Modeling Rayleigh-Taylor Instability of a Sedimenting Suspension Arising in Direct Numerical Simulation

R. Glowinski,* T.W. Pan* and D.D. Joseph†

* University of Houston, Houston TX 77024
† University of Minnesota, Minneapolis, MN 55455

Abstract

In this paper we study the sedimentation of 6400 circular particles in 2D using the method of distributed Lagrange multipliers for solid-liquid flow. The simulation gives rise to fingering which resembles Rayleigh-Taylor instability. The waves have a well-defined wavelength and growth rate which can be modeled as a conventional Rayleigh-Taylor instability of heavy fluid above light. The heavy fluid is modeled as a composite solid—liquid fluid with an effective composite density and viscosity. Surface tension cannot enter this problem and the characteristic short wave instability is regularized by the effective viscosity of the solid-liquid dispersion. The results of the simulation are in satisfying agreement with results predicted by the model using viscous potential flow without fitting parameters.

The data analyzed in this note is generated by the direct numerical simulation of solid-liquid flow by a distributed Lagrange multiplier/fictitious domain method. The method is based on a global variational formulation by Hesla (1991) and modified by Glowinski and Pan (see Glowinski *et al* 1997, 1998, 1999). The calculation is carried on fixed triangular mesh on which fluid equations are satisfied everywhere. Rigid motions of the portions of the fluid occupied by solids are accomplished by a strategic choice of a Lagrange multiplier field there. The method has a certain elegance in that the rigid motion constraint on the fluid is associated with a multiplier field in a manner analogous to the way in which the pressure in an incompressible flow is a multiplier field associated with the constraint on incompressibility. The details of the computation have been given in the cited references and will not be repeated here.

A 2D simulation of 6400 circles of diameter d = 1/12 cm, density ρ_p = 1.1 /cm^3 is arranged initially in a crystal of width 8cm and height 7.708cm shown in figure 1(a). The volume fraction of circles in this crystal is the ratio of the area A_P of the circles to the total area A_T of the crystal

$$\frac{A_P}{A_T} = \frac{6400\pi d^2/4}{8(7.708)} = 0.566 \tag{1}$$

The crystal of circles settles in water ρ = 1 g/cc, n_f = 0.01 poise. Between 13 and 14 waves develop at the bottom of the crystal (figure 1) and these waves remind us of Rayleigh-Taylor instability. The wavelength λ of these waves is therefore given by the inequality

$$8/14 = 0.571 \leq \lambda \leq 8/13 = 0.615\,\text{cm} \tag{2}$$

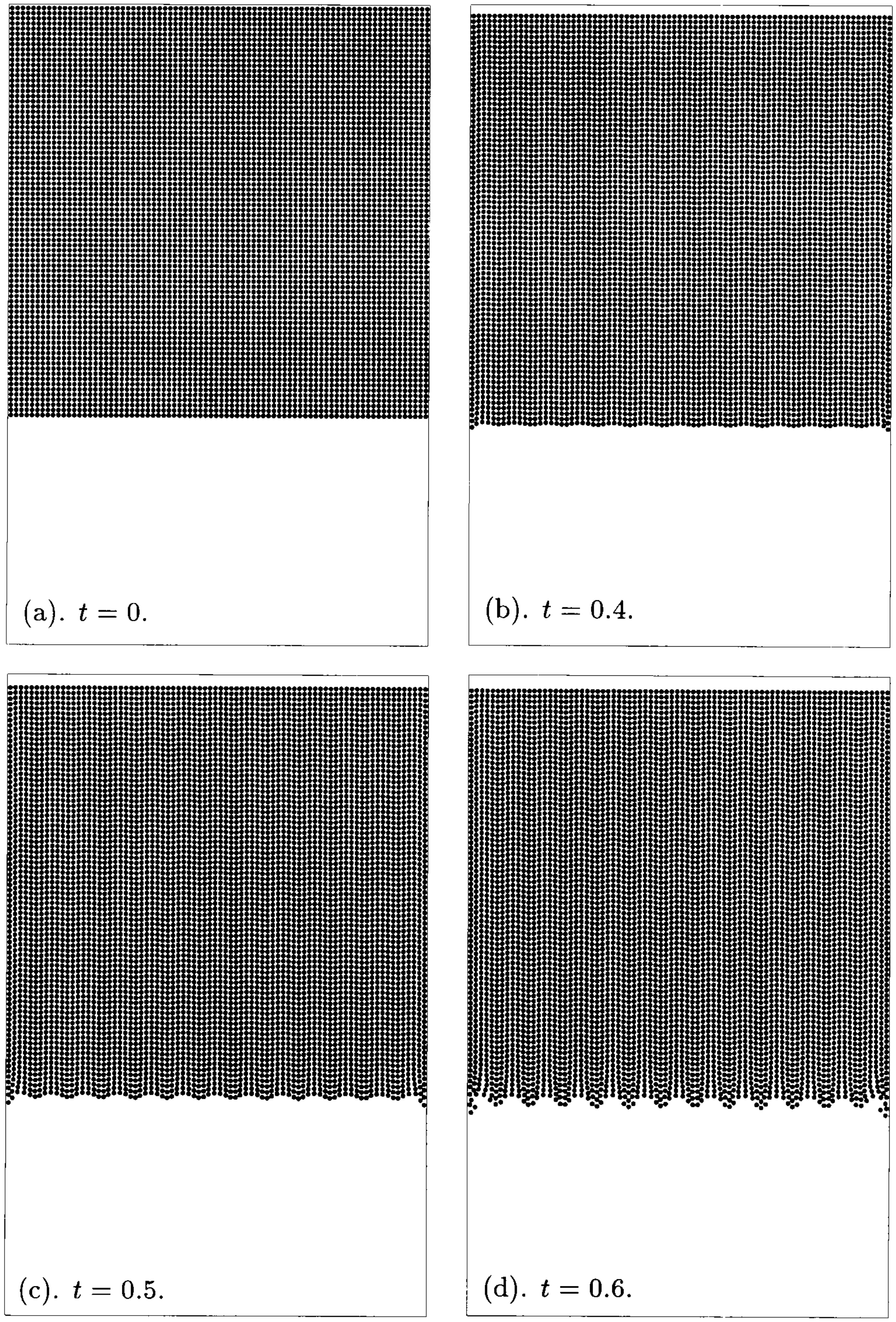

Figure 1. Snapshots of the simulation of 6400 circles in 2D. The arrangement of sedimenting particles is asymetric, flat on top and corrugated at the bottom. The drag on a single sphere is smaller than when it is among many so that isolated spheres on the bottom fall out of the crystal and isolated spheres on the top fall into the crystal.

ℓ	**t(sec)**	**A(cm)**	n_ℓ
1	0.101 0.151	3.65×10^{-4} 8.00×10^{-4}	17.31
2	0.201 0.251	1.62×10^{-3} 3.08×10^{-3}	14.15
3	0.301 0.351	5.63×10^{-3} 9.94×10^{-3}	12.81
4	0.401 0.451	1.66×10^{-2} 2.66×10^{-2}	9.50

Table 1. Growth rates for the wave amplitude $n_\ell = \log[A(t_{\ell+1})/A(t_\ell)](t_{\ell+1} - t_\ell)$.

The time step for this calculation was 0.001 sec; the growth of the wave amplitude at early times is given in table 1.

The average $\bar{n} = (n_1 + n_2 + n_3 + n_4)/4$ is

$$\bar{n} = 13.425\,\mathrm{sec}^{-1} \tag{3}$$

We turn next to the two-fluid modeling of the instability of the sedimenting suspension just described. The basic idea is to regard the particle laden portion of the sedimenting suspension shown in figure 1 as an effective fluid with an effective density ρ_2, an effective viscosity η_2 and, of course, zero surface tension γ; then we have two fluids, an effective one above and water below. The dynamics of this two-fluid problem is governed by viscous potential flow (Joseph & Liao 1994). Joseph, Belanger and Beavers (1999) showed that the wavelengths and growth rates obtained with viscous potential flow differ from those obtained from a fully viscous analysis by less than 1%. The success of the potential flow analysis arises from the fact that main action of viscosity is in the viscous part of the normal stress acting here in our problem through the effective viscosity of the solid-liquid suspension. Surface tension cannot enter into this problem so that the effective viscosity is the only mechanism which regularizes an otherwise ill-posed problem in which the growth rate increases like $1/\sqrt{\lambda}$, tending to infinity with ever shorter waves (Joseph & Saut 1990).

The analysis of Rayleigh-Taylor instability is carried out in an infinitely extended domain using the method of normal modes with disturbances proportional to

$$e^{\mathrm{int}} e^{i(k_x x + k_y y)} e^{\pm qz} \tag{4}$$

where, for viscous potential flow

$$q = k = \sqrt{k_x^2 + k_y^2} \tag{5}$$

where the z increases against gravity $g = 980\dfrac{\text{cm}}{\text{sec}}$ and the sign $\pm k$ chosen so that the amplitude decays at infinity. The analysis leads to the following dispersion relation (equation (25) of Joseph, Belanger & Beavers)

$$\rho_2 + \rho_f = \frac{k}{n^2}(\rho_2 - \rho_f)g - \frac{k^3\gamma}{n^2} - \frac{2k^2}{n}(\eta_2 + \eta_2) \tag{6}$$

Equation (6) depends on $q = k = \sqrt{k_x^2 + k_y^2}$ only through k in (5); hence (6) is valid in both two and three dimensions and it applies to the planar problem under discussion.

For the model under consideration the interfacial tension $\gamma = 0$; the effective density is given by

$$\rho_2 = \rho_P\phi + \rho_f(1-\phi) = 1.1\phi + (1-\phi) = 1 + 0.1\phi,$$
$$\rho_2 - \rho_f = 0.1\phi$$

and

$$\rho_2 + \rho_f = 0.1\phi + 2 \approx 2$$

in gm/cc. Inserting these values in (6), we get

$$2 = 0.1\phi\,\text{gk/n}^2 - 2k^2(\eta_2 + \eta_f)/n \tag{7}$$

where η_f = 0.01 poise, ϕ = 0.566 and

$$\eta_2 = \eta_f \Big/ \left(1 - \frac{\phi}{A_{2D}}\right)^2 \tag{8}$$

is the effective viscosity given by an empirical formula due to Kataoka *et al* (1978) for 3D, recommending A_{3D} = 0.680 as an effective maximum packing fraction. Metzner (1985) cites the 3D version of (8) with A_{3D} replacing A_{2D}, as the most successful of all empirical formulas the viscosity of concentrated suspensions. To obtain a value for A_{2D} we scaled using maximum packing fractions

$$\frac{A_{2D}}{A_{3D}} = \frac{V_{2D}}{V_{3D}} = \frac{0.9069}{0.74048} \tag{9}$$

where 0.74048 is the volume fraction of closed packed spheres in 3D and 0.9069 is the volume fraction of close packed discs in 2D. From (9) we find

$$A_{2D} = 0.8328 \tag{10}$$

and from (8) we get

$$\eta_2 = 0.01 \Big/ \left(1 - \frac{0.566}{0.8328}\right)^2 = 0.0950\,\text{poise}$$

Hence

$$\hat{\eta} \overset{def}{=} \eta_2 + \eta_f = 0.105\,\text{poise} \tag{11}$$

The dispersion relation (7) reduces to

$$2 = 55.37\,\text{k/n}^2 - 2k^2\hat{\eta}/n\,. \tag{12}$$

To get *k* which maximizes *n* we set *dn/dk* = 0 and find that

$$\frac{55.37}{n^2} - \frac{4k\hat{\eta}}{n} = 0$$

or

$$\text{k} = \frac{\alpha}{\hat{\eta}n},\ \alpha = \frac{55.37}{4} \tag{13}$$

Substituting *k* from (13) into (12) we get

$$1 = \frac{\alpha^2}{\hat{\eta}n^3}$$

The growth rate is given by

$$n = \left(\frac{\alpha^2}{\eta}\right)^{1/3} = 12.05\,\text{sec} \tag{14}$$

The associated wavelength is given by (13) and (14) as

$$k = \alpha^{1/3} \big/ \hat{\eta}^{2/3} = 10.49\,\text{cm}^{-1} \tag{15}$$

and

$$\lambda = 2\pi/k = 0.599\,\text{cm} \tag{16}$$

The values of n (14) and λ (16) from the two-fluid model are in good agreement with values (3) and (2) measured in the numerical simulation.

The construction of the model has involved certain choices, particularly with regard to the selection of an empirical formula for the effective viscosity of a concentrated solid-liquid dispersion and its scaling into two dimensions. All the choices made are conventional, perhaps not controversial, but they are model assumptions. No fitting parameters have been used to get agreement between the model and the numerical experiment; this could be as good as it gets.

Acknowledgement

This work was supported by the NSF under KDI/NCC grant NSF/CTS-9872326 and the University of Minnesota Supercomputing Institute. The paper is one of a sequence following from an NSF Grand Challenge Grant summarized on *www.aem.umn.edu/Solid-Liquid_Flows*. The work of Glowinski and Pan was also supported by the NSF under ECS 9527123, NSF DMS-9973318 and Dassault Aviation. Joseph's work was supported also by NSF GOALI grant and by DOE, Program of Engineering Research, Dept. of Basic Energy Sciences.

References

R. Glowinski, T.-W. Pan, T. Hesla, D.D. Joseph, and J. Périaux.
A distributed Lagrange multiplier/fictitious domain method for flows around moving rigid bodies: Application to particulate flow, *Int. J. Numer. Methods Fluids*, **30,** 1943-1066, 1999.

R. Glowinski, T.-W. Pan, T.I. Hesla, D.D. Joseph, and J. Périaux.
A Fictitious Domain Method with Distributed Lagrange Multipliers for Numerical Simulation of Particulate Flows, in *Domain Decomposition Method*s, **10**, (J. Mandel, C. Farhat, X.C. Cai eds.), American Mathematical Society, Providence, R.I., 121-137, 1998.

R. Glowinski, T.-W. Pan, T. I. Hesla, and D.D. Joseph.
A Distributed Lagrange Multiplier/Fictitious Domain Method for Flows Around Moving Rigid Bodies: Application to Particulate Flows, *Int. J. Multiphase Flow* **25**, 755-794, 1999.

T.I. Hesla. The Dynamical Simulation of Two-Dimensional Fluid/Particle Systems, unpublished notes (1991).

D.D. Joseph and T. Liao.
Potential flows of viscous and viscoelastic liquids, *J. Fluid Mech.,* **265**, 1-23, 1994.

D.D. Joseph and J.C. Saut.
Short-wave Instabilities and Ill-posed Initial-value Problems, *Theoretical Comput. Fluid Dynamics*, **1,** 191-227, 1990.

T. Kataoka, T. Kitano, M. Sasahara and K. Nishijima.
Viscosity of particle filled polymer melts, *Rheol. Acta*, **17,** 149-155, 1978.

A.B. Metzner.
Rheology of suspensions in polymeric liquids, *J. Rheol.* **29,** 739-776, 1985.