Transverse Velocity Components in Fully Developed Unsteady Flows

D. D. JOSEPH and L. N. TAO

It is known that if an incompressible fluid is confined to a straight pipe or channel, and if the axial velocity is steady and fully developed, then, under certain very general conditions, no transverse velocity components can exist. This conclusion is not valid for unsteady flows, and it is the purpose of this note to develop the appropriate restrictions for the unsteady case.

By fully developed we mean that the velocity components are two-dimensional functions of the transverse coordinates. It is commonly assumed that when a fully developed flow is confined to a straight channel, then the transverse components of the velocity are zero, i.e., \( u = (0, 0, u(x_1, x_2)) \), where \( x_1 \) is along the axial direction of the channel. Indeed, if one stipulates in addition that the transverse components of the body force should have a potential, and that there should be no relative motion of the boundaries so that the no-slip condition will require that all velocity components vanish at the walls, then it can be rigorously demonstrated that in steady flows the transverse velocity components must vanish. However, it does not follow that the transverse velocity components must vanish for motions which are fully developed and transient.

We note that the governing equations for a viscous incompressible fluid having constant material properties and being fully developed are

\[
\begin{align*}
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + F_i + \nu \nabla^2 u_i \\
\frac{\partial u_{\alpha}}{\partial t} + u_{\beta} \frac{\partial u_{\alpha}}{\partial x_{\beta}} &= P_{\alpha} + \nu \nabla^2 u_{\alpha}
\end{align*}
\]

where \( p, \rho, F, P, \) and \( \nu \) are the pressure, density, axial body force, combined pressure and transverse force potential, and kinematic viscosity, respectively.

We shall first demonstrate, by example, that equations (1) and (3) have nontrivial solutions which give transverse velocity components and satisfy the no-slip condition at the stationary walls bounding the cross section of the pipe. Let the pipe have a circular cross section and the pipe wall be located at \( r = a \). Initially, let the pipe and fluid contained rotate about the pipe axis with angular velocity \( \omega \). The initial flow is fully developed. The radial velocity component is initially not present and it may be assumed from symmetry that it will not develop. Let the pipe be brought impulsively to a state of rest. Equations (1) and (3) become

\[
\begin{align*}
\frac{\partial u_\theta}{\partial t} &= 0 \\
\frac{\partial u_\theta}{\partial t} &= \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} \right]
\end{align*}
\]

subject to the initial and boundary conditions

\[
\begin{align*}
t &= 0 \quad u_\theta(r, 0) = \omega r \\
t > 0 \quad u_\theta(a, t) = 0
\end{align*}
\]

1 Assistant Professor of Mechanical Engineering, Illinois Institute of Technology, Chicago, Illinois.
2 Professor of Mechanics, Illinois Institute of Technology, Chicago, Illinois.
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The solution of this initial boundary-value problem is
\[ u_\alpha = 2\omega P e^{\frac{-k\lambda l}{(kr)}} I_{1}((kr)/k'\lambda) \]
where the parameters \( k \) are the zeros of \( I_{1}(ka) \).

Hence, unlike the steady case, the assumption of a fully
developed velocity profile and the no-slip condition at stationary
walls does not necessarily imply that \( u_c(0,0,u_\alpha) \).

However, the foregoing assumptions do limit dramatically the
possible types of transverse velocity fields. In particular, it is
true that for fully developed transient flows in stationary pipes
or channels, (a) the transverse velocity components must be
monotonically decreasing function of the time; (b) if transverse
velocity components are not present initially, they will not de-
velop thereafter.

To prove these propositions, we introduce the stream function
\[ u_\alpha = e_\alpha \psi \]
where \( e_\alpha \) are the components of the two dimensional alternating
tensor. The stream function, so defined, satisfies equation (1)
identically. If one substitutes equation (4) into (3) and operates
on the resultant equation with \( e_\alpha \psi /e_\alpha \), one finds that the stream
function satisfies
\[ \frac{\partial \psi_{,\alpha\beta}}{\partial x_\beta} + e_\alpha \psi_{,\alpha\beta} = v \psi_{,\alpha\beta} \]
which is the two-dimensional equation governing the diffusion of
the axial component of vorticity.

Integration of equation (5) over a channel cross-sectional area
\( A \) bounded by a solid stationary curve \( C \) reveals that
\[ \frac{\partial}{\partial t} \int_A \psi_{,\alpha\beta} dA + \int_C e_\alpha \psi_{,\alpha\beta} dC = v \int_A \psi_{,\alpha\beta} dA \]
where Green's theorem has been used to effect the integration.
Since the no slip condition \( \psi_{,\alpha\beta} = 0 \) on \( C \), one concludes that
\[ \int_A \psi_{,\alpha\beta} dA = 0 \]

The integration of equation (5) multiply by \( \psi \) over the cross
section \( A \) shows, after some manipulations, that
\[ \frac{\partial}{\partial t} \left[ \psi(C) \int_A \psi_{,\alpha\beta} dA - \int_A (u_{\alpha\beta} + u_{\alpha\beta}) dA \right] \]
\[ + \frac{1}{2} e_\alpha \psi_{,\alpha\beta} dC \]
\[ = v \left[ \psi(C) \int_A \psi_{,\alpha\beta} dA - \int_C \psi_{,\alpha\beta} dC + \int_A (\psi_{,\alpha\beta}) dA \right] \]
which in view of equation (6) may be written as
\[ - \frac{\partial}{\partial t} \int_A (u_{\alpha\beta} + u_{\alpha\beta}) dA = 2v \int_A (\psi_{,\alpha\beta}) dA \]

Since the integral on the right is positive definite,
\[ \frac{\partial}{\partial t} \int_A (\psi_{,\alpha\beta}) dA \leq 0 \]
where \( q^1 = u_{\alpha\beta} + u_{\alpha\beta} \).

Furthermore, the equality sign holds only when \( q \equiv 0 \). Let
us suppose consistent with equation (8) that it is possible for the
kinetic energy of the transverse motion to be constant in time
\[ \int_A (q^1/2) dA = \text{const} > 0 \]
It follows from equation (7) that \( \psi_{,\alpha\beta} = 0 \) everywhere. Since,
on the boundary, \( \psi_{,\alpha\beta} = \psi_{,\alpha\beta} = 0 \) on \( C \); \( \psi \) can at most be a/