Given the cart with a pendulum attached shown in the figure below

\[ \begin{align*}
&M \quad \text{cart mass} \\
&m \quad \text{pendulum mass} \\
&l \quad \text{length of the pendulum} \\
&\theta \quad \text{pendulum angle from vertical} \\
&x \quad \text{cart position} \\
&y \quad \text{pendulum position} \\
&u \quad \text{force input} \\
&d \quad \text{disturbance force}
\end{align*} \]

The nonlinear rigid body equations describing the cart and pendulum are:

\[
\begin{align*}
(M + m)\ddot{x} + ml(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) &= u, \\
m(\ddot{x} \cos \theta + l\ddot{\theta} - g \sin \theta) &= d.
\end{align*}
\]

Answer the following questions about the system. Be descriptive with your answers.

1. Linearize the system about the two equilibrium positions: \((x, \theta) = (0, 0)\) and \((x, \theta) = (0, \pi)\), i.e., the pendulum either up or down.

2. Find the transfer functions from \(u\)-to-\(x\) and \(u\)-to-\(y\) for both linear systems.

3. Discuss the stability of the two systems and the location of the transfer function poles and zeros.

4. Consider the \(u\)-to-\(x\) transfer function for the up position of the pendulum (\(x\) is the measurement fed back to the control system).
   - (a) Discuss the relationship between the pendulum length \(l\) and the difficulty of stabilizing the system.
   - (b) Discuss the role the cart mass to the pendulum mass ratio, \(r := m/M\), plays in stabilizing the system.

5. Consider the \(u\)-to-\(y\) transfer function for the up position of the pendulum (\(y\) is the measurement fed back to the control system).
   - (a) Discuss how the stabilization problem has changed with \(y\) as the measurement signal as opposed to \(x\).

6. Discuss the role that sensor location can have on the difficulty of controlling a system and the ultimate achievable performance.

7. Discuss how right-half plane poles and zeros effect the control design of the cart-pendulum system.