - 2\phi_i^n + \phi_{i-1}^{n}}{\Delta x^2} \right)
\]

(a) Derive the modified equation for scheme II, determine its spatial and temporal order of accuracy, and establish consistency of the scheme.

(b) By inspection, determine the spatial and temporal order of accuracy of scheme I (you do not need to derive the modified equation).

(c) Perform von Neumann stability analyses for schemes I and II, and derive the amplification factors.

(d) Experience shows that for large \(\Delta t\) scheme I converges to the steady state solution while scheme II oscillates from one iteration to the next \(\phi_i^{n+1} = -\phi_i^n\). Prove this.

(e) The scheme
\[
\left( \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} \right) = \nu \left( \frac{\phi_{i+1}^{n+1} - 2\phi_i^n + \phi_{i-1}^{n+1}}{\Delta x^2} \right)
\]
is stable if \(\frac{\nu \Delta t}{\Delta x^2} \leq \frac{1}{2}\). Let’s say you’ve written a code that uses this scheme. You specify \(\Delta x\) and \(\Delta t\) as input. Your code blows up. What would you try to do to fix the problem and why?