Energy Stability of Hydromagnetic Flow

by

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The governing equations of motion for a viscous fluid with constant density $\rho$ and finite conductivity $\sigma$ flowing in a magnetic field are (see [3])

\[ \frac{d\mathbf{u}}{dt} = \frac{1}{\rho \mu} \mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{\rho \nu} (\mathbf{p} + \frac{1}{2} \mathbf{u}^2 |\mathbf{B}|^2) + \nu \nabla^2 \mathbf{u}, \]  

(1a)

\[ \frac{d\mathbf{B}}{dt} = \mathbf{B} \cdot \nabla \mathbf{u} + \frac{1}{\sigma \mu} \nabla^2 \mathbf{B}, \]  

(1b)

and

\[ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0. \]  

(1c)

where $\mathbf{B}$ is the magnetic flux density, $\mu$ is the magnetic permeability and, as before, $\nu$, $\mathbf{u}$ and $\mathbf{p}$ are viscosity, velocity and pressure. From (1b) one finds that

\[ \frac{d}{dt} \nabla \cdot \mathbf{B} = \frac{1}{\sigma \mu} \nabla^2 \nabla \cdot \mathbf{B}. \]  

The condition div $\mathbf{B} = 0$ is automatically guaranteed for solutions of (1b) which have div $\mathbf{B} = 0$ at time 0 and on the boundary $S$ at all times.

Here $\mathcal{V}$ is a bounded domain enclosed by a rigid surface on which $\mathbf{u}$ and $\mathbf{B}$ are assigned. As in [2], to analyze the stability of the basic motion $(\mathbf{u}_0, \mathbf{B}_0, p_0)$ we consider an altered motion $(\mathbf{u}_0^*, \mathbf{B}_0^*, p_0^*)$ which satisfies the same equations (1) and the same boundary conditions, but differs from the basic state initially. The differences
\( (U^* - U = u, \mathbf{B}^* - \mathbf{B} = b \) and \( p^* - p = \delta p \) are called disturbances and they satisfy the equations

\[
\frac{du}{dt} + u \cdot \nabla u + u \cdot \nabla u = \frac{1}{\rho \mu} (\mathbf{b} \cdot \nabla \mathbf{B} + \mathbf{b} \cdot \nabla \mathbf{b} + \mathbf{B} \cdot \nabla \mathbf{b})
\]

\[- \frac{1}{\rho} \left[ \delta p + \frac{1}{2 \mu} (|\mathbf{B}^*|^2 - |\mathbf{B}|^2) \right] + \nu \nabla^2 u, \quad (2a)\]

\[
\frac{db}{dt} + u \cdot \nabla b + u \cdot \nabla b = \mathbf{b} \cdot \nabla u + \mathbf{b} \cdot \nabla \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{u} + \frac{1}{\sigma \mu} \nabla^2 b, \quad (2b)\]

\[
\nabla \cdot u = \nabla \cdot b = 0 \quad (2c)\]

and

\[
\mathbf{u} = \mathbf{b} = 0 \bigg| \text{S} \bigg. \quad (2d)\]

Following CARMI & LALAS (1970) we next determine sufficient conditions under which the altered flow will tend asymptotically to the basic flow at \( t \to \infty \). Toward this end we form energy identities

\[
\frac{1}{2} \frac{d}{dt} \left< |u|^2 \right> = -\left< u \cdot \nabla \cdot u + \nu \nabla u : \nabla u \right> + \frac{1}{\rho \mu} \left< \mathbf{b} \cdot \nabla \mathbf{b} \cdot \mathbf{u} \right>
\]

\[
+ \mathbf{b} \cdot \nabla \mathbf{b} \cdot \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{b} \cdot \mathbf{u} \bigg>, \quad (3a)\]

and

\[
\frac{1}{2} \frac{d}{dt} \left< |b|^2 \right> = \left< \mathbf{b} \cdot \nabla \cdot \mathbf{b} \right> - \left< \frac{1}{\sigma \mu} \nabla b : \nabla b \right>
\]

\[
+ \left< \mathbf{b} \cdot \nabla \mathbf{u} \cdot \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{u} \cdot \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} \cdot \mathbf{b} \right> \bigg). \quad (3b)\]
where the angle brackets designate volume-averaged integrals and \( \mathcal{A} \) is the strain-rate tensor for \( \mathbf{U} \). In carrying out the integration, the integral \( \langle \nabla \cdot \mathbf{A}_1 \rangle = 0 \) which is added on the right of (3a) and the integral \( \langle \nabla \cdot \mathbf{A}_2 \rangle = 0 \) which added on the right of (3b) has been carried to the boundary by the divergence theorem. The vector fields

\[
\mathbf{A}_1 = \nu \nabla \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \frac{1}{2} |\mathbf{u}|^2 + \frac{\delta \mathbf{p}}{\rho} + \frac{1}{2\mu \rho} (|\mathbf{B}^*|^2 - |\mathbf{B}|^2) \]

and

\[
\mathbf{A}_2 = \frac{1}{\sigma \mu} \nabla \frac{1}{2} |\mathbf{b}|^2 - \mathbf{u} \frac{1}{2} |\mathbf{b}|^2
\]

vanish on \( \partial \mathbf{U} \).

The reader's attention is drawn to the fact that some of the cubic nonlinearities in the disturbance, which arise from the quadratic terms in \( \mathbf{u} \) and \( \mathbf{b} \) of (2), do not integrate to zero, that is, though

\[
\mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{b} \cdot (\mathbf{u} \cdot \nabla) \mathbf{b} = 0,
\]

the terms

\[
\mathbf{b} \cdot \nabla \mathbf{b} \cdot \mathbf{u} \quad \text{and} \quad \mathbf{b} \cdot \nabla \mathbf{u} \cdot \mathbf{b}
\]

are not necessarily zero and effect the energy balances (3a) and (3b).

There is a linear combination of (3a) and (3b) in which the cubic nonlinearities subtract out: thus,
\[ \frac{1}{2} \frac{d}{dt} \left( \rho \mu |\mathbf{u}|^2 + |\mathbf{b}|^2 \right) = - \rho \mu \langle \mathbf{u} \cdot \mathbf{D} \cdot \mathbf{u} \rangle + \langle \mathbf{b} \cdot \mathbf{D} \cdot \mathbf{b} \rangle \\
+ 2 \langle \mathbf{b} \cdot \Omega_B \cdot \mathbf{u} \rangle - \nu \mu \rho \langle \mathbf{v} \cdot \mathbf{u} \rangle \langle \mathbf{v} \cdot \mathbf{u} \rangle - \frac{1}{\sigma \mu} \langle \mathbf{v} \cdot \mathbf{b} \cdot \mathbf{v} \rangle , \] (4)

where

\[ \Omega_B = \text{antisymmetric part of } \mathbf{V} \mathbf{b} \)

This is the energy identity considered by CARMI & LALAS.

There are four fundamental measures of the basic flow; the strain rate tensor \( \mathbf{D} \), the vorticity tensor \( \Omega_B \), the symmetric part of the dyadic gradient of magnetic flow \( \mathbf{D}_B \) and the antisymmetric part \( \Omega_B \) of the same tensor; that is

\[ \mathbf{V} \mathbf{u} = \mathbf{D} + \Omega_B \quad \text{and} \quad \mathbf{V} \mathbf{b} = \mathbf{D}_B + \Omega_B . \]

Of the four measures only \( \mathbf{D} \) and \( \Omega_B \) appear in the identity (4).

The fact that (4) is homogeneous of degree two in the disturbance opens the possibility of finding a global result which is independent of the scale of the motion. This possibility is realized in the theorem of unconditional stability of CARMI * LALAS (1970).*

* This theorem is also proved in the paper of BHATTACHARYYA and JAIN (1972). These authors also consider hydro-thermoconvective flows.
In preparation for the statement of this theorem the energy identity (4) is made dimensionless by dividing \([x, t, u, b, B, D]\) by \([1, l^2/\nu, U_\infty, B_0P_m^{1/2}/A, B_0P_m^{1/2}/A, D_m]\) where \(U_\infty\) and \(B_0\) are typical values of the velocity and magnetic field, \(P_m = \mu \sigma \nu\) is the magnetic Prandtl number, \(A = B_0/U_\infty (\rho \mu)^{1/2}\) is the Alfvén number and the other symbols are as before. We will now work only with dimensionless variables which, for economy, are also designated as \(\tilde{u}, \tilde{b}, \) etc. In dimensionless variables we have
\[
\frac{d \tilde{E}}{dt} = RI_1 - \langle \nabla \tilde{u} : \nabla \tilde{u} + \nabla \tilde{b} : \nabla \tilde{b} \rangle ,
\]
where
\[
I_1 = - \langle \tilde{u} \cdot D \cdot \tilde{u} \rangle + P_m (\langle \tilde{b} \cdot D \cdot \tilde{b} \rangle + 2 \langle \tilde{b} \cdot \Omega \tilde{b} \cdot \tilde{u} \rangle ) ,
\]
\[
\tilde{E} = \frac{1}{2} \left\langle |\tilde{u}|^2 + P_m |\tilde{b}|^2 \right\rangle ,
\]
\[
R = \frac{U_\infty}{\nu}, \quad R_m = R P_m .
\]

Here the magnetic Reynolds number is ratio of the convection rate to the diffusion rate of the magnetic field. Large \(R_m\) implies a thin boundary layer in which dissipation occurs. Outside this region the magnetic field and the flow are "frozen" together. Small \(R_m\), on the other hand, implies that the total magnetic field of the flow is essentially equal to the imposed one, so that the induced field is small.
The magnetic Prandtl number $P_m$ is a measure of the ratio of the rate of diffusion of vorticity to the rate of diffusion of the magnetic field.

A direct consequence of (5) and the estimates given in [2] is the following theorem of unconditional stability of CARMI & LALAS.

Let $\mathcal{V} = \mathcal{V}(t)$ be a bounded domain. Let $\mathfrak{U}$ and $\mathfrak{B}$ be the velocity and magnetic flux density vectors satisfying prescribed condition on $\mathcal{V}$. Then $\mathcal{E} = \frac{1}{2} \langle |\mathfrak{u}|^2 + P_m |\mathfrak{b}|^2 \rangle$

satisfies the inequality

$$\mathcal{E}(t) < \mathcal{E}(0) \exp(-2MNt),$$

(6)

where

$$M = \hat{\Lambda} - (\hat{R} + \hat{R}_m), \quad \hat{R} = \max[R, P_m R], \quad \hat{R}_m = \hat{\mathfrak{B}}_{m R_m},$$

$$\hat{\mathfrak{B}}_{m} = \max \{ \text{curl } \mathfrak{B} \}_{i}, \quad N = \min(1, P_m^{-1}),$$

$$i = 1, 2, 3,$$

$$0 \leq t' \leq t$$

and $\hat{\Lambda}(\mathcal{V})$ is the decay constant defined by (5). If $M > 0$ for all $t$ then $\mathcal{E}(t) \to 0$ as $t \to \infty$ and the flow is globally and monotonically stable.

This is a theorem of a type first given by SERRIN [6]. CARMI & LALAS, following SERRIN, also show that if

$$\frac{1}{R_0} > \max_H \frac{I_1}{H} \langle |\nabla \mathfrak{u}|^2 + |\nabla \mathfrak{b}|^2 \rangle$$
where \( \mathcal{H} \) is the space of solenoidal vectors vanishing on \( \partial \Omega \) then \( \mathcal{E} \to 0 \) exponentially. This criterion also implies the uniqueness of steady flow.

There are a number of energy identities besides (3) which are of some value in treating the stability of hydromagnetic flows. For example, the three equations (2b) for the components of \( b \) each give an energy equation. This contrasts with the equations (2a) for the components of \( u \) which involve the pressure. Energy equations for the components of \( b \) are like the separate equations (3a) and (3b) in that they involve cubic nonlinearities. Energy analysis here requires special procedures, like those used in [5] to handle the cubic nonlinearities.

One special identity, the correlation identity, merits special attention. This identity is formed from the sum \( \langle b \cdot (2a) \rangle + \langle u \cdot (2a) \rangle \) and leads after integration by parts to

\[
\frac{d}{dt} \langle u \cdot b \rangle = -2 \langle u \cdot \Omega u \cdot b \rangle - \langle u \cdot D_B \cdot u \rangle
\]

\[
+ \frac{1}{\rho \mu} \langle b \cdot D_B \cdot b \rangle - (\nu + \frac{1}{\sigma \mu}) \langle \nabla b : \nabla u \rangle.
\]

(7)

The correlation identity is striking because it depends on the basic flow only through the measures \( D_B \) and \( \Omega_U \) of the "strain rate" of the magnetic flux and the vorticity of the basic motion; these measures are completely absent from the energy identity (4).
A linear combination of the energy identity (4) and the correlation identity (2) could form the basis for a modified energy analysis of the type considered in [4] for the convection problem in a fluid heated and salted from below.

After making (7) dimensionless, we may form a linear combination in the form

$$\frac{d\xi_\lambda}{dt} = RI_\lambda - D_\lambda,$$

where

$$\xi_\lambda = (\phi + \lambda) \xi + 2\langle u \cdot b \rangle,$$

$$\phi = \frac{1}{\beta_m} + 1, \quad \lambda > 0,$$

$$I_\lambda = (\phi + \lambda)I_1 + 2I_2,$$

$$I_2 = -2\langle u \cdot \Omega \cdot b \rangle - \langle u \cdot D_B \cdot u \rangle + \frac{\beta}{\mu_0} \langle b \cdot D_B \cdot b \rangle,$$

$$D_\lambda = \lambda \langle |\nabla u|^2 + |\nabla b|^2 \rangle + \phi \langle |\nabla (u+b)|^2 \rangle.$$

We remark that for all fixed values of $\phi > 1$ and $\lambda > 0$ there exist values

$$\frac{1}{R_\lambda} = \max_H \frac{I_\lambda}{D_\lambda},$$

and

$$\frac{1}{A_\lambda} = \max_H \frac{\xi_\lambda}{D_\lambda}.$$
These numbers define a stability limit and a decay constant in the energy stability theorem which is to be proved below. It is first necessary to establish a preliminary Lemma: \( \mathcal{E}_\lambda > 0 \) for all \( \lambda > 0 \).

To prove this, we note that

\[
\frac{1}{2} (\phi + \lambda) (|u|^2 + P_m |b|^2) + 2u \cdot b > 0
\]

\[
\frac{1}{2} (\phi + \lambda) 2 \rho \frac{v_h}{m} |u \cdot b| - 2 |u \cdot b|
\]

\[
= \rho \frac{v_h}{m} \{ (1 - \frac{1}{\rho \frac{v_h}{m}})^2 + \lambda \} |u \cdot b| > 0
\]

With this preliminary aside we may now establish the following:

**Energy theorem for hydromagnetic flow:** Let

\[
R < R_\lambda
\]

(11)

for any \( \lambda > 0 \). Then

\[
\mathcal{E}_\lambda(t) < \mathcal{E}_\lambda(0) \exp \{-\Lambda \_A t[1 - \frac{R}{R_\lambda}]\}.
\]

(12)

**Proof:** We may write (10) as

\[
\frac{d\mathcal{E}_\lambda}{dt} = -D_\lambda \{-R \frac{\Lambda \_A}{D_\lambda} + 1\} \leq -D_\lambda \{1 - \frac{R}{R_\lambda}\}
\]

where we have used (9) in forming the last inequality.
If $R < R_\Lambda$, by (10) we have

$$\frac{d \mathcal{E}_\Lambda}{dt} < -\Lambda \mathcal{E}_\Lambda \left(1 - \frac{R}{R_\Lambda}\right)$$

and (12) follows by integration.

**Corollary:** The largest $R$ domain of stability is associated with the "energy" $\gamma > 0$ where $\gamma$ is value of $\lambda$ for which

$$\frac{1}{R_\gamma} = \sup_{\lambda > 0} \frac{1}{R_\lambda}$$

It is easy to prove that the initial condition which solves the maximum problem (9) is also the one which makes $\mathcal{E}_\Lambda$ increase initially at the smallest $R$.

Computations for the criterion $R < R_\gamma$ for particular flows have not yet been carried out.
References


