Stokes Flow in a Driven Sector by Two Different Methods

A biorthogonal series expansion and a numerical finite-difference approximation are applied to the problem of steady Stokes flow in a driven sector of 10° total angle, providing mutual support of the theoretical techniques. For this problem the method of biorthogonal series is faster, cheaper, and more accurate.

Introduction

In this paper we model the Stokes flow in a long driven sector, using finite differences and a biorthogonal series expansion to compare the results. The problem is chosen from a modified Couette flow including a sector cavity [1]. Our aim is to examine closely the results of the approximate finite difference solution and to advertise the biorthogonal series for solving biharmonic boundary-value problems in domains where separation of variables is possible (a very common problem in fluid mechanics and elasticity). The analytic method is elucidated in [2, 3]. New aspects concerning the computation are developed here.

Mathematical Formulation

The slow motion of a Newtonian liquid, neglecting gravity (Stokes flow) for two-dimensional flow is described by

\[ \nabla^4 \psi = 0 \quad (1) \]

where \( \psi \) is the stream function and \( \nabla^2 \) is the Laplacian operator. Using polar coordinates \((r, \varphi, z)\),

\[ \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \]

and the velocity \( v = \text{rot} (\nabla \psi) \). In our model there shall be viscous nonlip at the solid walls \( \varphi = \pm \beta \) and \( r = r_0 \). At the outer radius \( r = r_1 = 1 \) we prescribe the vorticity \( \Omega = 1 - \sin^2 \left( \frac{\pi \psi}{2} \right) \), where \( \Omega = -\nabla^2 \psi \), and no flow through the surface shall be possible (Fig. 1). For our comparison we chose \( r_0 = 0.05 \) and \( 2\beta = 10^\circ \).

\[
\begin{align*}
\psi &= 0, \\
\Omega &= 1 - \sin^2 \left( \frac{\pi \psi}{2} \right), \\
\psi &= \psi_n = 0
\end{align*}
\]

Fig. 1 The biharmonic sector problem

The Series Solution

The theory of biorthogonal series for biharmonic functions as described in [2, 3] allows us to write the solution of (1) in the form
Table 1 The first five eigenvalues (note that $\lambda_{-n} = \lambda_n$ where overbar denotes complex conjugate)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25.141144141 + 12.86408537i</td>
</tr>
<tr>
<td>2</td>
<td>62.38088865 + 17.7499684i</td>
</tr>
<tr>
<td>3</td>
<td>98.82483851 + 20.3618729i</td>
</tr>
<tr>
<td>4</td>
<td>135.06293018 + 22.08005236i</td>
</tr>
<tr>
<td>5</td>
<td>171.21666479 + 22.42566135i</td>
</tr>
</tbody>
</table>

\[
\psi = \sum_{m=-n}^n \left( C_m \rho^{\lambda_m} + D_m \rho^{-\lambda_m + 2\beta} \right) \frac{\phi_1(n)(\rho)}{\lambda_n} \lambda_n (\lambda_n - 2) \psi_2(n) = \phi_1(n)(\rho) \psi_1(n) \]

where $\phi_1(n)(\rho) = \cos(\lambda_n - 2\beta) \cos \lambda_n \rho - \cos \lambda_n \beta \cos (\lambda_n - 2\beta) \rho$, the $\lambda_n$ are roots of $[2\beta(\lambda_n - 1) + (\lambda_n - 1) \sin 2\beta] = 0$ (see Table 1) and $C_0 = D_0 = 0$. The boundary conditions at $\rho = 0, \beta$ are already satisfied, so that the constants $C_n$ and $D_n$ will have to match the conditions at the inner and outer radius.

We introduce the biorthogonal sequence $\psi_1^{(n)}, \psi_2^{(n)}$, where

\[
\psi_1^{(n)} = \phi_1^{(n)}(\rho) \psi_2^{(n)} = \phi_1^{(n)}(\rho) / \lambda_n (\lambda_n - 2),
\]

for $n = 1, 2, 3, ...$ corresponding to adjacent $\psi^{(n)}$ with

\[
\psi_1^{(n)} = \frac{(\lambda_n - 2)}{\lambda_n} \cos(\lambda_n - 2\beta) \cos \lambda_n \rho - \frac{\lambda_n}{\lambda_n - 2} \cos \lambda_n \beta \cos (\lambda_n - 2\beta) \rho;
\]

such that

\[
(\psi_1^{(n)} A \psi_1^{(m)}) = \int_0^\beta \psi_1^{(n)}(\rho) \psi_1^{(m)}(\rho) d\rho = 0, \quad \text{for} \quad (\lambda_n - 1)^2 = (\lambda_m - 1)^2
\]

and the biorthogonality matrix

\[
A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.
\]

The “Fourier” coefficients $C_n$ and $D_n$ are determined by the biorthogonality condition

\[
\psi_1^{(n)} A \left( \begin{pmatrix} \psi_1^{(n)} \\ \psi_2^{(n)} \end{pmatrix} \right) |_{r=r_1} = - \psi_1^{(n)} A \left( \begin{pmatrix} 1 - \sin^2 \frac{\pi \rho}{2\beta} \\ 0 \end{pmatrix} \right)
\]

and

\[
\int_{r_1}^{r_0} \frac{\phi_1^{(n)}(\rho)}{\lambda_n} \phi_1^{(n)}(\rho) d\rho = 0.
\]

Further details of the theory can be found in [2]. We solve the linear system (4) by truncation, i.e., replace the “$\omega$” sign in (2) by a finite number. At this point it is interesting to look at equations (4a) in detail.

\[
(C_n + D_n)F_n + \sum_{m=-n}^n \left( \frac{2}{\lambda_m - 2} - \frac{2 D_m}{\lambda_m} \right) \phi_1^{(n)}(\lambda_m) \phi_1^{(m)}(\lambda_m) = - \psi_1^{(n)} A \left( \begin{pmatrix} 1 - \sin^2 \frac{\pi \rho}{2\beta} \\ 0 \end{pmatrix} \right)
\]

and

\[
\sum_{m=-n}^{n} \frac{r_0^\lambda C_m \phi_1^{(n)}(\rho) \phi_1^{(m)}(\rho)}{r_0 (\lambda_m - 2)} = \int_0^\beta \phi_1^{(n)}(\rho) \phi_1^{(m)}(\rho) d\rho.
\]

For the chosen $\beta$, the real parts of the eigenvalues $\lambda_n$ are very large (Table 1), so that for $r_0 = 0.05$ the coefficients in (5b) suggest that the $C_n$ are large compared with the $D_n$. Therefore (5a) or (4a) can be solved for the $C_n$, neglecting the $D_n$, which then can be easily found from (5b) or (4b) (or find $D_m = r_0^\lambda D_m$). Thus the system (4) or (5) is split into two systems that can be solved consecutively. This reflects the fact that the boundary condition at $r_0$ does not have any significant influence on the flow, except very close to $r_0$ where the $D_m = r_0^\lambda D_m$ term in (2) is dominating (even when the $D_n$ are small). Note that the $D_n = 0$ for $r_0 = 0$.

Result. Sufficient accuracy of the truncated series can be obtained for five terms in the series. The coefficients $C_n$ and $D_n$ converge rapidly as $n$ increases; see Table 2. The residual error in the boundary

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Table 2 Coefficients in the biorthogonal series (scientific notation: the second number is the power of ten)

<table>
<thead>
<tr>
<th>$C_n$</th>
<th>$D_n$</th>
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<tbody>
<tr>
<td>2.60839 + 0.1</td>
<td>0.21957 + 0.0</td>
</tr>
<tr>
<td>0.5928 + 0.0</td>
<td>0.4402 + 0.0</td>
</tr>
<tr>
<td>0.5928 + 0.0</td>
<td>0.59270 + 0.0</td>
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Table 3  Boundary values at the inner and outer radius; $a_1$, $b_1$, $c_1$, and $d_1$ are the indicated values at the boundary; they are compared to: $a_2$ the prescribed vorticity at $r = 1$, $b_2 = c_2$ the zero stream function values and $d_2$ the zero gradient value at $r = 0.05$

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$\Omega(r = 1)$</th>
<th>$1 - \sin^2 \varphi \pi/10^6$</th>
<th>$\Psi(r = 1)$</th>
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<th>$\Psi(r = 0.05)$</th>
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<tr>
<td>0°</td>
<td>0.9990</td>
<td>1</td>
<td>-0.659 10^{-7}</td>
<td>0</td>
<td>0.141 10^{-39}</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>1°</td>
<td>0.9056</td>
<td>0.9045</td>
<td>0.655 10^{-7}</td>
<td>0</td>
<td>-0.149 10^{-39}</td>
<td>0</td>
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<td>2°</td>
<td>0.6530</td>
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<td>0.204 10^{-7}</td>
<td>0</td>
<td>0.156 10^{-39}</td>
<td>0</td>
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<tr>
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<td>-0.452 10^{-7}</td>
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<tr>
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<td>0.0910</td>
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Fig. 3 Comparisons of the velocity and vorticity in the outer region: — biorthogonal series; —-0— approximate solution

conditions is insignificant; see Table 3. The $\Psi$ boundary conditions are satisfied exactly on the sidewalls, $\beta = \pm 5^\circ$. The stream function $\Psi$ (Fig. 2(a)) and vorticity (Fig. 3) show details of the solution.

The Numerical Solution

Now the same problem is solved numerically using finite differences. A successive over-relaxation method is used, alternating between $\Psi$ and $\Omega$ with a fixed relaxation factor for each as described in [4]. The relaxation factors were not optimized. In order to work in a rectangular plane a new radial coordinate $\eta = \ln r$ is introduced. Compromise between the desired accuracy and the cost of the computations, we use meshes of $h_\eta = 0.023404$ and $h_\eta = 0.005645$.

Result. The stream function $\Psi$ (Fig. 2(b)) and the vorticity $\Omega$ are calculated until their residual values are less than $10^{-9}$ and $10^{-4}$, respectively. Asymptotic theory, utilizing the first eigenvalue (after Moffatt [5] with Burgraff correction [6]), is used to fill in the inner part of the sector where $\Omega$ residuals exceed the functional values. Results are shown in Fig. 2(b) and Fig. 3.

Comparison

The profiles of the center-line velocity are compared in Fig. 3. The velocity at the center of the outside arc is 0.0201 for the approximate numerical solution and 0.0196 in the analytical result.

It is obvious that the result of the biorthogonal series solution is more accurate and because of the easy, straightforward computation its use should be preferred for similar problems. The computation of the numerical solution was carried out on an IBM 360/91 requiring about 60 sec of computing time compared to only fractions of a second for the series (on a Cyber 74). However, this test has shown that the numerical results may be good enough for many applications (within the two top vortices, where the liquid flows fastest, the streamline error lies within the mesh length) and the method can be applied to more general, nonseparable domains. The truncation error can be reduced by a finer mesh computation.

References


1 Neglecting the boundary condition at $r_0$ which cannot be satisfied.