Two plane rigid bodies are connected by a hinge at point A that permits free rotation of one body with respect to the other. The bodies move in a horizontal plane on a fixed, frictionless surface. Body 1 has mass $m_1$, centroidal moment of inertia $I_1$ about an axis perpendicular to the plane, and center of mass $C_1$ at distance $r_1$ from hinge A. The quantities $m_2$, $I_2$, and $r_2$ are defined in like fashion for body 2 and its center of mass $C_2$.

![Diagram of two rigid bodies connected by a hinge.](image)

**Figure 1.** The two rigid bodies

The four degrees of freedom for the system may be specified by generalized configuration coordinates \( \{x_A, y_A, \phi, \theta\} \). Here \( \{x_A, y_A\} \) are the coordinates of A relative to the inertially fixed coordinate frame Ox, \( \phi_1 = \phi \) is the angle from the x-direction to $C_1A$, and \( \theta \) is the relative position angle between $C_1A$ and $AC_2$ (so that $\phi_2 = \phi + \theta$ is the angle from the x-direction to $AC_2$).

Suppose the system is set in motion with no net linear momentum but with $\dot{\phi}_1$ and $\dot{\phi}_2$ possibly non-zero. The motion proceeds under zero resultant external force and external moment. Only the internal forces and moments at A between the bodies then affect the motion. The moment exerted by body 2 on body 1 at A is $u(t, \theta, \dot{\theta})$, i.e., the moment depends only on the relative angular position and velocity.

(a) Arguing on physical grounds, what first integral or integrals of the equations of motion
do you expect should exist?

(b) The kinetic energy of such a system is of course

\[ T = \frac{1}{2} m_1 |\vec{v}_{C_1}|^2 + \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} m_2 |\vec{v}_{C_2}|^2 + \frac{1}{2} I_2 \omega_2^2. \]

where \( \vec{v}_{C_1} \) and \( \vec{v}_{C_2} \) are the inertial velocities of the mass centers and \( \omega_1 = \omega_1 \vec{k} \) and \( \omega_2 = \omega_2 \vec{k} \) are the angular velocities.

Show that the use of the two first integrals allows the velocity of \( A \) and hence the velocities of \( C_1 \) and \( C_2 \) to be expressed in terms of \( \phi_1, \phi_2, \dot{\phi}_1, \) and \( \dot{\phi}_2 \). By substitution in the kinetic energy show that the kinetic energy \( T \) can be written as a function of \( \phi_1 \) and \( \phi_2 \) and their derivatives in the form

\[ T = \frac{1}{2} I_1 \dot{\phi}_1^2 + \frac{1}{2} I_2 \dot{\phi}_2^2 + \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) \left( r_1^2 \dot{\phi}_1^2 + 2 r_1 r_2 \cos(\phi_2 - \phi_1) \dot{\phi}_1 \dot{\phi}_2 + r_2^2 \dot{\phi}_2^2 \right). \]

(c) Using the principle of virtual work to deduce the proper generalized forces, derive Lagrange’s equations of motion for \( \phi \) and \( \theta \). (Note that the \( \phi_2 \) equation is not asked for here.)

(d) Find the equilibria of this system. Describe what these configurations correspond to physically. Which equilibria are stable and which are not?

(e) Suppose the moment \( u \) is a linear function of the angle \( \theta \) and its rate of change \( \dot{\theta} \), so that \( u = -k \theta - c \dot{\theta} \) with \( k > 0 \) and \( c > 0 \) constants. How does this affect the dynamics?

(f) With \( u = -k \theta - c \dot{\theta} \) as in (e), linearize the equations of motion about the steady rotation \( \phi(t) = \omega t \), and \( \theta(t) = 0 \). Find the mass matrix \( M \) and the stiffness matrix \( K \) for this system. Identify any terms which do not appear in \( M \) or \( K \) and explain them. Find the eigenvalues and eigenvectors for this system.

(g) Let now \( u = -k \theta - c \dot{\theta} + v(t) \) where \( k \) and \( c \) are as in (e) and \( v(t) \) is an additional torque we can apply to the joint. Is the system completely controllable from \( v(t) \)? If not, describe the largest controllable subspace.