DYNAMICS OF FLUIDIZED SUSPENSIONS OF SPHERES
OF FINITE SIZE

P. SINGH and D. D. JOSEPH
Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis,
MN 55455, U.S.A.

(Received 13 January 1993, in revised form 12 June 1994)

Abstract—We propose a one-dimensional theory of fluidized suspensions in which the fluids and solids momentum equations are decoupled by using a new mean drag law for the particles. Our mean drag law differs from the standard drag laws frequently used in that the drag is assumed to depend on the area fraction rather than the number density. For a monodisperse suspension of spheres of radius \( R \), the area fraction and the number density are related by a simple geometrical construction that takes into account the area of intersection of the spheres with a plane perpendicular to the flow. For the linearized theory uniformly fluidized suspension is unstable but not Hadamard unstable. However, there is a distinguished set of marginally stable modes belonging to a countable set of blocked wave numbers \( \alpha = 4.493/R, 7.725/R, 10.904/R, \ldots \). The nonlinear theory contains bounded solutions when a certain dimensionless "growth rate" parameter is below a critical value. The power spectrum of these bounded solutions is broad banded in both space and time, and is very low for the wave numbers that are marginally stable in the linear theory. These results agree with our experiments, as well as with the previous experimental results from diffraction studies.

Key Words: fluidized suspensions, radial and area-averaged distributions, particle phase theories, Hadamard instability, bubbling instability, bounded solutions

I. INTRODUCTION

We begin with a brief description of the nature of voidage fluctuations in fluidized suspensions as observed by various authors. We first discuss the experiments done in two-dimensional beds and then similar experiments done in three-dimensional beds. Photographs of the motion of solid spherical particles fluidized by water in our two-dimensional bed (see figure 1) are shown in figure 2.

Figure 1. Schematic diagram of the two-dimensional bed used to carry out experiments. Particle diameters are listed in table 1. The bed is actually three-dimensional but the particles are constrained to move in two dimensions.
voidage cracks which propagate by particles dropping out of the roof (see JFLS, FIl). At higher fluidizing velocities, on the other hand, the bed spreads much more uniformly and the neighboring particles move independently on paths that are quasi-random. In this regime, the fluctuations of the particle velocities and the number density are also quasi-random.

Even though the particle trajectories in three-dimensional beds appear to be qualitatively similar, the above quantitative visual observations can be made relatively easily only for two-dimensional beds. For three-dimensional beds, however, it is easier to use the diffraction techniques to obtain the spatial arrangement of the particles directly in terms of the power spectrums (also called the structure factors), i.e. the square of the magnitude of the Fourier transform of the number density. For example, Pusey (1981) used dynamic light scattering to measure the structure factor for a monodisperse suspension of spherical particles of 0.09 μm dia. This particular value of the particle diameter was used because the light scattering technique can be used only to probe wavelengths that are comparable to the wave length of the visible spectrum. Wai et al. (1987); Wignall et al. (1990) and Oettwill (1991), on the other hand, have used neutron scattering to obtain the structure factor for 0.1 μm uniform spheres, i.e. spheres that have the scattering density uniformly distributed within the particles, as well as for 0.1 μm spheres that have a neutron transparent shell and a neutron scattering core. A comparison of these experimental results with our results reported in section 5 suggests that the spatial power spectrum, and hence also the spatial arrangement of the particles for two- and three-dimensional beds, including the effects associated with wave number blockage, are similar (see section 3: Singh (1991), and Singh & Joseph (1990, 1991)).

In section 3 we consider a one-dimensional theory of fluidized suspensions in which the fluids and solids equations are decoupled, and the system is closed with a momentum equation for the particles alone. The simplest theory available is based on the mean drag acting on a typical particle of the fluidized suspension (see for example, Jackson 1963) except that the force that the fluid exerts on the particles is assumed to depend on the local area fraction rather than the local number density (or equivalently the local volume fraction, when the local volume fraction is defined to be the product of the local number density and the volume of one particle). In a monodisperse suspension of spheres of radius R, the area fraction and the number density are related by a simple geometrical construction that takes into account the area of intersection of the spheres with a plane perpendicular to the flow, even when their centers are not exactly on the plane (see section 3 for details). Our one-dimensional theory then has three unknowns, the number density, the area fraction and the particle velocity, rather than two. The term based on the gradient of the volume fraction which expresses the particle phase pressure is not included in the present analysis. The two-variable theory is recovered in the limit R → 0 or when the wavelength of the disturbance studied is much larger than R. The three-variable theory, however, is fundamentally different from the two-variable theory because for the linearized three-variable theory the uniform fluidized suspension is not Hadamard unstable. The uniformly fluidized state is linearly unstable even in the three-variable theory, but there is a distinguished set of marginally stable modes belonging to a countable set of blocked wave numbers generated by the relation between the number density and the area fraction (see section 3).

The initial value problems for the nonlinear three-variable theory are solved numerically in a periodic domain (see section 4 for details). The solutions are bounded when a certain dimensionless "growth rate" parameter is below a critical value. These bounded solutions are found to be independent of the initial conditions. However, when the "growth rate" parameter is larger than the critical value then the numerical solutions are unstable in the sense that the power contained in the fluctuations grows with time, without bound. For a bounded solution, when it exists, both the temporal and the spatial power spectrums are broad banded, but the power level is very low for the wave numbers in the blocked set which, as we have already noted, are marginally stable in the linear theory. These results are in good agreement with the experimental results reported in section 5, and also with the results obtained by Wai, Wignall et al. and Oettwill for the number density distributions.

In section 5 we will report our results for the spatial distribution of the area fraction as a function of time that are obtained by analyzing the digitized video recordings of two-dimensional fluidized beds. This data is then used to obtain the temporal autocorrelation and spectrum at a point, the spatial autocorrelation and spectrum at a fixed time, as well as the two-dimensional spectrum in space and time. The temporal autocorrelation decays monotonically to zero at all Reynolds
numbers. The spatial autocorrelation, on the other hand, becomes negative and then goes to zero for large spatial shifts. The Fourier transform of the spatial autocorrelation function (i.e., the structure factor), as suggested by the geometric relation between the number density and the area fraction, contains deep minima at the blocked wave numbers (see section 3). We will show later that these minima in the Fourier transform of the spatial autocorrelation of the area fraction arise because the fluidized bed contains discrete spherical particles. The number density distribution can be obtained from the area fraction by inverting the convolution that relates the number density to the area fraction. However, since the zeros associated with the convolution function introduce error in the inversion of the convolution, an independent method is needed to verify the results of deconvolution. This verification is accomplished by using the results of the neutron diffraction studies that give the spatial Fourier transform of the number density distribution of a fluidized suspension of particles directly when the diffraction radius of the particles is different from their mechanical radius. These experiments conclusively prove that the minima of the number density and of the area fraction are at the same set of blocked wave numbers (see Wai, Wignall et al. and Ottewill). Obviously, the minima of the number density distribution are created by a dynamical mechanism which at present is not fully understood.

2. RADIALY SYMMETRIC DISTRIBUTIONS

As indicated in the previous section, the results of diffraction techniques, e.g., the light and neutron scattering techniques, also show that the set of dimensionless minima of the spatial number density spectrum is the same as the set of zeros of the blockage function (see Wai, Wignall et al. and Ottewill). In their analysis, however, these latter authors have used the radially symmetric distributions, to describe their results, instead of the area averaged distributions we use in this paper. But, it is easy to show that the Fourier transform of a radially symmetric distribution, and that of an area averaged distribution obtained from a radially symmetric distribution, are the same. This proves that the two distributions are equivalent from a mathematical point of view, i.e., the same information is contained in both distributions.

The radial distribution functions are used in statistical mechanics to describe the spatial distribution of atoms or molecules in liquids. Given a particle at the origin of the coordinate system, the radial distribution function, g(r, t), gives the probability of finding another particle at distance r from the origin. Since for isotropic systems, the radial distribution function is independent of the orientation, we have

\[ g(r, t) = g(|r|, t). \]

In the Fourier transform space this implies that

\[ g(x, t) = g(|x|, t). \]

where \( g(x, t) \) is the Fourier transform of \( g(r, t) \). It is obvious that a radially symmetric distribution function can be described completely by its distribution along any one ray originating from the origin. So we need to know only, e.g., \( g(2k, t) \) in the real space and \( g(k, t) \) in the Fourier transform space, where k is the unit vector along the z-direction and z is the component of \( z \) along the z-direction. Next, we show that for any radially symmetric distribution there exists a unique area averaged distribution, independent of the direction of the plane used for averaging, the Fourier transform of which differs from the Fourier transform of the radial distribution function by a constant. We also show that the mapping between the area averaged distribution and the radially symmetric distribution function is invertible.

We begin with a radially symmetric distribution \( g(r, t) = g(|r|, t) \). Its Fourier transform is given by

\[ g(x, t) = \frac{1}{(2\pi)^{1/2}} \int g(r, t) \exp(i r x) \, dr. \]

Since \( g(|x|, t) \) depends only on \(|x|\), we may simplify the above expression by using \( x = a k \)

\[ g(|x|, t) = \frac{1}{(2\pi)^{1/2}} \int \left[ \int g(x, y, z, t) \, dy \right] \exp(i a z) \, dz \]

\[ = \frac{1}{(2\pi)^{1/2}} \int g(z, t) \exp(i a z) \, dz \]

where

\[ g(z, t) = \int g(x, y, z, t) \, dx \, dy \]

is the area averaged distribution obtained by averaging the radially symmetric distribution \( g(r, t) \) over the \( xy \)-plane. Since the left-hand side of [2.1] is independent of the orientation of the plane used for averaging, \( g(z, t) \) is also independent of the orientation of the plane used for averaging. Furthermore, the symmetry of the radially symmetric distribution implies that \( g(z, t) \) is an even function of \( z \), and hence its Fourier transform is real. The last relation can be simplified further by noting that the expression on the right-hand side is the Fourier transform of \( g(z, t) \) times \( 1/(2\pi) \), i.e.,

\[ g(|x|, t) = \frac{1}{2\pi} \int g(z, t) \, dz. \]

Therefore, the Fourier transform of the area averaged distribution \( g(z, t) \), that is obtained from a radially symmetric distribution function using [2.2], is equal to \( 2\pi \) times the Fourier transform of the radial distribution function. This result allows us to go from the Fourier transform of an area averaged distribution to the Fourier transform of the radial distribution function, and thus from a one-dimensional area averaged distribution to the corresponding radial distribution. Hence, the mapping [2.2] between the radial distribution function \( g(r, t) \) and the area averaged one-dimensional function \( g(z, t) \) is invertible. Therefore, the one-dimensional area averaged distribution \( g(z, t) \) is equivalent to the radially symmetric distribution function \( g(|x|, t) \). Therefore, the results of diffraction studies are directly relevant to our one-dimensional theory, and a comparison between the results can be made in the Fourier transform space without transforming the data.

3. TWO-VARIABLE AND THREE-VARIABLE THEORIES OF FLUIDIZED SUSPENSIONS

We begin this section with a brief review of one-dimensional particle phase theories of three-dimensional fluidized suspensions. In these one-dimensional theories only the effects of variations of fields averaged on the planes perpendicular to flow are modelled. The words "particle phase" mean that the momentum balance for the fluid and solid phases is decoupled, and thus it is possible \( a \) priori to model the effects of fluid on the particles. In this case we get a two-variable, one-dimensional theory of the type proposed by Jackson (1963), Anderson & Jackson (1967), Wallis (1969), Foscolo & Gibilaro (1984, 1987) and Batchelor (1988) for the number density \( N(t) \) and the particle velocity \( u(z, t) \). The number density satisfies the usual conservation law

\[ \frac{\partial N}{\partial t} + \frac{\partial \langle u N \rangle}{\partial z} = 0 \]

where \( z \) is in the direction of the fluidizing velocity \( u \), and both \( N \) and \( u \) denote their respective values for the particles whose centers are at \( z \), at time \( t \).

In order to close the system with a momentum equation for the particles, we need an estimate of the average drag force acting on a typical particle of the fluidized suspension. Wallis, Foscolo and Gibilaro, and Batchelor have proposed essentially similar forms for the average drag. The main idea used in obtaining the functional form of the average drag is that the drag acting on a particle in a uniformly fluidized state exactly balances the buoyant weight of the particle. However, these drag laws neglect the fact that in a real fluidized suspension the spatial distribution of particles is nonuniform, and thus the particles are constantly subjected to quasirandom forces. Therefore, the drag law, at best, captures the time average of the actual time dependent quasi-random drag force.
In this paper we will use the following expression proposed by Foscolo & Gibilaro (1987) for the average drag force acting on a single particle

$$F(u, \Phi) = m g \left( (1 - \Phi) + \frac{u_r - u}{U(0)} \right) \left( (1 - \Phi)^{1/2} \right).$$

(3.2)

Here \( m \) is the mass of a single sphere of radius \( R \),

$$g = \frac{\rho - \rho_f}{\rho}$$

is the reduced gravity, \( \rho \) is the density of the sphere, \( \rho_f \) is the density of the fluid, \( U(\Phi) \) is the steady fall velocity under gravity of a sphere in a uniform dispersion of spheres of solid fraction \( \Phi \), \( \Phi = 1 - R/N \), \( u_r = u(1 - \Phi) - \Phi \) is the fluidization velocity and \( u_f \) is the fluid velocity. In order to show that the above expression satisfies the Richardson–Zaki correlation, we note that in a uniform suspension: \( F = 0 \), \( u_f = 0 \), and therefore

$$u_r = U(\Phi) = U(0)(1 - \Phi)'$$

(3.3)

where \( U(0) \) is the velocity of one sphere in a pure liquid which can be expressed in terms of the Reynolds number using various empirical correlations, and \( n(Re) \) is the Richardson & Zaki exponent; it lies between 4.8 for small Reynolds numbers \( Re = u_2R/v \) and 2.4 for large \( Re \).

We note that the above expression for the drag law assumes that the drag depends on the local solids fraction, defined as \( \Phi = \frac{1}{R} R/N \). It also assumes that in a uniformly fluidized suspension the particles are uniformly distributed and have zero velocity. But, as we have noted earlier, in a real fluidized bed these two conditions are never satisfied. Another problem is that once we have accepted the functional form of the drag law, then it applies to all spatial distributions of the solids fraction, including the case where the solids fraction changes at scales comparable or smaller than the diameter.

The momentum equation of Foscolo & Gibilaro can be expressed as

$$m \Phi \left( \frac{\partial \Phi}{\partial t} + \frac{\partial u}{\partial z} \right) = N \Phi \left( K \frac{\partial \Phi}{\partial z} \right)$$

(3.4)

where \( K \) is a constant that depends on the parameters of the fluidized suspension. The gradient term proportional to \( \partial \Phi/\partial z \) takes into account the contribution of the particle phase pressure. This type of system has also been developed by Batchelor and the gradient term there interpreted in terms of diffusion against the gradient of concentration that causes empty places to fill up as a result of small fluctuations in the particle velocity. This effect is analogous to the effect of Brownian motion in gases. In order to have diffusion against the gradient \( K \) must be positive.

Equations (3.1) and (3.4) are then a system of unidimensional equations in two variables \( \Phi \) and \( u \). Uniform fluidization with a constant \( \Phi = \Phi_0 \) and \( u = 0 \) is a solution of [3.1] and [3.4], but it is Hadamard unstable when the gradient term in [3.4] is neglected (see section 4). The gradient term can regularize this instability and even introduce regions in the space of parameters where the uniform state is stable.

Furthermore, it is easy to show that in this case the criterion for the loss of stability is independent of the wave number of the perturbation, so that the system is unstable at all, it is unstable to long waves as well as to short waves [see Jones & Prosperetti (1985); Prosperetti & Jones (1987) and Prosperetti & Saiappe (1989)].

In the remainder of this section we develop a new theory in which the finite size of the particles is accounted for in the drag law. The term based on the gradient of the volume fraction which expresses the particle phase pressure is not included in the present theory. As we shall see, this introduction of the finite size makes it a three-variable theory. We call it a zeroth order theory because the gradient terms are not included.

Central to our three-variable theory is a little construction which relates the ensemble averaged number density \( N \), to the fractional area. Consider a plane at \( z = z_1 \), perpendicular to gravity, as shown in figure 3. Let us consider a sphere of area \( A = L^2 \) in this plane with \( L > R \), so many spheres intersect the plane at \( z_1 \). Let \( x \) be the distance from the plane \( z_1 \). All spheres whose centers are at \( |x| < R \) have a nonzero area of intersection with the plane \( z_1 \). Also note that spheres with \( |x| > R \) do not touch the plane \( z_1 \). Recall that \( N(z_1 + x, t) \) is the number of spheres per unit volume with centers at \( z = z_1 + x \), and that the area of intersection of one of these spheres with the plane \( z_1 \) is \( \pi(R^2 - x^2) \) (see figure 3). The differential number of spheres contained in a rectangular box of height \( dx \), centered at \( z = z_1 + x \), is \( N(z_1 + x, t)\pi \). Hence the total differential area of intersection of the spheres contained in this rectangular box with the plane \( z_1 \) is \( \pi(R^2 - x^2)N(z_1 + x, t) dx \). Therefore, the area of plane \( z_1 \) covered by the particles, \( A(z_1, t) \), is obtained by summing all of the areas of intersections coming from infinitesimal volumes centered on \( z_1 \) as \( x \) varies from \(-R \) to \( R \):

$$A(z_1, t) = \int_{-R}^{R} N(x + z_1, t)\pi(R^2 - x^2) dx.$$

(3.5)

The fractional area of the plane \( z_1 \) covered by the particles is, \( A(z_1)/A = \Phi \). It is convenient to substitute \( \Phi = \pi R^2 N \) in [3.5], then after dropping the subscript \( t \) we get

$$\Phi(t, z_1) = \int_{-R}^{R} N(x + z_1, t)\pi(R^2 - x^2) dx = \frac{3}{4R^2} \Phi(x + z_1, t)(R^2 - x^2) dx.$$

(3.6)

From this equation we note that when \( \Phi \) is independent of \( x \)

$$\phi(z_1, t) = \Phi(z_1, t) = \Phi_0 = \frac{3}{4R^2} N_0$$

(3.7)

where \( N_0 \) is the average number density. Therefore, in this case the area fraction and the solids fraction are equal. This relation holds approximately also when \( R \) is small compared to the distance over which \( N \) varies significantly. We remind the reader that this is the condition under which the drag law [3.2] is derived.

The next step in the construction of the three-variable theory is to replace the solids fraction \( \Phi \) in the force law [3.2] with the area fraction \( \phi \), i.e.

$$F(u, \phi) = mg \left( (1 - \phi) + \frac{u_r - u}{U(0)} \right) \left( (1 - \phi)^{1/2} \right).$$

(3.8)

We would state this as a hypothesis that the average value of the drag force acting on a particle depends on the blocked area normal to the flow direction. Of course, the actual functional form of the drag is expected to be far more complicated, but since at present it is not known we will proceed with the understanding that our drag law is only a crude approximation of the actual drag law. However, note that the modified drag law [3.8] reduces to [3.2] in a uniformly fluidized suspension, as can be seen by substituting [3.7] in [3.8]. The same is approximately correct when the wavelength of the disturbance perturbing the uniform state is much larger than the particle diameter (i.e. in the long wave limit, see section 4 for details). Therefore, since the original drag law [3.2] was derived for uniform fluidization and that the two laws are identical in this limit, we are only modifying its form in the regime where its true form is not known. However, we wish to stress that our objective in this paper is only to show that the dispersion relation for a drag law
based on the area fraction has some unique features that are also present in the power spectrums of the real fluidized suspensions. The correct form of the drag law is not the central issue in this paper, and therefore we will only compare qualitatively the forms of the spatial and temporal power spectrums for the theory with that for the experimental data.

The zeroth order three-variable theory for the area fraction based drag law is then given by

\[
\frac{\partial \phi}{\partial t} + \frac{\partial (u \phi)}{\partial z} = 0,
\]

where \(\phi = 1 - \phi_e\). Uniformly obvious fluidization, \(u = 0\), \(\phi = \phi_e = \phi_a\), is a solution of this system of equations.

At this point we want to state how the quantities defined in our three-variable model are to be obtained. We obtain \(N(z, t)\) and \(u(z, t)\) by ensemble averaging in the manner set down by Joseph & Lundgren (1990). It is necessary to think of ensemble averages rather than volume averages because we shall be looking at fields that vary over the length \(L\) of the microstructure. The fractional area \(\phi_e\) is the convolution of the ensemble averaged number density. The main idea of our model is to replace the volume fraction \(\phi\) in the force law \([3.2]\) by the area fraction \(\phi_e\), where the two are related by the simple construction \([3.6]\). We have constructed our model as a hypothesis which seems reasonable, but at present cannot be proved. However, as we shall see, the results obtained from this theory are in good agreement with the experiment data presented in section 5, as well as with the results obtained from diffraction techniques by Wal, Wignall et al. and Ottewill.

In the remainder of this section we obtain some mathematical properties of relation \([3.6]\) for obtaining these properties we will assume that the number density is in the Fourier transform class because in this case it is easy to integrate the integral in \([3.6]\). By definition, when \(N\) is the Fourier transform class

\[
N(x, t) = \frac{1}{2\pi} \int_{x}^{x} N(z, t) \exp(izx) \, dx.
\]

where \(N(z, t)\) is the Fourier transform of \(N(x, t)\). By substituting this form of the number density in \([3.6]\), we get

\[
\Phi_e(z, t) = \frac{1}{2} \int_{x}^{x} \int_{x}^{x} N(\beta, t) \exp(i(\beta x + z)) \, dx \, d\beta.
\]

By taking the Fourier transform of the above equation, we have

\[
\Phi_e(z, t) = \frac{1}{2\pi} \int_{x}^{x} \int_{x}^{x} N(\beta, t) \exp(i(\beta x + z)) (R^2 - x^2) \, dx \, d\beta.
\]

After changing the order of integration in the above, we get

\[
\Phi_e(z, t) = \frac{1}{2\pi} \int_{x}^{x} \int_{x}^{x} \exp(i(\beta x + z)) N(\beta, t) \, d\beta \, dx + \frac{1}{2\pi} \int_{x}^{x} \int_{x}^{x} \exp(i(\beta x + z)) N(\beta, t) \, dx \, d\beta.
\]

or

\[
\Phi_e(z, t) = \frac{2R^3}{3} \int_{x}^{x} \Theta(\beta R) \, d\beta \int_{x}^{x} \exp(i(\beta x + z)) N(\beta, t) \, dx - \frac{2R^3}{3} \int_{x}^{x} \Theta(\beta R) \, d\beta \int_{x}^{x} \exp(i(\beta x + z)) N(\beta, t) \, dx.
\]

From this relation we arrive at the following conclusions:

1. If for a monochromatic disturbance of wave number \(k\), \(\Theta(\beta R) = 0\), then it is obvious from \([3.15]\) that \(\Phi_e(z, t) = \frac{2R^3}{3} \Theta(\beta R) N(\beta, t)\) is constant. Therefore, for the blocked wave numbers the area fraction is constant in space, even though the number density distribution oscillates in space. This raises the following very important question—what effect do these blocked wave numbers have on the number density distribution, and in the overall dynamical behavior of the system?

2. The fact that \(\Theta(\beta R)\) is a rapidly decaying function of \(\beta R\) (see figure 4) implies that large wave numbers in the area fraction spectrum are strongly damped. Therefore, if the average drag acting on the particles depends on the area fraction, then the role played by large wave numbers in the dynamical response is greatly reduced.

3. Lastly, we note from figure 4 that \(\Theta(\beta R)\) is negative for certain \(\beta R\). This implies that for a monochromatic disturbance of wave number \(k\), with \(\Theta(\beta R)\) negative, \(\Phi_e\) is smaller than \(\frac{2R^3}{3} \Theta(\beta R) N(\beta, t)\) at places where \(N\) is larger than \(N_0\), and vice versa. Furthermore, if the drag acting on a particle depends on the area fraction, then the magnitude of the drag is larger than the average drag at places where \(N < N_0\), and vice versa. This result is counter intuitive, but can be easily understood if we note that the wavelengths for which \(\Theta(\beta R)\) is negative are smaller than \(D\).
We show next that the area fraction flux \( q(z, t) \), passing through the plane \( z \) at time \( t \), is also blocked. In order to obtain an expression for the area fraction flux, we note that the net area fraction flux contribution from the particles that are between \( x \) and \( x + dx \) is \( N(x + z, t)u(x + z, t)(R^2 - x^2) \) dx. Therefore, the total flux can be obtained by integrating from \( -R \) to \( R \).

\[
q(z, t) = \int_{-R}^{R} N(x + z, t)u(x + z, t)(R^2 - x^2) \text{dx}.
\]  

Since the convolution in the above relation is same as in [3.6], it is easy to show that the spatial distribution of the flux is also blocked, i.e.,

\[
q(z, t) = \frac{4\pi R^3}{3} \Theta(zR)(Nu)(z, t)
\]

where \((Nu)(x, t)\) is the Fourier transform of the product \( N(z, t) u(x, t) \), and \(q(z, t)\) is the Fourier transform of \(q(z, t)\).

For a monochromatic periodic disturbance of wave number \( z \) an alternative relation between the area fraction flux \( q(z, t) \) and \( Nu \) can be obtained by assuming:

\[
(Nu)_{1}(z, t) = (Nu)_{b} + (Nu)_{1}(t) \text{Re} \{e^{i\omega(t - z)}\},
\]

After substituting the above expression for \( Nu \) in [3.16] and evaluating the integral, we get

\[
q(z, t) = a_{0} + \frac{4\pi R^{3}}{3} \Theta(zR)(Nu)_{b} \text{Re} \{e^{i\omega(t - z)}\},
\]  

where \( q_{0} = \frac{4\pi R^{3}}{3} \Theta(zR)(Nu)_{b} \) is the average value of the area fraction flux. The above relations allow us to conclude:

1. For a monochromatic number density flux disturbance of wave number \( z \), with \( \Theta(zR) = 0 \), the spatial distribution of the area fraction flux is constant, even though the number density flux oscillates in space. Therefore, for blocked wave numbers there is a net transport of the number density, but there is no net transport of area fraction.

2. For a monochromatic disturbance of wave number \( z \), with \( \Theta(zR) \) negative, \( q(z, t) \) is smaller than the average value of \( q_{0} \) at places where \( Nu \) is larger than \((Nu)_{b}\), and vice versa.

### 4. Stability of Uniform Fluidization

In order to study the stability of uniform fluidization, we linearize [3.9]-[3.11] around the solution \((\mu, \Phi, \phi) = (0, \Phi_0, \phi_0)\) and find that the system of perturbed equations, denoted by the subscript 1, is

\[
\begin{align*}
\frac{\partial \Phi_1}{\partial t} + \Phi_0 \frac{\partial \mu_1}{\partial z} &= 0, \\
\frac{\partial \mu_1}{\partial t} - \frac{3}{4} \frac{\partial \Phi_1}{\partial z} - \Phi_0 \frac{1}{4R} \int_{-R}^{R} \Phi_1(z + x, t)(R^2 - x^2) \text{dx} = 0.
\end{align*}
\]  

where

\[
\begin{align*}
\frac{\partial \mu_1}{\partial t} &= \mu_1 = -a_{0} - \frac{4\pi R^{3}}{3} \Theta(zR)(Nu)_{b} \text{Re} \{e^{i\omega(t - z)}\}, \\
\frac{\partial}{\partial t} &= \frac{4\pi R^{3}}{3} \Theta(zR)(Nu)_{b} \text{Re} \{e^{i\omega(t - z)}\},
\end{align*}
\]

After eliminating \( \epsilon \) and \( u \), from [4.1]-[4.3], we find a single second order equation

\[
\frac{\partial^2 \Phi_1}{\partial t^2} - \frac{3}{4} \frac{\partial \Phi_1}{\partial z} - \Phi_0 \frac{1}{4R} \int_{-R}^{R} \Phi_1(z + x, t)(R^2 - x^2) \text{dx} = 0.
\]  

The stability of uniform fluidization may be determined by analysis of [4.6] using normal modes

\[
\Phi_1 = \Phi_1 e^{i\omega t}.
\]

We obtain a complex dispersion relation of the form

\[
\sigma^2 + \alpha + i\beta \Phi_0 \Theta(zR) = 0,
\]

where \( \Theta(zR) \) is given by [3.14].

We may solve this quadratic equation for

\[
\sigma = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta \Phi_0 \Theta(zR)}}{2}.
\]

where

\[
\Sigma = \frac{4\pi R^3 \Phi_0 \Theta(zR)}{\alpha^2}.
\]

We wish to draw the readers attention to the importance of the blockage function \( \Theta(zR) \) because when it is zero the growth rate \( \text{Re}(\sigma) \) is also zero. We have already listed the zeros of \( \Theta(zR) \), i.e., \( z = 0, 4.493, 7, 7253, 10, 904, 11, \ldots \) Also note that the blockage function approaches zero for large values of \( z \) like \( 10,000 \), and therefore the growth rate approaches zero also for large wave numbers. These results for the growth rate are not unique to the nonlinear drag law used in this study. In fact, any drag law based on the area fraction which reduces to [4.3] gives rise to the same results. Therefore, the zeros of the dispersion relation cannot be used to check the validity of the drag law. However, the form of the dispersion relation for the area fraction based drag laws is markedly different from that for the number density based drag law because in the former case there is a set of neutrally stable modes which shall be shown to have significantly less power in the saturated state. These features of the area fraction based model are also seen in the experimental data.

We next consider the nature of the instability of the uniform state for two different limits of short waves \( z \approx \infty \). In the first limit we have \( zR \approx 0 \) as \( z \to \infty \). This corresponds to the classical case in which finite size effects are neglected which can be recovered from our theory when

\[
2 \Theta(zR) \approx x.
\]

It then follows from [4.9] and [4.10] that the growth rate

\[
\text{Re} \sigma \to \sqrt{\text{Re} \Theta(zR)} z
\]

is unbounded for short waves of wavelength \( 2\pi/z, z \approx \infty \). In this case the uniform state of fluidization is Hadamard unstable.

On the other hand, if \( zR \approx z \approx \infty \), then the \( \Theta(zR) \approx 0 \) and \( z \to \infty \), and the uniform state is unstable, but not Hadamard unstable. We say that finite size effects regularize the Hadamard instability. Furthermore, when \( R > 0 \) is fixed then at each and every zero of \( \Theta(zR) \) we have

\[
\sigma = 0 \text{ or } -\alpha,
\]

otherwise

\[
\text{Re} \sigma > 0.
\]

The graph of \( \text{Re} \sigma(zR) \) is shown in figure 5. From this graph we conclude that for \( R > 0 \), however small, uniform fluidization is unstable, but not Hadamard unstable, i.e., the finite size of particles is a regularizer. We also note that there is a blockage of waves of wavelength \( 2\pi/z \) for \( z = 0, 4.493, 7, 7253 \), etc. which are neutrally stable and also do not propagate.

Finally, we note that the growth rate is maximum for \( z \approx 2.6/R \). By putting this value in [4.10], and using [4.4] and [4.5], we get

\[
\Sigma = \Sigma_{\infty} = 2.17n_{\text{eff}} \Phi_0 \frac{1}{\phi_0} \frac{\Phi_0}{\phi_0}.
\]

In section 5 we shall see that \( \Sigma_{\infty} \), which for the linearized theory determines the maximum value of the growth rate, determines the amplitude of the fluctuations for the solution of the nonlinear
5. EXPERIMENTS

In this section we briefly describe our experimental results for the average fraction in a two-dimensional bed obtained using the SPM Physics Motion Analysis System (SPMAS) (see Singh (1991) for details). The average area fraction \( \phi_a(t, z) \) is the fraction of horizontal plane \( z \) covered by the particles at time \( t \). We obtained \( \phi_a(t, z) \) at a discrete set of points at constant intervals of \( z \) and \( t \). The data obtained were stored in a two-dimensional array

\[
\phi_a(i, j) \quad i = 1, \ldots, N, \quad j = 1, \ldots, M,
\]

where \( \phi_a(i, j) = \phi_a(i, z), t = IT, z = iZ, T \) is the sampling time and \( Z \) is the sampling distance. The sample mean, \( \phi_m = \frac{1}{NM} \sum \phi_a(i, j) \), was removed from \( \phi_a(i, j) \). The new zero-mean array thus obtained is for convenience again denoted by \( \phi_a(i, j) \). For SPMAS, \( M = 239 \) but \( N \) is essentially unlimited.

For spherical particles listed in table 1 and fluidized in water, we have obtained the data arrays, \( \phi_a(i, j) \). The average solids fraction was held approximately constant around 0.25 for three cases listed in table 1. The Reynolds number was changed by changing the density of the fluidized particles. The goal was to analyze the two-dimensional array \( \phi_a(i, j) \) for the presence and properties of any spatial or temporal structure, traveling waves or any other distinct statistical structure, and to study how the structures change with the Reynolds number.

The temporal autocorrelation for a continuous time ergodic process \( \phi_a(t, z) \) is given by

\[
r_a(t, z) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi_a(t, z) \phi_a(t + \tau, z) dt
\]

where \( \tau \) is the time shift. The temporal autocorrelation function in the discrete case is given by

\[
r_a(n) = \frac{1}{N-n-1} \sum_{i=1}^{N-n} \phi_a(i, j) \phi_a(i + n, j)
\]

Table 1. Diameter, density and Reynolds number for four cases. The Reynolds number is \( Re = \frac{u A}{v} \) where \( v \) is the kinematic viscosity of water and \( u = \frac{Q}{A} \) is the superficial velocity, \( Q \) is the volume flow rate and \( A \) is the area of cross section.

<table>
<thead>
<tr>
<th>Particles</th>
<th>( D ) (mm)</th>
<th>Density (g cm(^{-3}))</th>
<th>( Re )</th>
<th>( \phi_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plastic</td>
<td>0.63</td>
<td>1.17</td>
<td>300</td>
<td>0.24</td>
</tr>
<tr>
<td>Glass</td>
<td>0.60</td>
<td>2.46</td>
<td>1650</td>
<td>0.23</td>
</tr>
<tr>
<td>Aluminium</td>
<td>0.69</td>
<td>2.70</td>
<td>2000</td>
<td>0.14</td>
</tr>
<tr>
<td>Rubber</td>
<td>0.68</td>
<td>4.22</td>
<td>2750</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Figure 6. The temporal autocorrelation function as a function of the temporal shift. The temporal sampling time is \( 0.1 t \). The plastic spheres which are fluidized at the smallest Reynolds number (see table 1) have the longest memory. Higher Reynolds numbers are associated with faster decay of the autocorrelation function.

Figure 7. The spatial autocorrelation function as a function of the spatial shift. The sampling distance \( Z = 0.075\) cm. The variance \( r_0(0) \) is a monotonic function of the Reynolds number (see table 1). \( r_0(0) \) minimum for a spatial shift of 0.64 \( D \) in dimensional terms. The recovery of \( r_0(m) \) to zero is faster when \( Re \) is small.
where the time shift $\tau$ is related to $n$ and the sampling time $T$, by $\tau = nT$. Similarly, we may compute the spatial autocorrelation function from

$$ r_s(\zeta) = \lim_{L \to \infty} \frac{1}{L} \int_0^L \phi_s(z) \phi_s(z + \zeta) \, dz. \tag{5.3} $$

where $\zeta$ is the spatial shift. However, in a practical problem, the spatial length $L$ over which the samples are available is usually too small to give an accurate estimate of the spatial correlation. This problem is easily resolved for a stationary ergodic process with finite temporal memory by averaging over the samples that are obtained after a long enough interval of time. Thus, for an ergodic process the resulting estimate is given by:

$$ r_s(\zeta) = \frac{1}{N^2} \sum_{i,j} \left[ \frac{1}{L} \int_0^L \phi_s(z) \phi_s(z + \zeta) \, dz \right] \tag{5.4} $$

Figure 9. The autoregressive power spectrum for the spatial autocorrelation function (5.3) as a function of nondimensional wave number $\alpha_D$. The value of $\alpha_D$ at the first maximum decreases with increasing $Re$.

Figure 10. Normalized temporal autocorrelation for the discrete equivalent of the time derivative of the solid fraction. Such changes are uncorrelated for random processes. This figure shows that increases in $\phi$ are followed on average by decreases; they are negatively correlated.

where $i$ runs over a set of statistically independent samples of $\phi_s$. It is straightforward to show that for the discrete case the spatial autocorrelation function is then given by

$$ r_s(m) = \frac{1}{N^2} \sum_{i,j} \left[ \frac{1}{M-m} \sum_{i,j,m} \phi_s(i,j) \phi_s(i,j + m) \right] \tag{5.5} $$

where $i$ runs over a set of statistically independent spatial distributions, and the spatial shift $\zeta$ is related to $m$ and the sampling time $Z_i$ by, $\zeta = mZ_i$.

Figure 11. Isovalues of two-dimensional power spectra as a function of the nondimensional wave number and temporal frequency: (a) plastic, (b) glass, (c) aluminium, (d) rubber (see Table 1).
In order to study the statistical behavior of the changes, we define the following differential process:

\[ \Phi(i, j) = \Phi(i + 1, j) - \Phi(i, j) \]

Its autocorrelation function

\[ r(i, j) = \frac{1}{N - i} \sum_{i=1}^{N} \Phi(i, i)\Phi(i + n, j) \]  

is obtained in a similar manner, and is shown in figure 10. From this figure, we note that \( r(i, j) \) is independent of \( j \), is negative, and increases monotonically until it becomes zero. As we have already noted in the study of the temporal spectra, since \( r(i, j) \) goes to zero faster at larger \( \Re \), the memory is shorter at larger \( \Re \). Also note that \( r(i, j) \) increases in magnitude with increasing \( \Re \). Therefore, the magnitude of the fluctuations \( \Phi(i, j) \) is larger at larger \( \Re \), and since the correlation is negative, the mechanism which counts the fluctuations of the area fraction is stronger at larger \( \Re \). Furthermore, since the correlation is nonzero, the fluctuations are not completely random. But, of course, the process has a short temporal memory.

We turn next to a detailed study of the nature of the time-averaged spatial autocorrelation (figure 7). For all cases considered the spatial autocorrelation function decreases rapidly and becomes negative. After reaching its maximum negative value it increases and approaches zero uniformly. It is zero for large spatial shifts. The length of the negative spatial memory increases with \( \Re \). This number is probably due to the waves getting bigger with increasing \( \Re \). The spatial shift at which the spatial autocorrelation function becomes zero gives us the length scale over which the particle positions are correlated. Therefore, in a fluidized suspension the nearby spheres arrange themselves in a somewhat organized way, but there is no such organization over large distances. Also note that both the length scale of the organized structures and the variance \( \sigma \) increase with increasing \( \Re \). Also, since the power spectrum of the temporal autocorrelation is broad banded, the fluctuations of the voidage from the average zero value are not periodic in time. If we assume that these fluctuations in the voidage are due to propagating waves then we can safely say that there are no dominant wave numbers.

The spatial spectrum shown in figure 9 is essentially a one-dimensional object, and thus cannot be used to analyze the propagation of structures. To study propagation we use the periodogram method to obtain the two-dimensional power spectrum of \( \Phi(i, j) \):

\[ F(\omega, z) = \left| \sum_{i=1}^{N} \sum_{k=1}^{M} \Phi(i, k)\exp(-2\pi ikx/N)\exp(-2\pi jz/M) \right|^2 \]

where \( j = \sqrt{-1}, \omega \) is the temporal frequency and \( z \) is the wave number. Again, since \( M \) is only 239, the large variance (or the error) of the above estimate is reduced by averaging over several such statistically independent estimates. The two-dimensional power spectrum \( F(\omega, z) \) gives a mathematical description of the dispersive properties of the large structures found in the bed. From figure 11 we note that \( F(\omega, z) \) is a decreasing function of \( \omega \) and contains valleys with relatively less power for the blocked wave numbers. In fact, the valleys in the two-dimensional spectra are so overwhelming that it is difficult to easily visualize any other feature. A detailed study of the spectrum, however, reveals that the blockage exists even in the following one-dimensional distribution. \( \omega_0(z) \) where \( \omega_0(z) \) is the frequency for which \( F(\omega, z) \) is maximum for a fixed value of \( z \); thus

\[ F(\omega_0, z) = \max F(\omega, z), \]

as can be seen in figure 12 where it is plotted as a function of \( z \). We note from this plot that for the blocked values of \( z \) the signal peaks at \( \omega = 0 \) and that for the unblocked \( \phi \) it peaks at \( \omega \neq 0 \). This shows that the propagation velocity of the waves is also related to the blockage function. In particular, the propagation velocity is zero for the blocked wave numbers and it is nonzero for the unblocked wave numbers. This dynamical feature is consistent with the linearized zeroth order theory (see section 4) where we have shown that the waves which correspond to the blocked wave numbers have zero propagation velocity.
6. Nonlinear Analysis

In this section we present numerical solutions of [3.9]-[3.11] obtained by integrating in time for different initial conditions. The Fourier-collocation method is used to spatially discretize the equations. The method assumes that $u$ and $N$ are periodic in space. All results presented in this paper were obtained by solving the equations in a domain, 0 ≤ $z$ ≤ $D$. For the Fourier collocation method $u$ and $N$ are given by

$$u(z, t) = \sum_{k=-j}^{j-1} \hat{u}_k(t) e^{i k z}$$

$$N(z, t) = \sum_{k=-j}^{j-1} \hat{N}_k(t) e^{i k z}$$

where $j$ is the number of collocation points. All results presented in this study are for $2j = 512$ or 1024, and have been verified for convergence with increasing $j$. The collocation points are uniformly distributed within the computational domain. We use a staggered grid for the velocity $u$ and the number density $N$. The velocities $u(z, t)$ are defined as $z = 0, \pi/l, \ldots, (2j-1)/l$, and the number densities $N(z, t)$ are defined at $z = \pi/2l, 3\pi/2l, \ldots, (4j-1)/2l$. A set of nonlinear ordinary differential equations (ODEs) for $u(x, t)$ and $N(x, t)$ is obtained by substituting the above representations for $u$ and $N$ in [3.9]-[3.11]. These nonlinear ODEs are then discretized using the fifth order implicit Adam–Moulton method. The system of nonlinear equations thus obtained is solved by using the Newton–Raphson method.

In the next few paragraphs we discuss some of the properties of our numerical method. We begin by showing that the use of exponentials as interpolation functions allows us to obtain the integral term in [3.10] exactly, i.e.

$$\phi_k(z, t) = 1 - e^{-z} \int_{-\infty}^{\infty} N(z + \xi, t) \pi(R^2 - \xi^2) d\xi$$

where $\pi(R^2 - \xi^2)$ is the probability density function of the random variable $R^2$. The expression above can be simplified to

$$\phi_k(z, t) = 1 - e^{-z} \int_{-\infty}^{\infty} N(z + \xi, t) \pi(R^2 - \xi^2) d\xi$$

as a function of time. Note that $|N(t)|^2$ and $|u(t)|^2$ are also the variances of the number density and the velocity distributions, respectively. Using our numerical results we would show that the power contained in the velocity and number density fluctuations increases or decreases with time, depending on the relative magnitude of its initial value to that of the bounded state. We call such solutions bounded. By the word "bounded" we mean that the solution is time dependent, but has power level bounded from both above and below.

In order to follow the progress of our numerical solution, we monitor the power contained in the velocity fluctuations.

$$|u(t)|^2 = \frac{1}{2j} \sum_{k=-j}^{j-1} |\hat{u}_k|^2,$$

and the number density fluctuations.

$$|N(t)|^2 = \frac{1}{2j} \sum_{k=-j}^{j-1} |\hat{N}_k|^2$$

as a function of time.
fluctuations for two different initial conditions, as a function of time. For the first case, the initial power is smaller than that for the bounded state, and for the second case the initial power is larger than that for the bounded state. For both cases the initial spatial distribution for $N$ and $u$ fields is assumed to be random. As expected, for the first case the power contained in the fluctuations grows with time until the nonlinear terms become comparable to the linear terms because the nonlinear terms in this problem are such that the growth of the fluctuations is stopped. The power contained in the fluctuations stops growing when it reaches the level equal to that of the bounded state, and this level is then approximately maintained. On the other hand, in the second case the fluctuations lose power with time until the power is down to the same approximate level as in the first case, and again this level is then approximately maintained. For these two different initial conditions, the power spectrum of the converged bounded solutions, for both the number density and velocity fields, are indistinguishable.

Next, we study the dependence of the bounded solutions on the model parameters. From table 2 we note that for fixed $n$ and $\Phi_0$, the power contained in the fluctuations increases with increasing $\Sigma_m$. But, for a given $n$ and $\Phi_0$, there is a maximum value of $\Sigma_m$ for which a bounded solution exists. When $\Sigma_m$ is larger than this maximum value the magnitude of the fluctuations of $W(z, t)$ becomes

Figure 13. The power contained in the fluctuations of (a) velocity and (b) solids fraction is plotted as a function of time for two different initial conditions.

Figure 14. For $\Sigma_m = 0.0004$, $\Phi_0 = 0.3$ and $n = 4.8$ the power spectrum of the bounded solution is shown. The blocked modes have very little power: (a) solids fraction, (b) velocity.
comparable to the average of $N$. This results in the failure of the numerical scheme because there are regions in the domain where $N(z,t)$ is zero (or even negative for numerical simulations). For example, for $n = 4.8$ and $\Phi_0 = 0.3$, the largest value of $\Sigma_n$ for which a bounded solution exists is 0.93. From the Richardson-Zaki correlation we know that $n = 4.8$ corresponds to small Reynolds numbers. We have picked this particular value of $n$ for our numerical study because the experimental data available for three-dimensional beds; the data by Pusey, Wai, Wignall et al. and Oettle, is for this regime of Reynolds numbers. However, as we have noted earlier, since the experimental data for two- and three-dimensional beds is qualitatively similar, our numerical results can be compared qualitatively with the data presented in section 5 for the two-dimensional bed. But, we must remember that the theory is for unbounded three-dimensional fluidized suspensions.

Although, the bounded solutions have nearly constant power their time evolution as well as their spatial distribution is complicated. Therefore, only the statistical nature of these bounded solutions, in terms of the temporal and spatial power spectra, is described here. Both spectrums, i.e. the temporal evolution at a fixed point and the spatial distribution for a fixed time, are broad banded [see figures 14(a)-(b)]. From figure 14(a) we note that the spatial power spectrum of the number density contains relatively small power for the blocked wave numbers. This is in general agreement with the experimental results we have presented in section 5, and also with the experimental results of Pusey, Wai, Wignall et al. and Oettle discussed earlier in this paper.

The Lagrangian acceleration for a typical particle and its spectrum are shown in figure 15. Since the spectrum is broad banded, the motion of an individual particle is also a complicated function of time. This agrees well with the observation of VMZ that the particles move on quasi-random paths.

We have noted earlier that for both experiments and numerical simulations, the power contained in the fluctuations and also the amplitude of the fluctuations, depend on the flow parameters. From a practical point of view, we want the power contained in the fluctuations to be as small as possible because then the heat and mass transfer rates between the fluid and the particles are maximum. In a real fluidized suspension, the judgement whether or not the fluctuations are sufficiently small is usually arrived at by looking at the spatial uniformity of the particles and their motion. Therefore, it is important to compare the statistical nature of the fluctuations for a real fluidized suspension with that for the model. We have already seen that the forms of spatial and temporal power spectrums are similar for the suspensions and the model. Now we note that for both the suspensions and the model, it is possible to control the amplitude of the fluctuations by controlling the parameters. In particular, when $\Sigma_n$ is smaller than one, the area fraction fluctuations are small compared to its average value. In experiments, the magnitude of the fluctuations increases with increasing flow rate which in turn is proportional to $\Sigma_n$ (or Re, see section 5). Therefore, when $\Sigma_n$ is small, the spatial distribution appears uniform because the uniform state dominates, see figure 16. On the other hand, when $\Sigma_n$ is of order one but smaller than one, the area fraction fluctuations are comparable to the average area fraction in the bed. In this case the spatial distribution does not appear uniform. Furthermore, when $\Sigma_n$ is larger than one, then the fluctuations are large enough to produce regions in the domain where the area fraction is very small (or even negative for numerical simulations). However, the numerical method failed in this regime. Therefore, we may conclude that as the parameter $\Sigma_n$ is increased, the spatial distribution of the particles becomes less uniform as the magnitude of the fluctuations increases (see table 2). This allows us to propose the criterion that when

$$\Sigma_n < 1$$

[6.5]

then the spatial distribution of the particles is sufficiently uniform. That is, there are no regions in the domain where the number density $N$ is too small or too large.

It is interesting to compare the above criterion with the criteria of Batchelor, and Foscolo & Gibilaro for predicting the stability of the fluidized suspensions. We note that their criteria are based on linear stability analysis. Here we will compare our criterion only with the Foscolo &

Figure 15. The Lagrangian acceleration, $\Sigma_n = 0.0094$, $\Phi_0 = 0.3$ and $n = 4.8$: (a) plotted as a function of time, and (b) the power spectrum. The power spectrum is broad banded, hence the particle motion is chaotic.
time dependent bounded solutions with nearly constant power when $\Sigma < 1$. These results are consistent with experimental results for two- and three-dimensional beds where one finds that there are always some fluctuations in the area fraction and number density distributions, and that the amplitude of the fluctuations increases with increasing flow rate (or $\Sigma$). In fact, it is possible to measure the magnitude, as well as the statistical nature of the fluctuations, as a function the parameters. The statistical nature of the fluctuations is described in terms of the temporal and spatial correlations, and their spectra. The spatial spectrum contains all wavelengths, including wavelengths that are an order of magnitude smaller than the particle diameter. In fact, the maximum of the power spectrum is at a wavelength that is comparable to the particle diameter, and wavelengths smaller than the diameter contain a significant portion of the total energy contained in the fluctuations. Therefore, any theory that attempts to describe the nature of fluctuations, must not only allow for wavelengths smaller than the diameter, but also predict their dynamical behavior correctly. This is, at least partially, accomplished in the present zeroth order theory because it allows all wavelengths to grow, and also correctly predicts the qualitative form of the spatial power spectrum, including blockage.

We also wish to note that there is no evidence of stable uniform fluidization, neither in the two-dimensional fluidized suspensions reported by VMZ and here, nor in three-dimensional fluidized suspensions of Pusey, Wai, Wignall et al. and Ottewill. Nor is there any indication of Hadamard instability.

We conclude by making the following additional observations:

- For a monodisperse suspension of spherical particles, if the number density is in the Fourier transform class, then the geometric relation [3.6] can be used to show that the Fourier transform of the area fraction contains blockage (see equation [3.13]). This equation also shows that if a particular wave number is missing in the number density spectrum, or contains very small power, then the same is true for the area fraction spectrum. The opposite is also true, except when the blockage function is small.
- Both, the experimental data and the results obtained numerically for the zeroth order theory, show that the fluctuations have a unique autocorrelation. The temporal autocorrelation is a monotonically decreasing function of the temporal shift, and the spatial correlation decreases to zero for a unique value of the spatial shift which decreases with increasing Reynolds number. The variance (or the magnitude) of the fluctuations at a point increases with increasing Reynolds number.
- The criterion $\Sigma < 1$ which we have found for numerical solutions with bounded power is similar to the Foscolo–Gibilaro criterion for bubbly beds which have obtained by linear stability analysis of uniform fluidization.

Acknowledgements—This research was supported under grants from the National Science Foundation, the US Army Research Office, Mathematics, the Department of Energy, AHP/CRC and the Supercomputer Institute of the University of Minnesota.

REFERENCES


