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Review Paper

Short-Wave Instabilities and Ill-Posed Initial-Value Problems[1]

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Abstract. We characterize ill-posed problems as catastrophically (Hadamard) unstable to short waves. The growth rate tends to infinity as the wavelength tends to zero. The mathematical description of ill-posed problems is framed in terms of instability. These problems cannot be integrated numerically; the finer the mesh, the worse is the result. The instability must be regularized. Ill-posed problems which arise in problems involving interfaces, oil recovery, granular media, and viscoelastic fluids are regularized in different ways, by adding effects of surface tension or viscosity or compressibility or by weakening the initial discontinuity. Problems which are stable as $t \to \infty$ for any fixed wavelength λ, no matter how small, can be Hadamard unstable with catastrophic instability as $\lambda \to 0$ for a fixed t, no matter how large. We stress the utility of freezing coefficients in nonlinear and quasilinear systems and prove that in general ill-posed problems cannot be solved unless the initial data is analytic. We show why the shock up of first-order systems which are nonlinear in first derivatives can be expected to lead to discontinuities in second, rather than first, derivatives.

1. Introduction

There are many problems for which the critical eigenvalue of linearized stability theory is proportional to some positive power of the wave number k

$$\sigma = k^m f, \qquad m > 0, \tag{1.1}$$

where $\operatorname{Re} \sigma$ is unbounded as $k \to \infty$. This means that the growth rate (Re = real part)

$$\operatorname{Re} \sigma = k^m \operatorname{Re} f \tag{1.2}$$

associated with exponential disturbances proportional to $e^{\sigma t}$ is unbounded as the disturbance wavelength $\lambda = 2\pi/k$ tends to zero.

Some interesting stability problems satisfying (1.1) are discussed in Sections 3, 4, and 6. The first example arises in Kelvin's (1871) analysis of the Helmholtz instability of vortex sheets and it seems to have been noticed first by Rayleigh (1896). He comments about the rapid growth of the wave amplitude h:

$$h = He^{\pm(1/2)kVt} \cos k(\tfrac{1}{2}Vt - x). \tag{34}$$

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> In (34) an arbitrary constant may of course be added to x. It appears that the waves travel in the same direction as the stream, and with *one-half* its velocity. In the case of the positive exponent, the rapidity with which the amplitude increases is very great. Since $k = 2\pi/\lambda$, the amplitude is multiplied by e^{π}, or about 23, in the time occupied by the stream in passing over a distance λ. If $\lambda = V\tau$, $\exp(\frac{1}{2}kVt) = \exp(\pi t/\tau)$, independent of V.

Equation (1.2) has curious implications. Suppose, following conventions used in linearized stability theory, that

$$\begin{aligned} \operatorname{Re}\sigma = k^m \operatorname{Re} f < 0 \quad & \text{means stability,} \\ \operatorname{Re}\sigma = k^m \operatorname{Re} f > 0 \quad & \text{means instability.} \end{aligned} \tag{1.3}$$

As $\lambda = 2\pi/k \to 0$ (short waves) stable disturbances are immediately squashed and unstable disturbances are massively unstable, growing without bound. In some cases $\operatorname{Re} f(p)$ depends on some parameter p and is such that $\operatorname{Re} f(p_c) = 0$ with $\operatorname{Re} f(p) < 0$ when $p < p_c$ and $\operatorname{Re} f(p) > 0$ when $p > p_c$. As p is increased past p_c, the flow loses stability in a catastrophe with exponentially unbounded growth of the shortest waves. This type of mathematical description of the underlying fluid mechanics is obviously unacceptable, but it occurs in good models of physically interesting problems.

We are obliged to consider what analysis leading to unbounded growth rates (1.1) tells us about physics and what is missed out. For example, this type of short-wave instability cannot lead to bifurcation in the usual sense because we are dealing with a continuum of unstable modes with strange properties. In fact, the usual equilibrated structures, steady, periodic, quasi-periodic, chaotic, and attracting flows do not appear to occur. Instead of these we may expect unsteady fingering instabilities leading to fibril structures.

The utility of maintaining a strict division between bounded and unbounded growth rates seems to have been noted first by Petrowsky (1938). His ideas were further developed by Birkhoff (1954). Birkhoff was considering how to set up partial differential equation problems so as to generate physically reasonable solutions. His was a work in the classical theory of well-set or well-posed problems that is generally attributed to Hadamard (1922). He made a good connection between problems that are ill-posed as initial-value problems and those that have catastrophic short-wave instabilities with unbounded growth rates, like (1.1). He argued that well-posedness was strongly tied to the selection of the class of functions in which a given problem is posed, to the choice of functions that might be considered "physically reasonable." He concluded that, at least for partial differential equations with constant coefficients, the functions with well-defined Fourier transforms constitute a sufficiently general class for many physical applications. In this class he then showed that unbounded growth rates imply the loss of well-posedness of the initial-value (Cauchy) problems.

A steady motion which is Hadamard unstable is also unstable in the sense of linear theory; the amplitude of the disturbance tends to infinity exponentially with the time. Unsteady motions may be "stable" in the sense of linear theory but Hadamard unstable. This is to say that the motion is "stable" because the amplitude of the disturbance eventually tends to zero or at least grows less rapidly than exponentially with time at any fixed k, no matter how large, but the motion is Hadamard unstable because the amplitude tends to infinity with k at any fixed time, no matter how large. Such situations are discussed at the end of Section 4 where we consider the unsteady Kelvin–Helmholtz instability for several cases.

Ill-posed problems are discussed briefly in the next section, from a mathematical perspective. For now it will suffice to emphasize three points about ill-posedness:

(1) Ill-posed problems are disasters for numerical simulations. Because such problems are unstable to ever shorter waves, the finer the mesh, the worse the result.
(2) Some techniques must be introduced to regularize the instability of shortest waves.
(3) Regularizing techniques are preferentially found from neglected physical effects, ordinarily small, which enter strongly at short wavelengths, like surface tension and viscosity.

2. Ill-Posed and Well-Posed Problems

Roughly speaking, well-posed problems are those for which the given data determines physically reasonable solutions. Physically reasonable solutions are defined relative to a mathematical class in which theorems of existence, uniqueness, and continuous dependence on the data can be proved. It is the data which is posed. A given equation (or system of equations) is compatible with some data and not with other data. For example, good data for hyperbolic or parabolic partial differential equations is associated with the prescription of initial data, while elliptic problems are ill-posed with the prescription of initial values and are well-posed with the prescription of boundary data. These considerations seem to have been introduced by Hadamard (1922). He gave the following example of how Laplace's equation is ill-posed as an initial-value problem.

Consider the half space

$$D = \{x, t; t > 0, -\infty < x < \infty\}$$

appropriate for an initial-value problem for Laplace's equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1}$$

with initial values

$$\begin{aligned} u(0, x) &= 0, \\ \frac{\partial u}{\partial t}(0, x) &= f(x). \end{aligned} \tag{2.2}$$

We may choose an oscillating function

$$f(x) = \frac{1}{k^p} \sin kx, \qquad p > 0, \tag{2.3}$$

which is bounded for all x and tends to zero for small wavelengths $2\pi/k \to 0$. The solution of this problem is

$$u(t, x) = \frac{1}{k^{1+p}} \sin kx \sinh kt. \tag{2.4}$$

Small data at $t = 0$ leads to huge, unbounded oscillations for any small $t > 0$ as the wavelength tends to zero. This lack of continuous dependence of the solution on the data is called *Hadamard instability*. This type of instability gets worse as the wavelength decreases, with ultimate catastrophe as $k \to \infty$.

Ill-posedness of the Cauchy problem can also be associated with the nonexistence of solutions of initial-value (Cauchy) problems ((2.1), (2.2)) for nonanalytic data. If u is a solution of (2.1) in $t > 0$, it is analytic there and, by reflection, to all of $\mathbb{R}^2$. But then $f(x)$ must be analytic. If $f(x)$ is not analytic, no solution is possible, even in a small interval of t around zero. We show in Section 21 that ill-posed problems generally cannot be solved outside a class of analytic initial data.

The backwards heat equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} \tag{2.5}$$

is also ill-posed as an initial-value problem. This problem may be regarded as the time reversal of the heat equation that is well-posed as an initial-value problem. The initial-value problem for the heat equation can be solved by Fourier series. Some implications of this type of solution were beautifully described by Maxwell (1875).

> If the body is originally heated in any arbitrary manner, Fourier shows us how to express the original temperature as the sum of a series of harmonic distributions. When the body is left to itself the part depending on the higher harmonic rapidly dies away, so that after a certain time the distribution of heat continually approximates to that due to the fundamental harmonic, which therefore represents the law of cooling of a body after the process of diffusion of heat has gone on for a long time.
>
> Sir William Thompson has shown, in a paper published in the 'Cambridge and Dublin Mathematical Journal' in 1844 how to deduce, in certain cases, the thermal state of a body in past time from its observed condition at present.
>
> For this purpose, the present distribution of temperature must be expressed (as it always may be) as the sum of a series of harmonic distributions. Each of these harmonic

> distributions is such that the difference of the temperature of any point from the final temperature diminishes in a geometrical progression as the time increases in arithmetical progression, the ratio of the geometrical progression being the greater the higher the degree of the harmonic.
>
> If we now make t negative, and trace the history of the distribution of temperature up the stream of time, we shall find each harmonic increasing as we go backwards, and the higher harmonics increasing faster than the lower ones.
>
> If the present distribution of temperature is such that it may be expressed in a finite series of harmonics, the distribution of temperature at any previous time may be calculated; but if (as is generally the case) the series of harmonics is infinite, then the temperature can be calculated only when this series is convergent. For present and future time it is always convergent, but for past time it becomes ultimately divergent when the time is taken at a sufficiently remote epoch. The negative value of t for which the series becomes ultimately divergent, indicates a certain date in past time such that the present state of things cannot be deduced from any distribution of temperature occurring previously to that date, and becoming diffused by ordinary conduction. Some other event besides ordinary conduction must have occurred since that date in order to produce the present state of things.
>
> This is only one of the cases in which a consideration of the dissipation of energy leads to the determination of a superior limit to the antiquity of the observed order of things.

Birkhoff (1954), following Petrowsky (1938), considered the problem of partial differential equations, elliptic, hyperbolic, and parabolic. He restricted his considerations to linear PDEs with constant coefficients, hoping that the analysis would extend to quasilinear PDEs with variable coefficients. He points out that different mathematicians have different definitions of the categories of the classification, "Maxwell's equations are hyperbolic in the sense of Courant–Hilbert, but not in the sense of Petrowsky." He advocates a scheme of classification based on Fourier transforms. "... functions with well-defined Fourier transforms constitute a sufficiently general class for many physical problems." Applying these ideas to the linear PDEs, he shows that the transforms are essentially superpositions of spatially periodic solutions expressed in normal modes proportional to

$$\exp\{\sigma t + i\mathbf{k}\cdot\mathbf{x}\}.$$

Of course, in the usual way the normal modes lead to solvability conditions of the form

$$F(\sigma, \mathbf{k}) = 0. \tag{2.6}$$

He then introduces the important idea of a regular eigenvalue $\sigma(\mathbf{k})$. He calls the underlying system stable if for all real values of $\mathbf{k}$ all of the roots $\sigma(\mathbf{k})$ have negative real parts, as usual. He calls the system regular if and only if $\operatorname{Re}\sigma(\mathbf{k})$ has a finite upper bound, independent of $\mathbf{k}$. In the regular case the initial-value problem is well-posed in the sense of Petrowsky, the rate of exponential growth of the Fourier transform is clearly bounded; in the other cases, it is unbounded. Birkhoff then proceeds to amplify the notion that regularity should be interpreted to mean that satisfactory existence and uniqueness theorems for the initial-value problem can be proved. In his 1964 paper he notes that

> Following Hadamard, most mathematicians would agree that a Cauchy problem ... should be called *well-set* when the solution at time t exists and is unique for given initial $\mathbf{u}(\mathbf{x}, 0)$. Unfortunately, this answer is highly ambiguous, until one has specified the *class* of functions admitted, together with a *topology* on the space of all "admissible" functions.
>
> In spite of this ambiguity, various interpretations support the conclusion that the Cauchy problem for (1) should be considered as *well-set* (properly posed) if and only if (1) if *regular* (see [**4**, p. 198]). (Here (1) is a linear partial differential equation with constant coefficients.) This conclusion was essentially reached by Hadamard, and, arguments supporting it have been given by Petrowsky, Garding, Hörmander, and others [**5**; **6**; **4**, pp. 330–1; **7**]. For the backwards heat equation ... it was already reached by Maxwell!

4 Friedman, A. "Generalized Functions and Partial Differential Equations." Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

5 Gelfand, I.M., and Schilow, G.E. "Verallgemeinerte Funktiones," 4 Vols. Deutscher Verlag, Wiss., Berlin, 1960.
6 Hörmander, L. "Linear Partial Differential Operators." Academic Press, New York, 1963.
7 Friedman, A. "Existence of smooth solutions of the Cauchy problem for differential systems of any type." J. Math. Mech. **12**, 335–74 [1963].

An interesting discussion of these matters is given by Hersh (1973).

In Section 21 we prove that for a fairly general class of systems with irregular eigenvalues the initial-value problem cannot be solved for initial data which is not analytic. The concept of a regular unstable eigenvalue also appears to be useful for applications in the theory of bifurcations. Problems with irregular eigenvalues, like those satisfying (1.1), do not lead to bifurcations but probably to fingering or filamentous solutions.

3. Interface Problems Which Are Hadamard Unstable

Some classical problems involving interfaces give rise to short-wave instabilities with unbounded growth rates. Rayleigh–Taylor and Kelvin–Helmholtz instabilities are of this type, as well as a Taylor–Saffman instability of a fluid interface in a porous media. In all three of these problems, there is a fluid interface

$$F = z - \zeta(x, y, t) = 0, \tag{3.1}$$

which is an identity in t, following the motion

$$\frac{dF}{dt} = w - \frac{\partial\zeta}{\partial t} - u\frac{\partial\zeta}{\partial x} - v\frac{\partial\zeta}{\partial y} = 0, \tag{3.2}$$

where (u, v, w) are velocity components corresponding to (x, y, z). The normal stress condition at the interface between two fluids is

$$2H\gamma\mathbf{n} = -[\![\mathbf{T}]\!]\cdot\mathbf{n} = ([\![\Phi]\!] - g\zeta[\![\rho]\!])\mathbf{n},$$

where $[\![\cdot]\!] = (\cdot)_1 - (\cdot)_2$ is the jump of $(\cdot)$,

$$\Phi = p + \rho g z \tag{3.3}$$

is the head, p is the pressure, $\mathbf{T} = -p\mathbf{1}$ is the stress,

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{e_z - e_x\zeta_x - e_y\zeta_y}{(1 + \zeta_x^2 + \zeta_y^2)^{1/2}},$$

where $\zeta_x = \partial\zeta/\partial x$, etc.

The normal stress condition for an interface between two fluids in a porous medium is usually framed in terms of composite fluids in which each fluid plus the porous solid is regarded as a composite fluid with "effective" material coefficients. The velocity in Darcy's law is such a composite velocity. It is called the superficial velocity and is defined as the volume flux across an area A fixed in the solid, over solid and voids, divided by A. The velocity components (u, v, w) are to be regarded as components of the superficial velocity in flow through a porous medium and

$$\begin{aligned} -[\![\Phi]\!] + [\![\rho]\!]g\zeta &= -[\![p]\!], \\ [\![p]\!] &= 2H\gamma + p_c \end{aligned} \tag{3.4}$$

holds on the macroscopic interface between two composite fluids in a porous medium where γ is the "effective" interfacial tension,

$$H = \tfrac{1}{2}\nabla_2\cdot\left\{\frac{\nabla_2\zeta}{(1 + |\nabla_2\zeta|^2)^{1/2}}\right\} \tag{3.5}$$

is the mean curvature of $z = \zeta$, and p_c is the capillary pressure difference between the two fluids due to microscopic curvature of the true fluid in the pores of the porous media.

4. Kelvin–Helmholtz and Rayleigh–Taylor Instability

We now confine our attention to the special situation shown in Figure 4.1. We are looking for Hadamard instabilities to short waves and hence neglect surface tension which would stabilize short waves.

The governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.1}$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla\Phi, \tag{4.2}$$

where $\mathbf{u} = (u, v, w)$, $\Phi = p + \rho g z$, and

$$\mathbf{u} = \mathbf{e}_x \begin{cases} U_2, & z \to \infty, \\ U_1, & z \to -\infty. \end{cases} \tag{4.3}$$

The pressure and normal component of velocity are continuous on $z = \zeta$. The normal component of velocity will be automatically continuous if (3.2) holds, and if $[\![p]\!] = 0$, then

$$[\![\Phi]\!] = g[\![\rho]\!]\zeta, \tag{4.4}$$

where the average value of $\zeta(x, y, t)$ on horizontal planes vanishes, $\bar{\zeta}(t) = 0$.

The basic flow is given by $\zeta = 0$,

$$\mathbf{u} = \mathbf{U} = \mathbf{e}_x \begin{cases} U_2, & z > 0, \\ U_1, & z < 0, \end{cases} \tag{4.5}$$

and $\phi = p_0$, $[\![p_0]\!] = 0$. Hence,

$$p = \begin{cases} p_0 - \rho_1 g z, & z > 0, \\ p_0 - \rho_2 g z, & z < 0. \end{cases} \tag{4.6}$$

The Bernoulli equation for this solution is given by

$$p + \frac{\rho}{2}|\mathbf{u}|^2 + \rho g z = c, \tag{4.7}$$

where c is the constant of integration which takes the value c_1 when $z > 0$ and c_2 when $z < 0$. Since the pressure is continuous across $z = 0$,

$$c_1 - \tfrac{1}{2}\rho_1 U_1^2 = c_2 - \tfrac{1}{2}\rho_2 U_2^2. \tag{4.8}$$

Kelvin (1871) solved the stability problem for (4.5)–(4.7), assuming irrotational flow

$$\mathbf{u} = \mathbf{U} + \nabla\phi, \tag{4.9}$$

where

$$\nabla^2\phi = 0 \tag{4.10}$$

and ϕ vanishes as $|z| \to \infty$. The Bernoulli equation is given by

$$p + \rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}(\mathbf{U} + \nabla\phi)^2 + \rho g z = c, \tag{4.11}$$

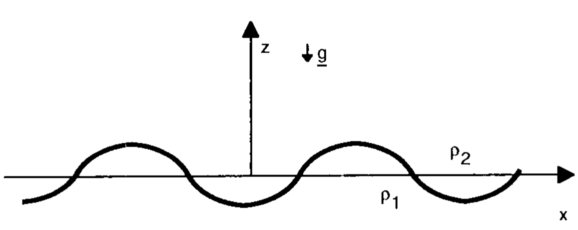

Figure 4.1. The interface at $z = \zeta$ separates two fluids of different density $[\![\rho]\!] \stackrel{\text{def}}{=} \rho_1 - \rho_2 \neq 0$ and different velocity.

where c_1 and c_2 are related by (4.8). The problem is now linearized and the interface conditions are expressed on $z = 0$. Equation (3.2) may then be written as

$$w = \frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x}. \tag{4.12}$$

This implies that the normal component of velocity is continuous across $z = 0$:

$$\frac{\partial \phi_1}{\partial z} - U_1\frac{\partial \zeta}{\partial x} = \frac{\partial \phi_2}{\partial z} - U_2\frac{\partial \zeta}{\partial x}. \tag{4.13}$$

Evaluation of the Bernoulli equation on either side of $z = 0$ with $p_1 = p_2$ gives

$$\rho_1\left[U_1\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial t} + g\zeta\right] = \rho_2\left[U_2\frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial t} + g\zeta\right]. \tag{4.14}$$

The problem (4.10), (4.12)–(4.14) is solved using normal modes

$$(\zeta, \phi_1, \phi_2) = (\hat{\zeta}, \hat{\phi}_1 e^{-\alpha z}, \hat{\phi}_2 e^{\alpha z})e^{i(kx+ly)}e^{\sigma t}, \tag{4.15}$$

where $\alpha = \sqrt{k^2 + l^2}$ and $\hat{\zeta}$, $\hat{\phi}_1$, $\hat{\phi}_2$ are constants. Elimination of these constants leads to

$$\sigma = -ik\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left\{\frac{k^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} - \frac{\alpha g(\rho_1 - \rho_2)}{\rho_1 + \rho_2}\right\}^{1/2}. \tag{4.16}$$

Only the positive root can lead to instability provided that

$$k^2\rho_1\rho_2(U_1 - U_2)^2 > \alpha g(\rho_1^2 - \rho_2^2). \tag{4.17}$$

The most rapidly growing wave is two-dimensional, $\alpha = k$. Thus we have instability when

$$k\rho_1\rho_2(U_1 - U_2)^2 > g(\rho_1^2 - \rho_2^2). \tag{4.18}$$

Equation (4.16) shows that the normal mode solution (4.15) is a wave traveling in the direction $\boldsymbol{\alpha} = k\mathbf{e}_x + l\mathbf{e}_y$ with speed $c = -\mathrm{Im}\,\sigma/|\alpha|$.

Various conclusions may be drawn from (4.16). The conclusions about instability to short waves can be determined by inspection for large wave number k. If $U_1 \neq U_2$ and k is large, we have an unstable eigenvalue with

$$\mathrm{Re}\,\sigma = k\frac{U_1 - U_2}{\rho_1 + \rho_2}\sqrt{\rho_1\rho_2}. \tag{4.19}$$

This is known as Kelvin–Helmholtz instability. If $U_1 = U_2$ and $\rho_2 > \rho_1$ (heavy above), then there is an unstable eigenvalue with

$$\mathrm{Re}\,\sigma = \left\{\frac{\alpha g(\rho_2 - \rho_1)}{\rho_1 + \rho_2}\right\}^{1/2}. \tag{4.20}$$

This is Rayleigh instability, sometimes called Rayleigh–Taylor instability because Taylor noticed that the same result would hold if g was replaced by any other acceleration.

Equations (4.19) and (4.20) show that Kelvin–Helmholtz and Rayleigh–Taylor instabilities are catastrophic short-wave instabilities of the Hadamard type.

The appearance of Hadamard instabilities of idealized problems should not be discounted as fundamental results in the study of physical systems. Their appearance shows that there is a kind of instability associated with short waves which brings into action terms which are small in the hydrodynamics of smoother motions. The relevance of these kinds of instabilities has been convincingly expressed by Birkhoff (1962).

> Helmholtz and Taylor instability are very real physical phenomena. They show up, at least qualitatively, in many familiar situations.
>
> Already in 1867,[1] Tyndall observed that *acoustic stimulation* could cause the vortex sheet surrounding a circular air jet to "roll up" into *periodic spirals*. This, and the related phenomenon of the roaring of a "sensitive gas jet," were exhaustively studied by Helmholtz

and Rayleigh [**7a**, (Section) 322, 70]. Rayleigh showed that resonance, as well as acoustic stimulation, could give rise to periodic instability.

High-speed liquid jets in air are also subject to Helmholtz instability; this is an important factor in the atomization of liquid fuel jets and sprays. But the phenomenon of atomization is extremely complicated. Thus, at low speeds, capillarity is more important than Helmholtz instability [**7a**, Chapter XX]. At higher speeds, atomization is influenced by a combination of Helmholtz instability, surface tension, viscosity, and turbulence,[2] a general mathematical description of whose combined action seems very difficult.

Many other natural phenomena have been attributed to Helmholtz instability. Most familiar is the generation by wind of waves in water, whose Helmholtz instability was first analyzed by Kelvin [**5**, pp. 76–85]. Helmholtz [**5**, p. 457] explained the formation of "mackerel clouds" as due to Helmholtz instability,[3] while Rayleigh [**7**, p. 367] attributed "the flapping of sails and flags" to the same cause.

Taylor instability helps to explain the loss of energy in successive pulsations of underwater explosion bubbles.[4] Near the minimum radius of a pulsating bubble, *the spherical interface postulated by Rayleigh* [**3**, p. 239] *is unstable, and so his solution of the equations of motion is unrealistic.* The Taylor instability of collapsing cavitation bubbles is more subtle [**3**, Chapter XI, (Section) 13)] but also important.

Taylor instability also helps to explain the observed instability of Humphreys pumps [**1**, p. 30], and the breakdown of film boiling.[5]

[1] Philos, Mag. vol. **33** [1867] pp. 92–99 and 375–391. Helmholtz' ideas were directly influenced by Tyndall's observations [**4**, p. 222].

[2] [3, pp. 328–331] and refs. given there. See also H.B. Squire, British J. Appl. Phys. vol. **4** [1953] pp. 167–169; N. Dombrowski and R.P. Fraser, Philos. Trans. Roy. Soc. London Ser. A. vol. **247** [1954] pp. 101–130; P. Eisenklam, N. Dombrowski, and D. Hasson, Imperial College Report JRL **44**, May, 1959.

[3] See also V. Bjerknes, J. Bjerknes, H. Solberg, and T. Bergeron, *Physikalische Hydrodynamik*, Berlin, Teubner, 1932.

[4] R.H. Cole, *Underwater Explosions*, Princeton, N.J., Princeton University Press, 1948, p. 304.

[5] S.S. Kutateladze, Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk vol. **4** [1951] pp. 529–536; Y.P. Chang, Trans. Amer. Soc. Mech. Eng. vol. **79** [1957] pp. 1501–1513; N. Zuber, ibid. vol. **80** [1958] pp. 711–720.

1 G. Birkhoff, *Taylor instability and laminar mixing*, Los Alamos Report LA-1862 [1954], with Appendices A–H issued as Report LA-1927 [1956].
3 G. Birkhoff and E.H. Zarantonello, *Jets, wakes, and cavities*, New York, Academic Press, 1957.
5 Lord Kelvin (Sir William Thompson), *Mathematical and physical papers*, vol. **4**, Cambridge University Press, 1910.
7 Lord Rayleigh (J.W. Strutt), *Scientific papers*, vol. 1, Cambridge University Press, 1899.
7a ——, *Theory of sound*, 2nd ed., 1896, vol. 2.

We could imagine different unsteady versions of the present problem. For example, if two ($l = 1, 2$) uniform body force fields $\mathbf{e}_x \rho_l f_l(t)$ are prescribed, then the basic flow satisfies $\mathring{U}_l = f_{l(t)}$ and can be unsteady. In this case, the stability problem has coefficients which depend on t and (4.15) would be replaced by

$$[\zeta, \phi_1, \phi_2] = \{\varepsilon(t), \hat{\phi}_1(t)e^{-\alpha z}, \hat{\phi}_2(t)e^{\alpha z}]e^{ikx}. \tag{4.21}$$

After eliminating $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ we find that

$$\ddot{\varepsilon} + 2iA(t)\dot{\varepsilon} + [-B(t) + iC(t)]\varepsilon = 0, \tag{4.22}$$

where

$$\begin{aligned} A &= k\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2}, \\ B &= k\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + k\frac{\rho_1 U_1^2 + \rho_2 U_2^2}{\rho_1 + \rho_2}, \\ C &= k\frac{\rho_1 \mathring{U}_1 + \rho_2 \mathring{U}_2}{\rho_1 + \rho_2}. \end{aligned} \tag{4.23}$$

We can have stability at any fixed k, no matter how large, if the evolution of A, B, C is such that $\varepsilon(t)$ is bounded as $t \to \infty$. Hadamard stability is different; we freeze coefficients, write $\varepsilon(t) \sim e^{\sigma t}$ and find that

$$\sigma = -iA \pm \sqrt{B - A^2 - iC}, \tag{4.24}$$

where

$$B - A^2 - iC = k\frac{\rho_1\rho_2}{\rho_1 + \rho_2} + k^2\frac{\rho_2\rho_1(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2}. \tag{4.25}$$

The frozen coefficient problem is Hadamard unstable if $U_1 \neq U_2$ or $\rho_1 > \rho_2$ (heavy above).

Moore and Griffith-Jones (1974) considered a Kelvin–Helmholtz problem with $\rho_1 = \rho_2$ for a circular vortex of radius $R(t)$ and a fixed circulation Γ. The basic flow for their problem has a source at origin with an outward radial component of velocity $U_r = \mathring{R}R/r$ and discontinuous tangential component of velocity $U_\theta = \Gamma/2\pi r$ when $r > R(t)$ and $U_\theta = 0$ when $r < R(t)$. They disturb this flow and consider the linearized stability relative to a disturbed interface at $r = R(t) + \varepsilon(t)e^{is\theta}$, s is a positive integer, and they find (4.22) with $C = 0$,

$$A = -i\frac{\mathring{R}}{R} + \frac{s\Gamma}{4\pi R^2},$$

$$B = \frac{s(s-1)\Gamma^2}{8\pi^2 R^4} - \frac{\mathring{\mathring{R}}}{R}.$$

They show that when $R(t) = R_0(at + 1)^n$, then $|\varepsilon(t)| \sim (at + 1)^{1-n}$ when $n > \frac{1}{2}$. Hence the stretched vortex flow is stable when the stretch rate $n > 1$ is large. However, an analysis of frozen coefficients like the one given above, or a WKB analysis leading to equation (4.1) of their paper, shows that the stable flows are Hadamard unstable for a fixed wave number k, however large, $\varepsilon(t, k) \to 0$ as $t \to \infty$ whilst for fixed t, however large, $\varepsilon(t, k) \to \infty$ exponentially with k.

5. Nonlinear Kelvin–Helmholtz Instability

The Kelvin–Helmholtz instability is sometimes described as the instability of a *vortex sheet*. The corresponding nonlinear initial-value problem can be framed as follows. Initially we are given some surface of concentrated vorticity in $\mathbb{R}^3$ outside of which curl $\mathbf{u} = 0$. We then seek to determine the evolution of this surface of discontinuity along solutions of Euler's equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p, \qquad \operatorname{div}\mathbf{u} = 0.$$

Birkhoff (1962) conjectured that this nonlinear problem has an analytic solution local in time. This problem has been studied by Sulem *et al.* (1981). They have proved some interesting results for the two-dimensional problem in which the surface of prescribed nonzero vorticity is replaced by a curve. They show that a weak solution of this problem exists for all time. We want to know if the vorticity of the fluid will remain concentrated on a smooth curve, if it is concentrated on a smooth curve initially. The interest then is in smooth solutions. They show that initial data in C^∞ functions can be found that become singular in arbitrarily short times. This is the Kelvin–Helmholtz instability, and it shows that the initial-value problem is not well-posed in a class of C^∞ functions. On the other hand, if all the data are analytic, then for a small time the vorticity of the fluid will remain concentrated on a smooth curve, but this process is highly unstable.

In the Kelvin–Helmholtz problem the amplification rate of a wave of wavelength k is inversely proportional to k. Thus short waves amplify faster than longer waves at all wavelengths, and so, except for some delay in actually exciting the short waves, the outcome is almost inevitably headed for a singularity of some kind. Analytic data is important because with such data the short waves may be avoided (certainly nonanalytic C^k data, even with k large or infinite, imply short waves; see Section 21). In linear problems, if short waves are not present initially, they will not develop. In nonlinear problems shorter and shorter waves develop from nonlinear interactions, leading to breakdown. Birkhoff and Fisher (1959) conjectured that an analytic solution of the nonlinear Kelvin–Helmholtz problem can stop being analytic at a finite time.

Support for this was given by Moore (1984, 1979) in an asymptotic analysis which predicted that to leading order in the initial amplitude ε, a singularity forms in the vortex sheet at a critical time $t_c(\varepsilon)$. The singularity which appears is an infinite jump discontinuity in the vortex sheet's curvature. Using Taylor series in time, Meiron *et al.* (1982) obtained results in agreement with Moore's. The evolution of a planar-vortex sheet was studied by Duchon and Robert (1986, 1988) using the Birkhoff–Rott equation. They constructed exact solutions of this equation that are analytic for all $t < 0$ but have a possible singularity at a finite time. Caflish and Orellana (1989) have shown that the vortex sheet problem is ill-posed in a Sobolev class H^n with $n > \frac{3}{2}$.

The picture we have developed is that an initially analytic distribution of the strength of a vortex sheet will evolve to a singularity in a finite time but that it is prey to Hadamard instability at each and every instant of its evolution.

6. Regularizing Mechanisms and Applications

An axiom of physics is that "there is always a cut-off;" in medicine, the same idea appears in the observation that "the bleeding always stops." From a more pragmatic point of view we must regard the emergence of a singularity as physically unacceptable, a feature that shows an inadequacy in the description of the problem. The notion that perturbations of arbitrarily short wavelengths grow arbitrarily fast cannot be a physically meaningful statement within the framework of hydrodynamic theory. Clearly, the basic equations must be augmented in some way to regularize the singularities that are associated with unstable short waves. In some problems it is clear how to so augment the equations, accounting for physical effects omitted in the idealized problems. For the Kelvin–Helmholtz or Rayleigh–Taylor problems we could add a viscous term, basing the analysis on the Navier–Stokes rather than the Euler equations, promoting regularization through diffusion. Rayleigh (1896) seems to have been the first to notice through diffusion that this type of regularization is appropriate to problems which, like gas jets, involve the stability of an interface between two domains of the same fluid. He notes that

> The investigations of (Section) 365 may be considered to afford an adequate general explanation of the sensitiveness of jets. In the ideal case of abrupt transitions of velocity, constituting vortex sheets, in frictionless fluid, the motion is always unstable, and degree of instability increases as the wave-length of the disturbance diminishes.
>
> The direct application of this result to actual jets would lead us to the conclusion that their sensitiveness increases indefinitely with pitch. It is true that, in the case of certain flames, the pitch of the most efficient sounds is very high, not far from the upper limit of human hearing; but there are other kinds of sensitive jets on which these high sounds are without effect, and which require for their excitation a moderate or even a grave pitch.
>
> A probable explanation of the discrepancy readily suggests itself. The calculations are founded upon the supposition that the changes of velocity are discontinuous—a supposition that cannot possibly agree with reality. In consequence of fluid friction a surface of discontinuity, even if it could ever be formed, would instantaneously disappear, the transition from the one velocity to the other becoming more and more gradual, until the layer of transition attained a sensible width. When this width is comparable with the wave-length of a sinuous disturbance, the solution for an abrupt transition ceases to be applicable, and we have no reason for supposing that the instability would increase for much shorter wave-lengths.

Kulikovski and Regirer (1968) have shown that electrohydrodynamic equations which change type in steady flow can become ill-posed as initial-value problems. Such solutions are Hadamard unstable and cannot be realized. They note that

> Owing to the rapid increase of perturbations, nonevolutionary equations cannot describe correctly changes of any physical quantity in time. Nonevolutionary solutions of the nonlinear equations in many cases can be regarded as an oversimplification in the derivation of these equations by discarding terms which are small for evolutionary solutions, but they can be essential for the perturbations which display a rapid increase. As the short wave disturbances increase most rapidly, then these could be the terms containing space derivatives of higher order or mixed derivatives with respect to space or time.

A similar point of view was adopted by Rutkevich (1970) in his discussion of loss of stability in the sense of ill-posed problems for viscoelastic fluids.

> In order to describe the development of small perturbations in the region where evolutionarity of the initial conditions is not possible, the effect of supplementary physical parameters should be taken into account. In a real system, these parameters can be extremely small, but they play a definite role in establishing a finite upper limit for the rate of buildup of perturbations.

In Sections 7–12 we consider some examples of Hadamard instability and introduce some methods which have been used to regularize the instability. These include the addition of physical effects associated with viscosity (Sections 7 and 9), surface tension (Section 8), capillarity of Korteweg's (1901) type (Section 9), compressibility (Section 11), and by weakening the discontinuity (Section 7).

7. Regularization of the Kelvin–Helmholtz Problem with Viscosity or by Replacing the Discontinuity in Velocity by a Discontinuity of Vorticity

The instability of the linearized Kelvin–Helmholtz problem can be regularized by taking viscous effects into account. In this case a discontinuity of velocity is not possible; instead there is a prescribed discontinuity of vorticity whose magnitude is determined by the requirement that the shear stress be continuous across the flat interface. Hooper and Boyd (1983) considered the problem of stability of the shear flow of superimposed immiscible viscous fluids with a linear velocity profile (constant vorticity) above and below the flat interface. They showed that this viscous analogue of the Kelvin–Helmholtz instability is unstable to surpassingly short waves $k \to \infty$. The instability is benign rather than catastrophic. Surface tension is especially effective in stabilizing this instability.

Another possible way to regularize the Kelvin–Helmholtz problem is to keep the inviscid approximation but to spread the vorticity over a finite layer. In this approximation, like the viscous problem, the undisturbed velocity is initially continuous but the vorticity is discontinuous. Rayleigh (1880) showed that the finite vortex layer is stable to long waves and is not unstable to short waves; the maximum growth rate occurs for wavelengths approximately eight times the layer thickness. The finite layer problem is therefore not ill-posed in a linearized approximation. Pozrikidis and Higdon (1985) did numerical studies of the nonlinear Kelvin–Helmholtz problem for a finite vortex layer. The growth rate of the disturbances is strongly affected by the layer thickness; however, the finite amplitude of the disturbance is relatively insensitive to the thickness and reaches a maximum value of approximately 20% of the wavelength. This might imply that the maximum amplitude is unbounded in the limit of ill-posed problem $k \to \infty$. Actually the crux of the well-posedness of the finite vorticity layer is not the layer, but resides in the diminution of the order of discontinuity, as the following argument shows. We consider the problem of the stability of a plane shear flow of two inviscid fluids with the same density separated by a flat interface. The velocity of the shear flow is continuous, with constant but different vorticity above and below $z = 0$, as shown in Figure 7.1.

The linearized equations governing the perturbation of the plane shear flow are

$$\rho u_t + \rho U u_x + \rho w U' = -p_x, \tag{7.1}$$

$$\rho w_t + \rho U w_x = -p_z, \tag{7.2}$$

$$u_x + w_z = 0. \tag{7.3}$$

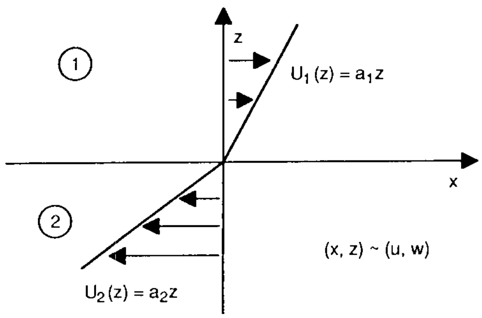

Figure 7.1. Plane shear flow with continuous velocity and discontinuous vorticity

Of course, the integral $\mathbf{u}\cdot\mathbf{n}$ over a pillbox control volume centered on $z = 0$ shows that the normal component of velocity is continuous on $z = 0$

$$[\![w]\!] = w_1 - w_2 = 0. \tag{7.4}$$

Moreover,

$$w = \delta_t \tag{7.5}$$

and

$$p_1 - p_2 = T\delta_{xx}, \tag{7.6}$$

where T is surface tension and δ is the amplitude of the perturbation.

We may eliminate p and u from (7.1)–(7.3). Thus

$$[U\partial_x + \partial_t]\nabla_2^2 w = 0. \tag{7.7}$$

Now we eliminate p from the interface conditions by forming the jump of (7.1) across $z = 0$:

$$\rho[\![u_t]\!] + \rho w[\![U']\!] = -[\![p_x]\!] = -T\delta_{xxx}. \tag{7.8}$$

After differentiating (7.8) with respect to x, we may eliminate u, using (7.3)

$$[\![w_{zt}]\!] - w_x[\![U']\!] = \frac{T}{\rho}\delta_{xxxx}. \tag{7.9}$$

Now we solve (7.5), (7.7), and (7.9) using normal modes

$$\begin{aligned} \delta &= \hat{\delta}e^{\sigma t}e^{ikx}, \\ w &= \hat{w}(z)e^{\sigma t}e^{ikx}. \end{aligned} \tag{7.10}$$

We find that

$$\hat{w}(0) = \sigma\hat{\delta} \tag{7.11}$$

and

$$(iUk + \sigma)(\hat{w}'' - k^2\hat{w}) = 0. \tag{7.12}$$

Hence there is a continuous spectrum with

$$\sigma = -iU(z)k \qquad \text{for all} \quad z \in \mathbb{R}, \tag{7.13}$$

where $U(z)$ is linear in z, as in Figure 7.1. Another solution is

$$\begin{aligned} \hat{w}_1 &= Ae^{-kz}, \\ \hat{w}_2 &= Ae^{kz}. \end{aligned} \tag{7.14}$$

Hence $\hat{w}_1(0) = \hat{w}_2(0) = \hat{w}(0) = A$ and

$$A = \sigma\hat{\delta}. \tag{7.15}$$

We next evaluate (7.9):

$$-2\sigma kA - kiA[\![U']\!] = \frac{T}{\rho}k^4\hat{\delta},$$

where

$$[\![U']\!] = a_1 - a_2.$$

Hence

$$2\sigma^2 k + ik\sigma[\![U']\!] = -\frac{T}{\rho}k^4.$$

It follows that the shear flow with discontinuous vorticity is stable even when the surface tension $T = 0$ and in any case is not Hadamard unstable.

8. Fingering Instabilities in Porous Media. Regularization with Interfacial Tension

Fingering instabilities are an important topic for the dynamical description of flowing multicomponent systems. They may be related to Hadamard instabilities, when regularizing mechanisms are neglected. The mathematical analysis of fingering is not well developed; it is perhaps best developed for the case of displacement of one fluid by another in a porous media. The analysis of stability of this problem is evidently due to Chouke *et al.* (1959) (see Homsy (1987) for a historical note). According to Saffman (1986),

> About 1956, Sir Geoffrey Taylor paid a visit to the Humble Oil Company and became interested in problems of two phase flow in porous media. He worked out the macroscopic instability which can arise when a less viscous fluid drives a more viscous one and which is at least partly responsible for the coreing in processes of secondary recovery in oil fields. He also realized that two-dimensional flow in a porous medium is modelled by flow in a Hele–Shaw [1898] apparatus consisting of two flat parallel plates separated by a small gap b. Then the average two-dimensional velocity $\mathbf{u}$ of a viscous fluid in the space between the plates is related to the pressure by the formula
>
> $$\mathbf{u} = -\frac{b^2}{12\mu}\,\mathrm{grad}\,p, \qquad \mathrm{div}\,\mathbf{u} = 0,$$
>
> where μ is the viscosity. This is identical to Darcy's law for motion in a porous medium of permeability $b^2/12$. But it is, of course, an approximation valid when the gap or transverse dimension b is small compared with variations of scale a, say, in the lateral dimension parallel to the plates.

It is necessary to add that nonlinear effects in the flow in porous media which are described by (8.3) may not be well modeled by averaging inertial terms in a small gap of a Hele–Shaw cell.

The famous fingering instability result of Chouke *et al.*, given in Saffman and Taylor (1958), explains why it is so difficult to push oil out of the ground with water. The water fingers through the oil. They considered stability of a plane interface between the fluids when the two fluids are advancing against gravity with a speed W. They assumed that the flow of each of the two fluids is governed by Darcy's law

$$\nabla\Phi = -\frac{\mu}{\kappa}\mathbf{u}, \tag{8.1}$$

where $\Phi = p + \rho g z$ is the head, p is the pressure, ρ is the density, z is against gravity, μ is the viscosity, κ is the permeability, and $\mathbf{u}$ is the velocity. They linearized their problem around the motion with a flat moving interface and considered disturbances periodic in x (along the interface) with period $2\pi/\alpha$ proportional to $e^{\sigma t}$. They found that

$$\frac{\sigma}{\alpha}\left[\frac{\mu_1}{\kappa_1} + \frac{\mu_2}{\kappa_2}\right] = g(\rho_1 - \rho_2) + \left[\frac{\mu_1}{\kappa_1} - \frac{\mu_2}{\kappa_2}\right]W, \tag{8.2}$$

where the subscript two is for the lower fluid and the subscript one is for the upper fluid, α is a wave number. In the formula the upper and lower fluids have different permeabilities κ_1 and κ_2. When water advances into oil, some oil is left behind so that the permeability of solid to oil is different than the permeability of the solid plus some oil, to water. Equation (8.2) shows that Taylor's fingering instability is a Hadamard instability.

Darcy's law is an asymptotic law, valid for very low Reynolds numbers. The effects of inertia manifest themselves in a quadratic drag law which arises from the pressure loss in the wakes of solid particles, averaged over many particles. This law is expressed by the quadratic term introduced by Dupuit (1863) and Forcheimer (1901). We express this law in vector notation as

$$\nabla\Phi = -\frac{\upsilon}{\kappa}\mathbf{u}, \qquad \upsilon(|\mathbf{u}|) \overset{\text{def}}{=} \mu + c\rho|\mathbf{u}|\sqrt{\kappa}, \tag{8.3}$$

where c is the "form drag" constant.

The effect of inertia on the fingering instability in the flow of two fluids in a saturated porous medium is to increase the resistance by a term that is independent of viscosity and is proportional to the density times the square of the velocity. This effect can stabilize the displacement of more viscous fluids by less viscous ones and destabilize the displacement of less viscous fluids by more viscous ones.

It is of interest to repeat the analysis of Chouke *et al.* using (8.3) rather than (8.2), following the work of Saville (1969). We must satisfy (8.3), div $\mathbf{u} = 0$, $\mathbf{u} \to \mathbf{e}_z W$ as $|z| \to \infty$. At the interface $z = \zeta$ we have (3.2), implying that the normal component of velocity is continuous. The origin $z = 0$ moves upward with constant velocity $\mathbf{W}$. The normal "stress" condition at the interface is expressed by (3.4) which reduced to the continuity of the pressure over $z = \zeta$ when surface tension effects are neglected. These equations can be satisfied by a traveling flat interface

$$(\mathbf{u}, \Phi, \zeta) = (\mathbf{e}_z W, \Phi^0(z), \zeta_0), \qquad \zeta_0 = 0, \tag{8.4}$$

where $W = d\zeta_0/dt$ is constant and

$$\Phi_1^0 = -\frac{\upsilon_1(W)}{\kappa_1} Wz + \pi_1, \qquad z > 0, \tag{8.5}$$

$$\Phi_2^0 = -\frac{\upsilon_2(W)}{\kappa_2} Wz + \pi_2, \qquad z < 0, \tag{8.6}$$

$$[\![\Phi^0]\!] = \Phi_1^0 - \Phi_2^0 = [\![\pi]\!] = p_c. \tag{8.7}$$

To study stability we extend the basic solution; (8.5) holds for $z > \zeta$ and (8.6) for $z < \zeta$. We then consider a perturbation

$$(\mathbf{u}, \Phi, \zeta) = (\mathbf{u} + \mathbf{e}_z W, \phi + \Phi^0, \zeta) \tag{8.8}$$

of the extended basic flow. The evolution of ζ is governed by (3.2), ϕ satisfies

$$\nabla\phi = -\frac{\upsilon(|\mathbf{e}_z W + \mathbf{u}|)}{\kappa}(\mathbf{u} + \mathbf{e}_z W) - \frac{\upsilon(W)}{\kappa}\mathbf{e}_z W \tag{8.9}$$

and after accounting for Φ^0 in $[\![\phi + \Phi^0]\!]$, we get the normal stress equations on $z = \zeta$ in the form

$$[\![\phi]\!] - [\![\rho]\!]g\zeta - \left[\!\!\left[\frac{\upsilon(W)}{\kappa}\right]\!\!\right] W\zeta = \gamma\nabla_2\cdot\left\{\frac{\nabla_2\zeta}{(1 + |\nabla_2\zeta|^2)^{1/2}}\right\}. \tag{8.10}$$

After linearizing for small disturbances $|\mathbf{e}_z W + \mathbf{u}| \to W + w$, we replace (8.9) with

$$\nabla\phi = -\frac{\upsilon(W)}{\kappa}\mathbf{u} - \frac{c\rho}{\sqrt{\kappa}} wW\mathbf{e}_z, \tag{8.11}$$

(3.2) with

$$w = \frac{\partial\zeta}{\partial t} \tag{8.12}$$

on $z = 0$, and (8.10) with

$$[\![\phi]\!] - [\![\rho]\!]g\zeta - \left[\!\!\left[\frac{\upsilon(W)}{\kappa}\right]\!\!\right] W\zeta = \gamma\nabla_2^2\zeta \tag{8.13}$$

on $z = 0$. It is convenient to write (8.11) as

$$\left[\frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right] = -a[\xi w, u, v], \tag{8.14}$$

where

$$a = \frac{v(W)}{\kappa},$$

$$b = \frac{v(W)}{\kappa}, \tag{8.15}$$

$$\xi a = a + b = \frac{1}{\kappa}(\mu + 2c\rho W\sqrt{\kappa}).$$

After using (8.14) we get

$$\frac{\partial\zeta}{\partial t} = w = -\frac{1}{\xi a}\frac{\partial\phi}{\partial z} \tag{8.16}$$

and, using div $\mathbf{u} = 0$,

$$\frac{1}{\xi}\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{8.17}$$

We may now solve (8.16) and (8.17) using normal modes:

$$[\zeta, \phi_1, \phi_2] = [\tilde{\zeta}, \tilde{\phi}_1 e^{-q_1 z}, \tilde{\phi}_2 e^{q_2 z}]e^{\sigma t - i(\alpha x + \beta y)}, \tag{8.18}$$

where

$$(q_1, q_2) = k(\sqrt{\xi_1}, \sqrt{\xi_2}), \qquad k^2 = \alpha^2 + \beta^2. \tag{8.19}$$

The normal mode reduction of (8.16) gives

$$\sigma\tilde{\zeta} = \frac{q_1}{\xi_1 a_1}\tilde{\phi}_1 = \frac{-q_2}{\xi_2 a_2}\tilde{\phi}_2. \tag{8.20}$$

Hence,

$$\tilde{\phi}_1 - \tilde{\phi}_2 = \sigma\tilde{\zeta}A^2/k, \tag{8.21}$$

where

$$A^2 = k\left\{\frac{\xi_1 a_1}{q_1} + \frac{\xi_2 a_2}{q_2}\right\} = a_1\sqrt{\xi_1} + a_2\sqrt{\xi_2}$$

and

$$a\sqrt{\xi} = \left(\frac{\mu}{\kappa} + \frac{c\rho W}{\sqrt{\kappa}}\right)^{1/2}\left(\frac{\mu}{\kappa} + \frac{2c\rho W}{\sqrt{\kappa}}\right)^{1/2}.$$

The growth rate σ may be determined from the normal stress balance (8.13), now written as

$$\tilde{\phi}_1 - \tilde{\phi}_2 = [\![\rho]\!]g\tilde{\zeta} + \left[\!\!\left[\frac{v(W)}{\kappa}\right]\!\!\right]W\tilde{\zeta} - k^2\gamma\tilde{\zeta}. \tag{8.22}$$

Hence

$$\sigma = \frac{k}{A^2}\left\{[\![\rho]\!]g + \left[\!\!\left[\frac{v(W)}{\kappa}\right]\!\!\right]W\right\} - \frac{k^3}{A^2}\gamma. \tag{8.23}$$

Equation (8.23) is the main result of the analysis. The flat advancing displacement front is stable when

$$[\![\rho]\!]g + \left[\!\!\left[\frac{v(W)}{\kappa}\right]\!\!\right]W = (\rho_1 - \rho_2)g + \left(\frac{v_1}{\kappa_1} - \frac{v_2}{\kappa_2}\right)W \stackrel{\text{def}}{=} \Gamma < 0, \tag{8.24}$$

where

$$v_1 = \mu_1 + c_1\rho_1 W\sqrt{\kappa_1}.$$

The displacement front is stable if the heavy and more viscous fluid is below. When there is no surface tension and $\Gamma > 0$, then $\sigma = k\Gamma/A^2$ is unbounded, irregular, tending to infinity with k. In the absence of stabilization by gravity, it is not possible to displace oil with water; the water will finger through. The effects of inertia are stabilizing when

$$\frac{\rho_1 c_1}{\sqrt{\kappa_1}} < \frac{\rho_2 c_2}{\sqrt{\kappa_2}}.$$

Consider the case in which the less viscous heavy liquid advances $W > 0$ into a more viscous liquid ($\mu_1 > \mu_2$, $\rho_1 < \rho_2$), say, water displaces oil. Assume, only for simplicity, that the form drag constant $c_1 = c_2$ and permeability $\kappa_1 = \kappa_2$ are the same on either side of the advancing front. Then the term

$$v_1 - v_2 = \mu_1 - \mu_2 - cW(\rho_2 - \rho_1)\sqrt{\kappa}$$

is destabilizing when

$$W < \frac{\mu_1 - \mu_2}{c(\rho_2 - \rho_1)\sqrt{\kappa}}$$

and is stabilizing when

$$W > \frac{\mu_1 - \mu_2}{c(\rho_2 - \rho_1)\sqrt{\kappa}}.$$

When surface tension is positive, $\gamma > 0$ and $\Gamma > 0$, small waves with large k are stable but there is always a band $k^2 < \Gamma/\gamma$ of unstable wave numbers. Moreover, $\sigma(k)$ is bounded with a maximum growth rate $\sigma = \frac{2}{3}\Gamma\sqrt{(\Gamma/3\gamma)}/A^2$ at $k^2 = \Gamma/3\gamma$.

The aforementioned results show that fingering in the oil displacement problem, using the Darcy–Forcheimer law, leads to Hadamard instability when surface tension is neglected and is regularized by surface tension. It is natural to think about what might actually happen to this problem if the experiment could be carried out with two liquids possessing vanishing interfacial tension. Nittman *et al.* (1985) did an experiment to answer this question. They asked, "What happens when one attempts to push water through a fluid of higher viscosity? Under appropriate experimental conditions, the water breaks through in the form of highly branched patterns called viscous fingers. Water was used to push a more viscous but miscible, non-Newtonian fluid in a Hele–Shaw cell. The resulting viscous finger instability was found to be a fractal growth phenomenon."

9. Instability of Phase-Change Models Based on Reclining S-shaped Curves. Regularization by Viscosity and Capillarity

Mathematical models of phase changes are sometimes based on material behavior based on reclining S-shaped curves (Figure 9.1). A classical example of this behavior is the phase diagram for the van der Walls gas.

This type of constitutive assumption leads to Hadamard instability in the so-called spinoidal region. This instability can be regularized by viscosity and by capillarity.

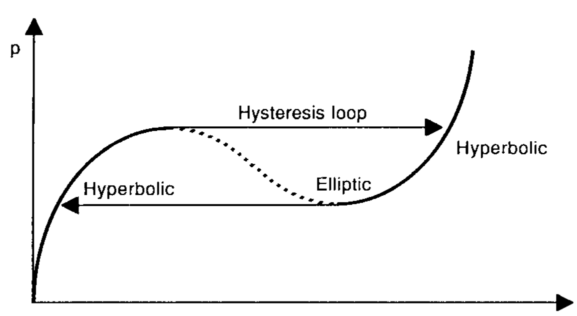

Figure 9.1. The system (9.1) is hyperbolic when $p'(v) > 0$. The elliptic branch is unstable in the sense of Hadamard. This region is called "spinoidal."

Consider the quasilinear system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial p(v)}{\partial x}, \\ \frac{\partial v}{\partial t} &= \frac{\partial u}{\partial x}, \end{aligned} \tag{9.1}$$

where $p(v)$ lies on a reclining S-shaped curve shown in Figure 9.1

The curve $p(v)$ is a constitutive assumption giving, say, the stess p is a function of strain or velocity for different problems. Equations (9.1) are a first-order quasilinear system which is equivalent to a single second-order equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 p(v)}{\partial x^2}. \tag{9.2}$$

It is clear from (9.2) that wherever $p'(v)$ is negative, (9.2) is like Laplace's equation and is Hadamard unstable. The resulting PDE is nonlinear:

$$\frac{\partial^2 v}{\partial t^2} + |p'(v)|\frac{\partial^2 v}{\partial x^2} = p''(v)\left(\frac{\partial v}{\partial x}\right)^2.$$

The sign of p' is important!

Let us do something a little artificial to show how Hadamard instability arises on frozen coefficients of a stability problem for (9.2). We could solve

$$\frac{\partial^2 p(v)}{\partial x^2} = 0$$

for $v(x)$ such that $v(0) = 0$ and $v(1) = 1$. Call this solution v_0 and let w be a small perturbation of v_0. For w we get

$$\frac{\partial^2 w}{\partial t^2} = 2p''(v_0)\frac{\partial v_0}{\partial x}\frac{\partial w}{\partial x} + p'(v_0)\frac{\partial^2 w}{\partial x^2} + \left[p'''(v_0)\left(\frac{\partial v_0}{\partial x}\right)^2 + p''(v_0)\frac{\partial^2 v_0}{\partial x^2}\right]w, \tag{9.3}$$

where $v' = 0$ at $x = 0, 1$. In general, (9.2) has variable coefficients and does not admit stability studies using normal modes.

Now we look at short wavelength disturbances. If the wavelength of a disturbance is short enough, v_0 will be nearly constant over the whole length of the wave. This leads us to freeze the coefficients in (9.3), treating v_0 as a constant in a small neighborhood of each and every point. In each neighborhood we write

$$w = \text{const}\, e^{\sigma t + ikx} \tag{9.4}$$

and get

$$\sigma^2 = p'(v_0)k^2 + O(|k|).$$

Of course we cannot satisfy the boundary condition for (9.3); the analysis of stability here is purely local and it may hold only if $k \to \infty$, for short waves. Hence,

$$\frac{\sigma^2}{k^2} = p'(v_0) \tag{9.5}$$

asymptotically and we have Hadamard instability wherever v_0 is such that $p'(v_0) > 0$.

In another version of this problem we could try to regularize with a Newtonian viscosity μ. Then we would consider the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial p(v)}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial u} &= \frac{\partial u}{\partial x}. \end{aligned} \tag{9.6}$$

The system (9.6) can be reformulated as an equation of third order

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 p(v)}{\partial x^2} + \mu\frac{\partial^3 v}{\partial x^2 \partial t}.$$

Let $v_0(x - ct)$ be a traveling wave. Now write

$$v = v_0 + w$$

and linearize to get

$$\frac{\partial^2 w}{\partial t^2} = 2p''(v_0)\frac{\partial v_0}{\partial x}\frac{\partial w}{\partial x} + p'''(v_0)w\left(\frac{\partial v_0}{\partial x}\right)^2 + p'(v_0)\frac{\partial^2 w}{\partial x^2} + p''(v_0)w\frac{\partial^2 v_0}{\partial x^2} + \mu\frac{\partial^3 w}{\partial x^2 \partial t}. \tag{9.7}$$

We next freeze the coefficients and write

$$w = \text{const } e^{\sigma t + ikx}.$$

Then (9.7) reduces to the algebraic expression

$$\sigma^2 = 2p''(v_0)\frac{\partial v_0}{\partial x} ik + p'''(v_0)\left(\frac{\partial v_0}{\partial x}\right)^2 + p''(v_0)\frac{\partial^2 v_0}{\partial x^2} - p'(v_0)k^2 - \mu\sigma k^2. \tag{9.8}$$

Divide (9.8) by k^2, and simplify the result for large k. Thus

$$\sigma^2 + \mu\sigma k^2 \sim -p'k^2.$$

Hence, to lowest order

$$\frac{\sigma}{k} = -\frac{\mu k}{2} \pm \sqrt{\frac{\mu^2 k^2}{4} - p'}.$$

We clearly have stability when $p' > 0$. Now consider $p' = -|p'|$. Then

$$\frac{\sigma}{k} = -\frac{\mu k}{2} + \sqrt{\frac{\mu^2 k^2}{4} + |p'|} = -\frac{\mu k}{2}\left[-1 + \sqrt{1 + \frac{4|p'|}{\mu^2 k^2}}\right].$$

For large μk this reduces to $\sim |p'|/\mu k$. Hence $\sigma \sim |p'|/\mu$. The solution is unstable, but not Hadamard unstable.

Hadamard instability "occurs" when

$$\mu^2 k^2 \ll |p'|$$

and

$$k \to \infty.$$

So in this case we get a formal regularization of the equations, but the Hadamard instability persists in a practical sense if μ is small enough.

Of course, there is some form of continuity between ill-posed problems as the regularizing parameter tends to zero.

Korteweg (1901) introduces the notion that the stress in a fluid should depend on the density. He derived the invariant form for the quadratic approximation of this stress. In Newtonian–Korteweg fluids this stress can be expressed, in notation used by Truesdell and Noll (1965), as

$$\mathbf{T} = (-p + \lambda \operatorname{tr} \mathbf{D} - \alpha|\nabla\rho|^2 + \gamma\nabla^2\rho)1 + 2\mu\mathbf{D} - \beta\nabla\rho \otimes \nabla\rho + \delta\nabla(\nabla\rho), \tag{9.9}$$

where p is to be determined from the equations as in an incompressible fluid. In fact Korteweg's theory has been applied to liquid vapor transitions in which sharp interfaces are replaced by narrow layers with strong gradients of density. The theory has been applied by Slemrod (1983) and Hagen and Slemrod (1983) to a problem of admissibility of shock solutions. Without going into their application we can use their equations to introduce the notion of Korteweg regularization of ill-posed problems:

$$\frac{\partial u}{\partial t} = \frac{\partial \tau}{\partial x},$$

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x},$$

where u is the velocity and v is the density. In the notation of Hagan and Slemrod

$$\tau = -p(v) + B(v)v_x^2 - C(v)v_{xx} + \mu(v)u_x$$

and

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 \tau}{\partial x^2} = \frac{\partial^2}{\partial x^2}[-p + Bv_x^2 - Cv_{xx} + \mu v_t].$$

We now repeat the analysis which was constructed for the case $B = C = 0$ when μ, B, C are constants, possibly not zero. We write

$$v = v_0 + w,$$

where v_0 is a traveling wave solution and w is small. Then, neglecting some terms which will be negligible in the analysis of short waves on frozen coefficients we get

$$\frac{\partial^2 w}{\partial t^2} = p'(v_0)\frac{\partial^2 w}{\partial x^2} + 2Bv_{0x}\frac{\partial^2 w}{\partial x^2} - cw_{xxxx} + \mu w_{txx}.$$

We may reduce this using normal modes (9.4) to

$$\sigma^2 = \mu\sigma k^2 = -(p'(v_0) + 2Bv_{0x})k^2 - ck^4,$$

$$\frac{\sigma}{k} = -\frac{\mu k}{2} \pm \sqrt{\left(\frac{\mu^2}{4} - c\right)k^2 - (p' + 2Bv_{0x})}.$$

This problem is ill-posed when c is negative, with unbounded positive growth rate $\sigma \sim k\sqrt{|c|}$ for sufficiently large k. This result could be used to argue that $c > 0$ in good models.

We leave it to the reader to formulate the cut-off condition for which the problem with small, positive c and μ is exactly well-posed but effectively ill-posed.

10. Regularization with Nonlinear Viscosity

Beale and Schaeffer (1988) have exhibited equations which they interpret as being linearly ill-posed and nonlinearly well-posed. Their results can be interpreted in terms of a nonlinear viscosity which vanishes with the trivial solution. Consider a Cauchy–Riemann equation perturbed by a viscosity μ

$$u_t + iu_x = \mu u_{xx}.$$

When $\mu = 0$, disturbances of the form $e^{ikt}e^{\sigma t}$ give rise to

$$\sigma = k,$$

that is, the Hadamard instability. When $\mu \neq 0$

$$\sigma = -\mu k^2 + k$$

which is Hadamard stable. Beale and Schaeffer introduced a nonlinear viscosity $\mu = |u_x|^2$ so that the linearization of

$$u_t + iu_x = (|u_x|^2 u_x)_x$$

around zero is the left-hand side alone and is Hadamard unstable. On the other hand, linearization on any solution for which $u_x \neq 0$ will give an effective visocity which will regularize the problem. Beale and Schaeffer have shown that the nonlinear equation does possess a solution in a suitable Sobolev class. The analysis of short waves using the method of frozen coefficients shows that the flow will be Hadamard unstable in the neighborhood of any point for which $u_x = 0$. This shows again that the concept of ill-posed problems is rigorously attached to the solutions, and not to the equation.

11. Instability in the Evolution Equations Describing Granular Flow. Regularization by Compressibility

This section summarizes some results of Schaeffer (1987) and Pitman and Schaeffer (1988). Their studies show that compressibility regularizes ill-posed problems in granular materials. The type of problem they consider arises in the application of mathematics to silo design. They obtain a system of governing evolutions using constitutive modeling, which is to specify a flow rule, a yield condition, and a relation between density and stress. They then freeze the coefficients and determine the conditions for Hadamard instability.

They find that when compressibility is not taken into account, the equations governing granular flow can be Hadamard unstable, depending on geometric and material parameters. The instability which develops is analogous to that for

$$u_t = u_{xx} - u_{yy}. \tag{11.1}$$

(If we write $u = e^{\sigma t} e^{i(\alpha x + \beta y)}$, then $\sigma = -\alpha^2 + \beta^2$ gives Hadamard instability when $\beta^2 - \alpha^2 > 0$.) This instability they say is to be expected for parameter values arising in most industrial applications, but the instability can be suppressed by compressiblity.

It is of interest to look at their derivation of the system of evolution equations. Equations for conservation of mass and momentum are

$$\frac{\partial \rho}{\partial t} + \rho \operatorname{div} \mathbf{u} = 0, \tag{11.2}$$

$$\rho \frac{\partial u_i}{\partial t} + \frac{\partial \Sigma_{ij}}{\partial x_j} = \rho g_i, \tag{11.3}$$

where ρ is the density, u_i is a velocity component, Σ_{ij} $(= -T_{ij})$ is a stress tensor, positive in compression, and g is gravity.

Constitutive assumptions which lead to ill-posedness are: (1) incompressibility; (2) a yield stress condition of Von-Mises type

$$\sum_{i=1}^{3} (\Sigma_{ij})^2 \leq k^2 \sigma^2, \tag{11.4}$$

where $\sigma = \frac{1}{2} \operatorname{tr} \Sigma$ is the mean normal stress (in two dimensions, $\sigma = \frac{1}{2} \operatorname{tr} \Sigma$), Σ_{11}, Σ_{22}, Σ_{33} are eigenvalues of Σ, and k^2 is a constant which depends on the material; (3) a flow rule relating to the velocity

$$\mathbf{D}[\mathbf{u}] = q \operatorname{dev} \Sigma, \tag{11.5}$$

where

$$\operatorname{dev} \Sigma = \Sigma - \sigma \mathbf{1} \tag{11.6}$$

is the stress deviator, $\mathbf{D}[\mathbf{u}]$ is the negative of the symmetric part of grad $\mathbf{u}$, and q is a positive scalar. The grains flow, following the flow rule, when the material yields, equality holds in (11.4). In this case,

$$\sum_{i=1}^{3} (\Sigma_{ij} - \sigma)^2 = |\operatorname{dev} \Sigma|^2 = k^2 \sigma^2, \tag{11.7}$$

where $|\mathbf{A}|^2 = \operatorname{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}})$, for any matrix $\mathbf{A}$. It follows that $|\mathbf{D}[\mathbf{u}]|^2 = q^2 k^2 \sigma^2$ which may be solved for q, eliminating q in the flow rule

$$\Sigma = \sigma \left\{ k \frac{\mathbf{D}[\mathbf{u}]}{|\mathbf{D}[\mathbf{u}]|} + \mathbf{1} \right\}. \tag{11.8}$$

The governing evolution equation

$$\begin{aligned} &\operatorname{div} \mathbf{u} = 0, \\ &\rho \frac{\partial u_1}{\partial t} = -k \frac{\partial}{\partial x_j} \left\{ \sigma \frac{D_{ij}}{|\mathbf{D}|} \right\} - \frac{\partial \sigma}{\partial x_j} + \rho g_i \end{aligned} \tag{11.9}$$

follows after substituting (11.8), assuming incompressibility and neglecting inertia. An interesting consequence of these equations is that the dissipation does not increase with the velocity, $D_{ij}/|\mathbf{D}|$ is homogeneous of degree zero. Schaeffer (1987) remarks that

> A striking illustration of this point occurred when mechanical plows replaced draught animals on farms: it was found, to everyone's surprise, that plowing at greater speeds does not require greater forces.

The analysis of (11.9), using frozen coefficients, leads to conical regions of flow in which, depending on parameters, the problem is ill-posed. All granular materials are at least slightly compressible. The equations which describe granular flow when compressibility is included are significantly more complicated than (11.9). Schaeffer notes that

> ..., if these equations are expanded in an asymptotic series in powers of the compressibility, then (11.9) emerges as the zeroth order term in the expansion. Unfortunately, the subsequent terms in the expansion are singular perturbations of (11.9). For example, the first order perturbation of (div $\mathbf{U} = 0$) may be written
>
> $$\operatorname{div} \mathbf{U} + \varepsilon P\left(\frac{\partial \sigma}{\partial t}, \frac{\partial^2 \sigma}{\partial x_i \partial x_j}, \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_g}\right) = 0,$$
>
> where the arguments of P indicate the highest order terms in the perturbation.

Pitman and Schaeffer (1988) studied the effect of small compressibility for flows in two dimensions. They found that even though the magnitude of the density changes in real granular flows is quite small, nevertheless compressibility effects greatly regularize the equations. If the material is loosely packed, then the equations are still linearly well-posed. If the material is tightly packed, the equations are still linearly ill-posed, but the instability is greatly changed and corresponds to the shear banding which is observed in shearing of consolidated granular media. There are two specific directions in wave number space in which disturbances will amplify uncontrollably; other modes are bounded. The two singular directions are characteristic of the hyperbolic steady-state equations. This instability appears to be related to the tendency for overconsolidated material to shear discontinuously along characteristics of the steady-state equations.

Pitman and Schaeffer also considered elastic effects and they say that they showed that elastic effects alone are insufficient to regularize the equations.

12. Quasilinear Systems and Nonlinear Systems

It is always possible to define systems of PDEs of any order as a system of first-order equations in more unknowns. We restrict attention to systems of first-order PDEs which may be quasilinear or nonlinear. For example,

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = f(u)$$

is quasilinear, linear in $\partial u/\partial t$ and $\partial u/\partial x$ while

$$\frac{\partial u}{\partial t} + u\left(\frac{\partial u}{\partial x}\right)^2 - f(u) = 0 \tag{12.1}$$

is nonlinear. We may write (12.1) as

$$F \stackrel{\text{def}}{=} p + uq^2 - f(u) = 0, \qquad p = \frac{\partial u}{\partial t}, \qquad q = \frac{\partial u}{\partial x}. \tag{12.2}$$

We can reduce a system of N first-order nonlinear PDEs in γ independent variables to a system of $(\gamma + 1)N$ quasilinear equations. A general system of N first-order PDEs in two independent variables can be expressed as

$$F_i(x, y, u_1, \ldots, u_n, p_1, \ldots, p_n, q_1, \ldots, q_n) = 0, \qquad i = 1, \ldots, n, \tag{12.3}$$

where

$$p_i = \frac{\partial u_i}{\partial x}, \qquad q = \frac{\partial u_i}{\partial y}$$

are introduced as additional unknowns. We have $3N$ variables and $3N$ equations but one of the equations is nonlinear rather than quasilinear. The system can be reduced to a quasilinear one by differentiation but the reduction is not unique. One symmetric reduction is: $F_i = 0$ is an identity in x and y jointly, hence

$$\frac{dF_i}{dy} = \frac{\partial F_i}{\partial y} + \frac{\partial F_i}{\partial u_l} q_l + \frac{\partial F_i}{\partial q_l}\frac{\partial q_l}{\partial y} + \frac{\partial F_i}{\partial p_l}\frac{\partial p_l}{\partial y} = 0, \tag{12.4}$$

$$\frac{dF_i}{dx} = \frac{\partial F_i}{\partial x} + \frac{\partial F_i}{\partial u_l} p_l + \frac{\partial F_i}{\partial q_l}\frac{\partial q_l}{\partial x} + \frac{\partial F_i}{\partial p_l}\frac{\partial p_l}{\partial x} = 0. \tag{12.5}$$

Equation (12.3) implies that

$$\frac{\partial F_i}{\partial p_j} p_j + \frac{\partial F_i}{\partial q_j} q_j = \frac{\partial F_i}{\partial p_j}\frac{\partial u_j}{\partial x} + \frac{\partial F_i}{\partial q_j}\frac{\partial u_j}{\partial y}. \tag{12.6}$$

We put this system into a symmetric form by writing $\partial p_l/dy = \partial q_l/dx$ in (12.4) and (12.5). Then we put the principal part on the right and the lower-order terms on the left:

$$-\left(\frac{\partial F_i}{\partial y} + \frac{\partial F_i}{\partial u_j} q_j\right) = \frac{\partial F_i}{\partial p_j}\frac{\partial q_j}{\partial x} + \frac{\partial F_i}{\partial q_j}\frac{\partial q_j}{\partial y}, \tag{12.7}$$

$$-\left(\frac{\partial F_i}{\partial x} + \frac{\partial F_i}{\partial u} p_i\right) = \frac{\partial F_i}{\partial p_j}\frac{\partial p_j}{\partial x} + \frac{\partial F_i}{\partial q_j}\frac{\partial q_j}{\partial y}, \tag{12.8}$$

Equations (12.6)–(12.8) are $3N$ equations for the $3N$ unknowns.

The principal parts of each of the equations (12.6)–(12.8) are identical. Each one determines the same characteristic directions. We have

$$\frac{\partial F_i}{\partial p_j}\frac{\partial a_j}{\partial x} + \frac{\partial F_i}{\partial q_j}\frac{\partial a_j}{\partial y} = \text{lot}. \tag{12.9}$$

Hence the characteristics $\lambda = dy/dx$ are determined from

$$\det\left[\lambda \frac{\partial F_i}{\partial p_j} - \frac{\partial F_i}{\partial q_j}\right] = 0. \tag{12.10}$$

(The value of λ for (12.2) is $\lambda = 2uq$.) Equation (12.10) has N roots. The N nonlinear first-order PDEs give rise to N characteristic roots for the quasilinear system arising from differentiating the nonlinear system once with respect to each independent variable.

If we generate (12.10) by the method of simple jumps we can state that real characteristic directions are the loci for discontinuities in the derivatives of p_i and q_i. This means that the second derivatives of u_i suffer jumps in the nonlinear case and the first derivatives suffer jumps in the quasilinear case. The first derivatives are smooth when second derivatives jump so that we get one more derivative of smoothness in the nonlinear case.

It appears that a more far-reaching conclusion following along lines of the last paragraph may be true. Compare quasilinear and nonlinear first-order systems which allow blow up in finite time. The solutions are smooth before the blow-up time. To find blow up we look for intersecting characteristics. First derivatives blow up in quasilinear systems, second derivatives in nonlinear systems. This conjecture is true for some special one-dimensional models of flow of a viscoelastic fluid which have been studied by Slemrod (1985) and Renardy *et al.* (1987).

13. Characteristic Surfaces and Simple Jumps

Let (12.3) be a quasilinear problem and write it in a direct notation as

$$\sum_{l=0}^{n} \mathbf{A}_l \frac{\partial \mathbf{u}}{\partial x_l} = \mathbf{f}, \qquad \mathbf{x} = (t, x_1, x_2, \ldots, x_n). \tag{13.1}$$

A surface S, defined by the equation $\phi(t, x_1, x_2, \ldots, x_n) = 0$, is characteristic with respect to (13.5) at $\mathbf{x} = (t, x_1, x_2, \ldots, x_n)$ if

$$\det\left(\sum_{l=0}^{n} \mathbf{A}_l \frac{\partial \phi}{\partial x_l}\right)(\mathbf{x}) = 0. \tag{13.2}$$

If $\phi = x_n - f(x_0, \ldots, x_{n-1})$, then

$$\det\left(\mathbf{A}_n - \sum_{l=0}^{n-1} \mathbf{A}_l \frac{\partial f}{\partial x_l}\right) = 0. \tag{13.3}$$

Any one of the $n+1$ quantities $\partial\phi/\partial x_l$ in (12.6) may be regarded as an eigenvalue. We say (13.1) is hyperbolic if $\mathbf{A} = \mathbf{A}_\mu$ is nonsingular and, for any choice of the real parameters (γ_l, $l = 0, 1, \ldots, n$; $l \neq \mu$), the roots α of

$$\det\left(\alpha\mathbf{A} - \sum_{\substack{l=0 \\ l\neq\mu}}^{n} \gamma_l \mathbf{A}_l\right) = 0 \tag{13.4}$$

are real and are associated with k linearly independent characteristic vectors $\mathbf{v}$:

$$\alpha\mathbf{A}\mathbf{v} = \sum_{\substack{l=0 \\ l\neq\mu}}^{n} \gamma_l \mathbf{A}_l \mathbf{v}. \tag{13.5}$$

If all the roots are complex, the system (13.1) is said to be elliptic. Equations (13.1) may also be of parabolic type, but we do not state the conditions for this. In general, the roots of the polynomial (13.4) are neither all real (hyperbolic) nor all complex (elliptic). In this case, the quasilinear system (13.1) is neither totally hyperbolic nor entirely elliptic and it is said to be a "quasilinear system of composite type."

Most of our applications are framed in one or two space dimensions. For these, it is enough to consider quasilinear systems of the form

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{C}\frac{\partial \mathbf{u}}{\partial y} = \mathbf{f}. \tag{13.6}$$

At this point the identification of t as a time variable and (x, y) as space variables is entirely arbitrary. The identification is in our mind when we think of an initial-value problem for which the initial data is some prescribed function of (x, y). It is useful at first to think of (13.6) in $\mathbb{R}^2(x, y)$ with initial data in a Fourier transform class and with $t \geq 0$ and to modify this thought when the occasion arises. Now we identify characteristic surface $\phi(x, y, t) = 0$ as the locus of S of points across which derivatives of $\mathbf{u}$ may be discontinuous, even though $\mathbf{u}$, hence $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{f}$, is continuous across S. In this case the jump $[\![\mathbf{u}]\!]$ of $\mathbf{u}$ across depends on only the coordinate normal to $\phi = 0$. The equations satisfied by $[\![\mathbf{u}]\!](\phi)$ are

$$\left[\mathbf{A}\frac{\partial\phi}{\partial\tau} + \mathbf{B}\frac{\partial\phi}{\partial x} + \mathbf{C}\frac{\partial\phi}{\partial y}\right]\cdot\frac{\partial[\![\mathbf{u}]\!]}{\partial\phi} = 0, \tag{13.7}$$

where $d\phi = (\partial\phi/\partial t)\,dt + (\partial\phi/\partial x)\,dx + (\partial\phi/\partial y)\,dy = 0$.

The eigenvalues for (13.7) are the roots of

$$\det\left[\mathbf{A}\frac{\partial\phi}{\partial\tau} + \mathbf{B}\frac{\partial\phi}{\partial x} + \mathbf{C}\frac{\partial\phi}{\partial y}\right] = 0 \tag{13.8}$$

and $\upsilon = \partial[\![\mathbf{u}]\!]/\partial\phi$ are the eigenvectors.

Characteristic surfaces can be generated from defining statements which do introduce the notion of discontinuities. These defining statements all lead to the same characteristic surfaces S and eigenvectors, up to normalization. This part of the theory is classical and well developed in standard books on PDEs. A defect in standard books in that they confine attention to purely hyperbolic, parabolic, or elliptic problems for which the mathematical theory is well developed and do not treat problems of composite type.

14. Systems of Composite Type, and Mixed Type

In general, matrices have real and complex eigenvalues which lead to systems of composite type. Many systems arising in applications are of composite type. In such problems it is necessary to identify which variables are hyperbolic, etc. In the case of irrotational water waves the velocity potential is elliptic but the height function is governed by a hyperbolic equation giving rise to water waves.

For plane steady flow the time derivative vanishes and (13.8) may be written as

$$\det\left[\mathbf{B}\frac{dy}{dx} - \mathbf{C}\right] = 0, \tag{14.1}$$

where real-valued roots $dy/dx = -(\partial\phi/\partial x)/(\partial\phi/\partial y)$ $(d\phi = 0)$ give the projection of a tangent vector on the characteristic surface S. Complex roots indicate that there are elliptic variables in the quasilinear system. Plane steady flow of inviscid incompressible fluids are governed by a quasilinear system of mixed type (see Joseph, Renardy, and Saut (1985), hereafter called JRS, for a discussion). Plane steady flow of viscoelastic fluids like Maxwell's (Section 17) form a system of quasilinear equations (18.4) of composite type in which the streamlines are doubly characteristic, the stream function is elliptic, and the vorticity can be either hyperbolic or elliptic, depending on the flow. In quasilinear problems the matrices $\mathbf{B}$ and $\mathbf{C}$ depend on $\mathbf{u}$ and $\mathbf{x}$ and the values of their components change from point to point, solution to solution. So we may have a **change of type** with ellipticity in one region of the flow and hyperbolicity in another. This type of behavior is characteristic of the velocity potential in gas dynamics and leads to transonic flow, supersonic in some regions, and subsonic in others. The same behavior is characteristic of the vorticity in plane steady flows of viscoelastic fluids like Maxwell's (see JRS and Joseph and Saut (1986), hereafter called JS).

Sometimes fields with mixed hyperbolic and elliptic regions are said to be of mixed type. This can be confused with quasilinear problems of composite type in which reference is made to real and complex eigenvalues. We have agreed that "mixed type" refers to mixed fields, as in transonic flow.

The concept of **change of type** is tied strongly to nonlinearity. In transonic flow a change of type occurs when the local speed exceeds the sound speed, and a similar criterion (23.2) applies to "transonic vorticity" of a viscoelastic fluid.

At this point in our analysis there is no reason to think that a change of type could not occur in the evolution part of the problem. In fact the eigenvalue problem

$$\det\left[\mathbf{A}\frac{dx}{dt} - \mathbf{B}\right] = 0 \tag{14.2}$$

is not formally different from (14.1). The difference lies in our understanding the frame in which these problems are set. Evolution problems also can and do change type so that *there are two types of **change of type***, one for steady problems leading to transonic fields and the other to a change of type in the time-dependent problem, leading, as we shall show, to Hadamard instability and ill-posed problems. Ill-posed problems need regularization, at least, no matter what the truth of the underlying physics they expose. Transonic problems can be perfectly normal and correct statements of the underlying physics. It is necessary to understand the difference between these two types of change of type.

15. Higher-Order Problems, Symbols

Though it is always possible to reduce a quasilinear system of PDEs of order higher than one to a first-order system, it is not always convenient or necessary to do so. The theory of classification of PDEs of higher order can be formulated in terms of the symbol. Actually the symbols is the matrix of wave numbers and coefficients after analyzing systems with constant coefficients (or frozen coefficients) using normal modes when lower-order terms are neglected. The symbol is a convenient way to express the result of computations we would do anyway.

Consider the linear differential operator

$$P = \sum_{|\alpha|=m} a_\alpha(\mathbf{x}, t)\partial^\alpha + \sum_{|\alpha|<m} a_\alpha(\mathbf{x}, t)\partial^\alpha, \tag{15.1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and t are space and time coordinates, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \Sigma\, \alpha_i$, m is the highest order of derivative in P, and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}. \tag{15.2}$$

The equation

$$\sum_{|\alpha|=m} a_\alpha(x,t)\sigma^\alpha = 0, \qquad \sigma = (\sigma_0, \dots, \sigma_n), \qquad \sigma^\alpha = \sigma_0^{\alpha_0} \dots \sigma_n^{\alpha_n} \tag{15.3}$$

is called the characteristic equation for P. Only the principal part of P, the terms of highest order, appears in (15.3).

A surface S in (x, t) space is characteristic for P at a point $s \in S$ if the normal vector to S at s satisfies the characteristic equation. If $\sigma = (\sigma_0, \dots, \sigma_n)$ is a unit normal vector at s, S is characteristic for P if and only if

$$\sum_0^n \sigma_k^2 = 1 \quad \text{and} \quad \sum_{|\alpha|=m} a_\alpha(x)\sigma^\alpha = 0. \tag{15.4}$$

The characteristic equation for Laplace's equation $\Delta u = \sum_{k=1}^n \partial^2 u/\partial x_k^2 = 0$ is $\sum_{k=1}^n \sigma_k^2 = 0$. There are no real characteristics because $(15.4)_1$ is not satisfied. The characteristic equation $(15.4)_2$

$$\sigma_0^2 - c^2 \sum_{l=1}^n \sigma_k^2 = 0$$

for the n-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

satisfies the characteristic equation $(15.4)_1$ when $\sigma_0 = \pm c/\sqrt{c^2+1}$. Therefore a surface is characteristic for the wave equation if and only if its normal makes an angle β, $\cos\beta = c/\sqrt{c^2+1}$, with the t axis. For the one-dimensional wave equation $\Delta = \partial^2/\partial x^2$, this implies that the family of lines $x \pm ct = \text{const}$ are characteristic. The classic example of a parabolic equation is the heat equation, $\partial u/\partial t = \Delta u$. The characteristic equation $(15.4)_2$ is

$$\sum_{l=1}^n \sigma_l^2 = 0. \tag{15.5}$$

Hence, from $(15.4)_1$, $\sigma_0^2 = 1$ and the characteristic surfaces are the hyperplanes $t = \text{const}$. Operators of the form $\partial u/\partial t + Lu$, where L, like $-\Delta$, is a positive definite elliptic operator, are parabolic. These operators are strongly dissipative and lead to diffusion rather than to propagation. Unlike hyperbolic operators, parabolic operators will smooth initially discontinuous Cauchy data.

We define the

$$\text{Symbol of } P = P(\mathbf{x}, t; i\xi_0, i\xi_1, \dots, i\xi_n), \tag{15.6}$$

where $i = \sqrt{-1}$. To form the symbol we replace the arguments $\partial/\partial t$, $\partial/\partial x_1$, …, $\partial/\partial x_n$ of P with the Fourier variables $i\xi_0$, $i\xi_1$, …, $i\xi_n$. In this way we obtain a polynomial in the real variables x. The symbol of the Laplace operator $-\Delta$ is $\sum_{i=1}^n \xi_i^2$; the symbol of the wave operator $(\partial^2/\partial t^2) - \Delta$ is $-\xi_0^2 + \sum_{i=1}^n \xi_i^2$; the symbol of the heat operator $(\partial/\partial t) - \Delta$ is $i\xi_0 + \sum_{i=1}^n \xi_i^2$. The symbol for a system of equations is defined in a similar fashion and is a matrix with polynomial entries. For example, the matrix symbol for the system of differential equations

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial x_1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial x_2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \tag{15.7}$$

is

$$A \stackrel{\text{def}}{=} \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{bmatrix}. \tag{15.8}$$

If all the roots of the determinant of the principal part of the matrix symbol are real and distinct, the system is strictly hyperbolic; it is hyperbolic if the roots are real, and complex if the roots are all complex, as in (15.8), or of composite type if some roots are real and others complex.

It can happen, especially in cases in which higher-order systems have been reduced to first-order systems, that the determinant of a matrix symbol does not give up n roots. For example, we could have just as well introduced a velocity potential ϕ, with $\mathbf{u} = \nabla\phi$ for (15.7). Then instead of (15.7) we could just as well write these equations as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x_1} \begin{bmatrix} u_1 \\ u_2 \\ \phi \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{\partial}{\partial x_2} \begin{bmatrix} u_1 \\ u_2 \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix}. \tag{15.9}$$

The determinant of the principal matrix symbol

$$\mathbf{A} = \begin{bmatrix} \xi_1 & \xi_2 & 0 \\ 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \end{bmatrix}$$

for (15.9) does not give any nonzero roots and is not equivalent to (15.8). To deal with situations like this we need to introduce the notion of weights (e.g., see Agmon *et al.*, 1964; Pitman and Schaeffer, 1988). The assignment of weights appears to be a delicate art for which we can give no prescription.

Two homogeneous scalar operators are said to be of the same type if, up to a transformation of the independent variables, their symbols have the same asymptotic behavior at infinity. If the asymptotic behavior of the symbol changes, then we say that the equation changes type. For example, the Tricomi equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic when $y < 0$ and elliptic when $y > 0$. Another example is the quasilinear system (8.1) which is hyperbolic for $p'(v) > 0$ and elliptic for $p'(v) < 0$. These problems all involve a change in the sign of the symbol and Hadamard instabilities, which occur if the solution of the Cauchy problem with initial data in the hyperbolic region enters the elliptic region.

In Figure 9.1 the solid lines, where $p'(v) > 0$, lead to a hyperbolic equation and the dashed lines lead to an elliptic equation. The elliptic portion is rejected because it will exhibit Hadamard instabilities; and actual solutions are required to operate only on the hyperbolic parts of the curve. This leads to spartially segregated solutions, separated by lines of discontinuity, each part operating on a different hyperbolic branch of the curve. There is hysteresis and abrupt transitions in the response of such models. These features are all present in the recent study of Hunter and Slemrod (1983), which attempts to explain some observations of Tordella (1969) of a type of melt fracture called ripple. This phenomenon shows hysteresis loops, double-valued shear rates at certain stresses, and spatially segregated flow regimes. Similar ideas have also been used to explain the phenomenon of necking occurring in cold drawing of polymers.

Regirer and Rutkevich (1968) have considered fluids of the Reiner–Rivlin type which exhibit change of type. Their constitutive law is

$$\mathbf{T} = -p\mathbf{l} + \eta f(\mathrm{II})\mathbf{D},$$

where $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}})$, $\mathrm{II} = \mathrm{tr}\,\mathbf{D}^2$. Written in terms of a stream function $\psi = (u, v) = (\psi_y, -\psi_x)$, the equation governing steady two-dimensional flows is as follows:

$$L\psi \stackrel{\text{def}}{=} a_1\left[\frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4}\right] + 2a_2\frac{\partial^4\psi}{\partial x^2\partial y^2} + 4a_3\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) = H(\psi), \tag{15.10}$$

where $H(\psi)$ is a nonlinear third-order operator and the coefficients a are nonlinear functions of the second derivatives of ψ. The characteristic curves $y(x)$ are solutions of

$$a_1 y_x^4 + 4a_3 y_x^3 + 2a_2 y_x^2 - 4a_3 y_x + a_1 = 0. \tag{15.11}$$

There are three cases:

(i) $f + 2\Pi f' > 0$ (no real roots, elliptic),
(ii) $f + 2\Pi f' = 0$ (parabolic),
(iii) $f + 2\Pi f' < 0$ (four real roots, hyperbolic).

The hyperbolic regions are those where the stress decreases as a function of shear rete, and the elliptic regions are those where it increases. The unsteady problem corresponding to (15.10) is

$$\rho \frac{\partial}{\partial t}(\Delta\psi) + H(\psi) = L(\psi).$$

When the right-hand side is elliptic, this problem is parabolic and well-posed. When the right-hand side changes type, the problem is neither parabolic nor well-posed and Hadamard instability occurs. Changes of type and Hadamard instabilities can occur in rheological problems which are not one-dimensional and they need not be associated with nonmonotone constitutive equations. An interesting case of this type arises in a stability analysis of plane Couette flow by Akbay *et al.* (1980). In order to obtain a manageable equation, they introduce the "short memory approximation." This means that, in the memory integrals occurring in the equation for the disturbances, only terms of first order in the relaxation time of the fluid are kept. Proceeding thus, they find (15.10) with

$$H(\psi) = \rho\kappa y \frac{\partial}{\partial x}\Delta\psi, \tag{15.12}$$

where κ is the rate of shear of the Couette flow,

$$\begin{aligned} a_1 &= \tau', \\ 2a_2 &= -2\tau' + \frac{4\tau}{\kappa}, \\ 4a_3 &= N_1' - \frac{N_1}{\kappa}, \end{aligned} \tag{15.13}$$

and $\tau(\kappa)$, $N_1(\kappa)$ are the shear stress and first normal stress function of the rate of shear. The problem is posed on the strip.

$$-\infty < x < \infty, \qquad 0 < y < h,$$

$$\psi = \frac{\partial\psi}{\partial y} = 0 \qquad \text{on} \quad y = 0, h.$$

The paper by Ahrens *et al.* (1984) reports a study of the stability of viscometric flow using the type of short memory introduced by Akbay *et al.* The instability found by Akbay *et al.* can be identified as a loss of evolution leading to the catastrophic short-wave instability of Hadamard type whenever

$$\frac{[(N_1(\kappa)/\kappa)']^2\kappa^3}{\tau(\kappa)\tau'(\kappa)} > 16. \tag{15.14}$$

If we consider the symbol of the differential operator, i.e., if we formally set $\partial/\partial t = \sigma$, $\partial/\partial x_1 = i\alpha$, $\partial/\partial x_2 = i\beta$, then the left-hand side of (15.10) becomes

$$\rho(\sigma + \kappa x_2 i\alpha)(-\alpha^2 - \beta^2),$$

and the right-hand side becomes

$$-\left(N_1' - \frac{N_1}{\kappa}\right)(\alpha^2 - \beta^2)\alpha\beta + \tau'(\alpha^2 - \beta^2)^2 + \frac{4\tau}{\kappa}\alpha^2\beta^2.$$

When (15.14) holds, Re σ becomes arbitrarily large as the wavelength tends to zero.

Catastrophic instabilities to short waves of this type may be characteristic for some of the types of instability called "melt fracture." Ahrens *et al.* (1984) addressed the question of justification for the short memory assumption and finds that it cannot be justified for some of the more popular rheological models. The left-hand side of (15.14) reduces to the square of the recoverable shear (N_1^2/τ^2) when the variation of N_1/κ^2 and τ/κ is small. Gleissle (1982) found that flow instabilities (melt fracture) commenced in 14 very different types of polymer melts and solutions when the recoverable shear varied 4.36–5.24 with a mean 4.63. This seems to be in rather astonishing agreement with the criterion (15.14).

16. Second-Order Scalar Equations in Two Dimensions

The equation

$$
\begin{aligned}
A(p, q, \phi)\frac{\partial^2\phi}{\partial x^2} + 2B(p, q, \phi)\frac{\partial^2\phi}{\partial x\partial y} + C(p, q, \phi)\frac{\partial^2\phi}{\partial y^2} &= f(p, q, \phi),\\
p &= \frac{\partial\phi}{\partial x},\\
q &= \frac{\partial\phi}{\partial y},\\
\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} &= 0,
\end{aligned}
\tag{16.1}
$$

arises in many problems, especially in gas dynamics. We may write this as a first-order quasilinear system for p and q:

$$
\begin{bmatrix} A & 2B \\ 0 & -1 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 0 & C \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{16.2}
$$

Following now the development leading to (13.11) we find the characteristic equation

$$
\det\begin{bmatrix} A\dfrac{dy}{dx} & 2B\dfrac{dy}{dx} - C \\ 1 & \dfrac{dy}{dx} \end{bmatrix} = A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0, \tag{16.3}
$$

where dy/dx is the projection of the tangent vector of $\phi = 0$, $d\phi = (\partial\phi/\partial x)\, dx + (\partial\phi/\partial y)\, dy = 0$ onto the (x, y) plane. We find that

$$
\frac{dy}{dx} = \frac{B}{A} + \frac{\sqrt{B^2 - AC}}{A}. \tag{16.4}
$$

There are two real characteristic at all points for which the discriminant $B^2 - AC > 0$. When $B^2 - AC = 0$ there is one real characteristic and when $B^2 - AC < 0$ there are no real characteristics. Hence

$$
\begin{aligned}
B^2 - AC &> 0 \qquad \text{(hyperbolic)},\\
B^2 - AC &= 0 \qquad \text{(parabolic)}.
\end{aligned}
\tag{16.5}
$$

There is a relation between ill-posed problems and the classification of type of the equation

$$
L\phi \stackrel{\text{def}}{=} A\frac{\partial^2\phi}{\partial x^2} + 2B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2}. \tag{16.6}
$$

The problems

$$
\frac{\partial^2\phi}{\partial t^2} = L\phi,
$$

$$
\frac{\partial\phi}{\partial t} = L\phi
$$

are well-posed when L is elliptic, say $A = C = 1$, $B = 0$, and are ill-posed when L is hyperbolic, say

$$
AC = -1, \qquad B = 0 \qquad \text{as in (11.1)}.
$$

It is necessary to caution the reader at this point against overinterpreting the result just exhibited. A superficial statement of this result could be formulated as follows: any $L\phi$ which undergoes a transonic change of type will be ill-posed, because there will be some region of flow in which $L\phi$ is hyperbolic. In fact the definition of type for $L\phi$ depends on how the time derivative is defined. In

Section 1 we showed that in the theory of plane flow of fluids of Maxwell's type we may write

$$L\xi = \rho L_1 \xi + L_2 \xi, \tag{16.7}$$

where $\xi = \phi$ is the vorticity and L_2 alone is relevant for ill-posedness.

17. Viscoelastic Fluids like Maxwell's

Every important feature of the general theory arises in the study of equations governing the flow of viscoelastic fluids. We use these equations to motivate, organize, and develop further relations between Hadamard instability, ill-posed problems, problems of numerical simulation, and problems of transonic type.

We assume that the part $\boldsymbol{\tau}$ of the stress $\mathbf{T} = -p\mathbf{1} + \boldsymbol{\tau}$ in an incompressible fluid satisfies a constitutive equation like Maxwell's

$$\lambda \frac{\mathscr{D}\boldsymbol{\tau}}{\mathscr{D}t} = 2\eta \mathbf{D}[\mathbf{u}] + l[\mathbf{u}, \boldsymbol{\tau}], \tag{17.1}$$

where $\mathbf{D}[\mathbf{u}]$ is the symmetric part of $\nabla\mathbf{u}$, $\mathbf{u}$ is the velocity, λ is the relaxation time, and η is the "elastic" viscosity.

$$\frac{\mathscr{D}\boldsymbol{\tau}}{\mathscr{D}t} \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{\tau} - a(\mathbf{D}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{D}), \tag{17.2}$$

where a $(-1 \le a \le 1)$ is a real number, $\boldsymbol{\Omega}[\mathbf{u}]$ is the skew symmetric part of $\nabla\mathbf{u}$, and $l[\mathbf{u}, \boldsymbol{\tau}]$ is of lower order; it does not depend on derivatives of $\mathbf{u}$ of $\boldsymbol{\tau}$. Models like Maxwell's differ in lower-order terms but have the same principal part. The Oldroyd–Maxwell models of $l = -\boldsymbol{\tau}$; $a = 1$ is an upper-convected Maxwell model, $a = -1$ is a lower-convected Maxwell model, and $a = 0$ is a corotational model. A model of Giesekus is associated with $l = -\boldsymbol{\tau} - c_1\boldsymbol{\tau}^2$, $a = 1$, where c_1 is constant. A model of Phan Thien–Tanner is associated with $l = -\boldsymbol{\tau} - c_2\boldsymbol{\tau}\,\mathrm{tr}\,\boldsymbol{\tau}$ and $a = 1$, where c_2 is a constant. Models like Maxwell's are rendered nonlinear when λ and η are made to depend on the second (quadratic) invariant Π of the rate of strain tensor $\mathbf{D}[\mathbf{u}] = \mathrm{sym}\,\nabla\mathbf{u}$. These nonlinear problems can be made quasilinear by differentiation, as in Section 12.

The upper and lower convected Maxwell models are special in that there is a restriction on the allowed range of stress. The upper convected model ($a = 1$) can be written as an integral model

$$\boldsymbol{\tau} = \frac{\eta}{\lambda^2}\int_{-\infty}^{t} \exp\left[\frac{-(t-\tau)}{\lambda}\right][\mathbf{C}_t^{-1}(\tau) - \mathbf{1}]\,d\tau, \tag{17.3}$$

where $\mathbf{C}_t^{-1}(\tau)$ is the finger tensor. The lower convected model ($a = -1$) can be written as

$$\boldsymbol{\tau} = \frac{\eta}{\lambda^2}\int_{-\infty}^{t} \exp\left[\frac{-(t-\tau)}{\lambda}\right][\mathbf{1} - \mathbf{C}_t(\tau)]\,d\tau, \tag{17.4}$$

where $\mathbf{C}_t(\tau)$ is the right relative Cauchy–Green tensor. The finger and Cauchy–Green tensors have positive eigenvalues, implying a restriction on the range of $\boldsymbol{\tau}$. For $a = 1$, in plane flow, this condition may be expressed as

$$\det\begin{bmatrix} \sigma + \dfrac{\eta}{\lambda} & \tau \\ \tau & \gamma + \dfrac{\eta}{\lambda} \end{bmatrix} \ge 0, \tag{17.5}$$

where

$$[\boldsymbol{\tau}] = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix}. \tag{17.6}$$

These restrictions keep the upper and lower convected models from becoming ill-posed on smooth solutions. The other values of $a \neq \pm 1$ do not rule out Hadamard instability, as we see in Section 18, and even the upper and lower convected models can become ill-posed on discretized solutions.

The dynamical equations governing the motion of fluids like Maxwell's are quasilinear; they are nonlinear, but linear in derivatives. We can write the dynamical equations

$$\mathbf{G}(\mathbf{Q}_t + \mathbf{u}\cdot\nabla\mathbf{Q}) + \mathbf{H}\mathbf{Q}_x + \mathbf{J}\mathbf{Q}_y = l[\mathbf{Q}], \tag{17.7}$$

where $\mathbf{G}$ is not invertible (e.g., there is no p_t in this system), $\mathbf{Q}$ is a system vector whose components are the velocity, stresses, and pressure, and $\mathbf{G}$, $\mathbf{H}$, $\mathbf{J}$, and l depend on $\mathbf{Q}$, but not on the derivatives of $\mathbf{Q}$.[2] In two dimensions we have velocity components $u = (u, v)$ corresponding to x, y stress components (18.6) and

$$\mathbf{Q} = [u, v, \sigma, \gamma, \tau, p].$$

There are six quasilinear equations for the six scalar fields, $[u, v, \sigma, \gamma, \tau, p]$, linear in derivatives with lower-order right-hand sides, l_1, l_2, and l_3:

$$\begin{aligned}
\sigma_t + u\sigma_x + v\sigma_y + \tau(v_x - u_y) - a[2\sigma u_x + \tau(u_y + v_x)] - 2\mu u_x &= l_1,\\
\tau_t + u\tau_x + v\tau_y + \tfrac{1}{2}(\sigma - \gamma)(u_y - v_x) - \tfrac{1}{2}a(\sigma + \gamma)(u_y + v_x) - \mu(u_y + v_x) &= l_2,\\
\gamma_t + u\gamma_x + v\gamma_y + \tau(u_y - v_x) = a[2\gamma u_y + \tau(u_y + v_x)] - 2\mu v_y &= l_3,\\
\rho(u_t + uu_x + vu_y) + p_x - \sigma_x - \tau_y &= 0,\\
\rho(v_t + uv_x + vv_y) + p_y - \tau_x - \gamma_y &= 0,\\
u_x + v_y &= 0.
\end{aligned} \tag{17.8}$$

18. Hadamard Instability and Ill-Posed Problems for the Flow of Viscoelastic Fluids

For the moment it is useful to think about how we might carry out an analysis of stability. First we assume that $\hat{\mathbf{Q}}$ is a solution of (17.7). It could even be an unsteady solution. Then we write

$$\mathbf{Q} = \hat{\mathbf{Q}} + \mathbf{q}$$

and assume that $\mathbf{q}$ is small. The linearized equations are

$$\hat{\mathbf{G}}(\mathbf{q}_t + \hat{\mathbf{u}}\cdot\nabla\mathbf{q}) + \hat{\mathbf{H}}\mathbf{q}_x + \hat{\mathbf{J}}\mathbf{q}_y = \hat{l}\mathbf{q}, \tag{18.1}$$

where $\hat{\mathbf{H}}$, $\hat{\mathbf{J}}$, $\hat{l}$ depend on $\hat{\mathbf{Q}}$, derivatives of $\hat{\mathbf{Q}}$ but not on $\mathbf{q}$. We can imagine trying to solve (18.1) plus boundary conditions as an initial-value problem for the stability of $\mathbf{Q}$. If $\hat{\mathbf{Q}}$ is steady, we could write $\mathbf{q} = c^{\sigma t}\hat{\mathbf{Q}}(x)$ and determine stability from eigenvalues σ. To get the eigenvalues we would have to solve a complicated set of PDEs over the whole field of flow, satisfying boundary conditions.

Now we treat the problem of stability for a special class of disturbances which lead to simple but deep results. We consider short waves, tending to zero noting that $\hat{\mathbf{Q}}$, hence $\hat{\mathbf{G}}$, $\hat{\mathbf{H}}$, $\hat{\mathbf{J}}$, and $\hat{l}$, is nearly constant on any sufficiently small neighborhood $|x - x_0| < \varepsilon$ of any point x_0.

The coefficients of (18.1) are constant on such a small neighborhood and we may try for a solution in terms of normal modes

$$\mathbf{q} = \mathbf{a}\exp\{-i\omega t + i\alpha(x - x_0) + i\beta(y - y_0)\}, \tag{18.2}$$

where $\mathbf{a}(x_0)$ is an amplitude, ω is a frequency, and α and β are wave numbers. Of course, we cannot hope to satisfy boundary conditions with a solution of this form. It is a strange form for the solution because it applies at each and every point x_0, so we may find stability at some points and instability at others.

[2] Equation (17.7) with an invertible G is a canonical quasilinear problem in two space dimensions. The lack of invertibility for the viscoelastic system does not introduce special problems.

To set some notations, we define a wave vector:

$$\mathbf{k} \stackrel{\text{def}}{=} \mathbf{e}_x\alpha + \mathbf{e}_y\beta, \\ |\mathbf{k}| = \sqrt{\alpha^2 + \beta^2}. \tag{18.3}$$

Since $e^{-i\omega t} = e^{\omega_i}e^{-\omega_r t}$ where $\omega = \omega_r + i\omega_i$ we may define a growth rate $\sigma = \omega_i$.

Instability to Short Waves

After putting the normal modes (18.2) into (18.1) we get

$$\hat{L}(\omega, \mathbf{k})\mathbf{a} = \hat{l}\mathbf{a}, \tag{18.4}$$

where

$$\hat{\mathbf{L}} = (-\omega + \hat{\mathbf{u}}\cdot\mathbf{k})\hat{\mathbf{G}} + \alpha\hat{\mathbf{H}} + \beta\hat{\mathbf{J}}.$$

We divide (18.4) by $|\mathbf{k}|$ and let $|\mathbf{k}| \to \infty$. Since $\hat{l}$ is independent of α and β, $\hat{l}a/|k| \to 0$ and

$$\hat{\mathbf{L}}\left(\frac{\omega}{|\mathbf{k}|}, \frac{\mathbf{k}}{|\mathbf{k}|}\right)\mathbf{a} = 0. \tag{18.5}$$

Equation (18.5) represents six linear, homogeneous equations for the six unknown components of a. Hence

$$\Delta \stackrel{\text{def}}{=} \det \hat{\mathbf{L}} = -\rho c^2 + \hat{f} = 0,$$

where

$$c = \frac{(\omega - \mathbf{u}\cdot\mathbf{k})}{|\mathbf{k}|} \tag{18.6}$$

and

$$\hat{f} = \frac{\eta}{\lambda} - \frac{\hat{\gamma}}{2}(1 - a) + \frac{\hat{\sigma}}{2}(1 + a). \tag{18.7}$$

The expression (18.7) for $\hat{f}$ has been simplified by choosing x so that $\beta = 0$, $\mathbf{k} = \alpha\mathbf{e}_x$, and $\hat{\sigma} \geq \hat{\gamma}$. The growth rate ω_i is given by

$$\text{imaginary } c = \frac{\omega_i}{|\mathbf{k}|} = \pm\text{imaginary } \sqrt{\frac{\hat{f}}{\rho}}.$$

It follows that

$$\omega_i = 0 \qquad \text{if} \quad \hat{f} > 0$$

and there is a positive growth rate if $\hat{f} < 0$.

We may phrase the condition for stability to short waves in terms of the wave speed c, with stability only if c^2 is positive corresponding to real wave speeds.

The condition

$$0 > \hat{f} = \frac{\eta}{\lambda} - \frac{\hat{\gamma}}{2}(1 - a) + \frac{\hat{\sigma}}{2}(1 + a) \tag{18.8}$$

for instability to short waves is framed as a condition on the values of the normal stresses $\hat{\gamma}$ and $\hat{\sigma}$ in a coordinate system in which $\mathbf{k} = \alpha\mathbf{e}_x$, $\hat{\sigma} > \hat{\gamma}$. If the solution enters into this region of forbidden stress, a very ugly instability will ensue.

Suppose $\hat{f} < 0$, then

$$\omega_i = \pm|\mathbf{k}|\sqrt{|\hat{f}|\rho}. \tag{18.9}$$

This is a strange formula. When $\hat{f} = 0$, $\omega_i = 0$ but when $f < 0$ and small, $\omega_i \to \infty$ with $|\mathbf{k}|$. We can get stability to short waves at some points and instability at others, depending on the values of $\hat{\gamma}$ and $\hat{\sigma}$ through frozen coefficients. This kind of catastrophic short-wave number instability is the so-called Hadamard instability.

We may express the condition (18.8) in a general coordinate frame (see JS) as a condition for stability against short waves:

$$\left[\frac{\eta}{\lambda}-\frac{1}{2}\gamma(1-a)+\frac{1}{2}\sigma(1+a)\right]\left[\frac{\eta}{\lambda}-\frac{1}{2}\sigma(1-a)+\frac{1}{2}\gamma(1+a)\right]-\tau^2>0,$$
$$\frac{1}{2}\gamma(1-a)-\frac{1}{2}\sigma(1+\alpha)-\frac{\eta}{\lambda}<0. \tag{18.10}$$

When $a = 1$, this reduces to

$$\left(\frac{\eta}{\lambda}+\sigma\right)\left(\frac{\eta}{\lambda}+\gamma\right)-\tau^2>0,$$
$$\frac{\pi}{\lambda}+\sigma>0. \tag{18.11}$$

Conditions (18.11) are always satisfied as (17.3) and (17.5) show. Hence smooth solutions of an upper convected Maxwell model are always well-posed. A similar argument leads to the same result for the lower convected Maxwell models, but not for the others at $a \neq \pm 1$.

19. The Vorticity Equation, Short-Wave Instabilities, and Problems of Transonic Type

For models like Maxwell's it is possible to frame the discussion of hyperbolicity in terms of a second-order PDE for the vorticity (see equation (4.1) in JS).

In plane flow there is one nonzero component of vorticity satisfying

$$\rho\frac{\partial^2\zeta}{\partial t^2}+2\rho(\mathbf{u}\cdot\nabla)\frac{\partial\zeta}{\partial t}-A\frac{\partial^2\zeta}{\partial x^2}-2B\frac{\partial^2\zeta}{\partial x\partial y}-C\frac{\partial^2\zeta}{\partial y^2}+l=0, \tag{19.1}$$

where l is of lower order, and A, B, C are defined by

$$A=-\rho u^2+\mu+\tfrac{1}{2}\sigma(1+a)-\tfrac{1}{2}\gamma(1-a),$$
$$B=\tau-\rho uv,$$
$$C=-\rho v^2+\mu-\tfrac{1}{2}\sigma(1-a)+\tfrac{1}{2}\gamma(a+1),$$
$$\mu=\frac{\eta}{\lambda}.$$

Analysis of (19.1) using the method of short waves leads directly to the criterion of (18.8). The same criterion, positive wave speed $c^2 > 0$, is sufficient to guarantee that (19.1) is hyperbolic (see JS).

Two conclusions follow from the foregoing comparison:

1. The quasilinear system (17.8) is well-posed or ill-posed if the vorticity equation (19.1) is well-posed or ill-posed, respectively.
2. The quasilinear system (17.8) is well-posed if and only if the vorticity equation (19.1) is hyperbolic. It is useful here to remark that the property of well-posedness of an initial-value problem is a more general one than hyperbolicity but in the present case there is a sense in which the two concepts coincide.

Now we put the time derivatives to zero. This means that we have left behind the problem of ill-posed problems and short-wave instabilities. The problem now is to find the regions of steady flow in which the vorticity equation gives rise to real characteristic directions. In general, the analysis of the characteristics of (10.1) when the time derivatives vanish will lead to conditions for the emergence of transcritical flow, like transonic flow in aerodynamics, elliptic in some regions of flow and hyperbolic in others. Elementary analysis of the problem of characteristics in steady flow leads to the formula

$$\frac{dy}{dx}=\frac{B}{A}\pm\frac{\sqrt{B^2-AC}}{A},$$

where A, B, C are defined under (19.1). Clearly there are real characteristics whenever the discriminant

$$\begin{aligned} B^2 - AC = &-\mu^2 + \rho[\mu + \alpha\sigma + a\gamma](u^2 + v^2) + \tfrac{1}{2}\rho(\gamma - \sigma)(u^2 - v^2) \\ &\tau^2 + \tfrac{1}{4}\sigma^2(1 - a^2) + \tfrac{1}{4}\gamma^2(1 - a^2) - \mu a(\sigma + \gamma) - 2\rho\tau v > 0, \end{aligned} \tag{19.2}$$

and the vorticity equation is elliptic wherever $B^2 - AC < 0$. The criterion (19.2) depends on the inertia through the terms multiplying the density ρ but the criterion $\hat{f} < 0$ for ill-posedness is independent of ρ (see (18.8)). Usually regions of high-speed steady flow will go hyperbolic when the velocities are large enough. However, it is possible for a steady flow of an inertia-less fluid with $\rho = 0$ to change type. The following theorem proved by JS relates the criterion for ill-posed problems, basically defined for evolution, to the criterion for change of type in steady flow. *The quasilinear system* (17.8) *is well-posed if and only if the vorticity equation* (19.1) *of steady flow is hyperbolic. If the vorticity equation of steady flow is hyperbolic when* $\rho = 0$, *then the unsteady vorticity equation is elliptic and the quasilinear system ill-posed. Conversely, if the vorticity of an inertia-less steady flow is elliptic and* $A > 0$ *the system* (17.8) *is well-posed.*

It is easiest to examine the criterion just given in principal coordinates for the stress, $\tau = 0$. Then putting ρ to zero we have a well-posed problem for σ and γ such that

$$A = \mu + \tfrac{1}{2}\sigma(1 + a) - \tfrac{1}{2}\gamma(1 - a) > 0$$

and

$$B^2 - AC = -\mu^2 + \tfrac{1}{4}\sigma^2(1 - a^2) + \tfrac{1}{4}\gamma^2(1 - a^2) - \mu a(\sigma + \gamma) < 0.$$

For upper convected models, $a = 1$, this criterion reduces to

$$A = \mu + \sigma > 0, \qquad B^2 - AC = -\mu(\mu + \sigma + \gamma) < 0.$$

For lower convected models, $a = -1$, this criterion reduces to

$$A = \mu - \gamma > 0, \qquad B^2 - AC = -\mu(\mu - \sigma - \gamma) < 0.$$

These inequalities are always satisfied because of restrictions on the range of τ implied by the constitutive equations when expressed in integral form (see JRS, equations (5.8) and (5.9)). Dupret and Marchal (1986) used the differential form of the upper and lower convected Maxwell models to show that if the criterion for well-posed problems is satisfied initially, it will not fail subsequently.

The loss of well-posedness and transonic change of type are indirectly related by the theorem of comparison just proved. Transonic change of type involves inertia, ill-posedness involves only a condition on stresses, not inertia. A more direct comparison could have been made in terms of a transonic type of change of type of the operator

$$L_2\zeta = A_2\frac{\partial^2\zeta}{\partial x^2} + 2B_2\frac{\partial^2\zeta}{\partial x\partial y} + C_2\frac{\partial^2\zeta}{\partial y^2}$$

to which we alluded in (16.7). The decomposition of $L = \rho L_1 + L_2$ into a part L_1 depending on inertia and a part L_2 depending on stress acknowledges the fact that the part L_1 arises in one way or another on acceleration which of course does not vanish in steady flow. Tracing back, we can verify that all terms with ρ as a coefficient arise from the substantial time derivative containing a steady part $\rho\mathbf{u}\cdot\nabla$.

A large number of examples of problems which change type, using different constitutive models and different flows, were considered by JRS and JS. Some models are always evolutionary (well-posed) and do not change type in unsteady flow. The vorticity equation for steady flow of such models can and does change type. Other models can become ill-posed and undergo instability to short waves. Some flows of all these models, like simple shear or Poiseuille flow, are always well-posed while other flows, like plane extension or sink flow, can become ill-posed. Sink flows of upper convected and lower convected Maxwell models change type in steady flow, but cannot be ill-posed. On the other hand, sink flows of corotational Maxwell models change type in steady flow and are also ill-posed. Nearly every possibility is realized for some flow of some model.

20. Hadamard Instability and the Failure of Numerical Simulation

Hadamard instabilities are a disaster for numerical simulations. If a flow is Hadamard unstable, the finer you make the mesh the worse is the result. Shorter waves grow uncontrollably. These instabilities arise in many fields and they are a serious problem since they frustrate the computation of results. The case of viscoelastic fluids with instantaneous elasticity is a case in point. It has not been possible until recently to compute flows in complex geometries at high Weissenberg numbers. For the models exhibited in Section 17 we can define a Weissenberg number as a ratio of times, $W = \lambda/t_0$ where t_0 is an approximation of the process time for externally given data and λ is a relaxation time, a material parameter. Large numbers W means that the material retains its elasticity for a long relative time. The numerical simulations would break down for large values, and even fairly small values of W. The failure was only weakly dependent on the choice of the model and the choice of numerical method. This suggested that the root cause was associated with mathematical problems that do not vary from model to model and method to method.

There are various reasons to believe that this problem of failure of simulations at high W is associated with Hadamard instability. First, the results get worse and worse with increasing mesh refinement. The simulations readily break down in corners where the stress levels get high, evidently entering into the region of forbidden stresses. This type of failure of numerical simulation can occur in nonsteady and steady flow; the growth rate tells us how numerical errors are amplified, and the amplification is uncontrollable in the case of ill-posed problems.

We have already noted in Section 19 that smooth solutions of flows of an upper or lower convected Maxwell model ($a = \pm 1$) never become ill-posed. Dupret *et al.* (1985) showed that discretization errors could introduce a Hadamard instability for these well-posed models. Discretization for such problems allows us to step into forbidden and otherwise inaccessible regions of the addded stress. A recent work by Marchal and Crochet (1987) seems to have partially solved the problem of "false" discretization-induced instability by upwinding on streamlines. They introduced technical improvements and the addition of an artificial diffusivity which goes to zero with mesh refinement. These methods, of course, would be unavailing in problems exhibiting true Hadamard instability. For such problems, it is necessary to regularize the equations. The addition of Newtonian viscosity to the constitutive model for fluids with instantaneous elasticity is the natural way to regularize ill-posed problems. Many models, like Oldroyd B, already have a Newtonian contribution, expressed by a retardation time. Physically we expect a Newtonian velocity to arise from the decay of rapidly decaying modes associated with small molecules. All this puts forward the rheometrical problem of measurements of an "effective" Newtonian viscosity. This new problem needs a solution.

21. Ill-Posed Initial-Value Problems Cannot Be Solved unless the Initial Data is Analytic

We now establish the connection between Hadamard instability and ill-posed problems in the class of functions having Fourier transforms. The connection is particularly suited to problems which arise from freezing coefficients. Problems which are Hadamard unstable have no solutions in any class of initial data more general than analytic. For example, the Cauchy problem of $\mathbb{R}^2$ which follows from freezing the coefficients has no solution with initial data in the C^m class. This result is well known for the Laplace equation treated in the celebrated example by Hadamard. The solution of Laplace's equation is analytic, say in the half (x, y) plane with $x > 0$, and it can be extended to $x < 0$ by reflections. Hence the initial data on $y = 0$ must also be analytic. This "nonexistence" result is valid generally.

We prove the result just mentioned in two ways. The first one uses the Paley–Wiener theorem; the second one is elementary and shows that no solution can exist in a C^k class. Suppose, for example, that the coefficients of (18.1) are constant (as will, in fact, be implied by frozen coefficients) and that $\mathbf{q}$ lies in a Fourier transform class;

$$\mathbf{p}(\alpha, \beta, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\alpha x + \beta y)} \mathbf{q}(x, y, t)\, dx\, dy \tag{21.1}$$

is the transform of $\mathbf{q}(x, y, t)$ and $\mathbf{p}(\alpha, \beta, 0)$ is the transform of the Cauchy initial data $\mathbf{q}(x, y, 0)$ for

$\mathbf{q}(x, y, t)$ and

$$\mathbf{q}(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\alpha x + \beta y)} \mathbf{p}(\alpha, \beta, t)\, d\alpha\, d\beta. \tag{21.2}$$

The transform $\mathbf{p}(\alpha, \beta, t)$ satisfies the following ordinary differential system:

$$\hat{\mathbf{G}}\left(\frac{dp}{dt} + i\hat{\mathbf{u}} \cdot k\mathbf{p}\right) + i(\alpha\hat{\mathbf{H}} + \beta\hat{\mathbf{J}})p = 0 \tag{21.3}$$

with a prescribed $\mathbf{p} = \mathbf{p}(\alpha, \beta, 0)$ at $t = 0$. Assuming semi-simple eigenvalues (the argument is not essentially changed in the general case) we can reduce the system to a diagonal one. Then (keeping the same notation as for the transformed variables)

$$\mathbf{p}(\alpha, \beta, t) = \mathbf{\Omega}(\alpha, \beta, t)\mathbf{p}(\alpha, \beta, 0), \tag{21.4}$$

where $\mathbf{\Omega}$ is the diagonal matrix with diagonal entries $-i\omega_l t$ (the ω_l's are the eigenvalues of (18.2)).

Let $\omega_j = \xi_j + i\sigma_j$ be an irregular eigenvalue, that is to say the imaginary part $\sigma_j = \sigma(\alpha, \beta)$ is unbounded as $\alpha^2 + \beta^2 \to \infty$. Since (18.1) is time reversible we can assume $\sigma(\alpha, \beta) > 0$.

Then

$$q_j(\alpha, \beta, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\alpha x + \beta y - \omega_j t)} p_j(\alpha, \beta, 0)\, d\alpha\, d\beta. \tag{21.5}$$

The usual factor under the integral is

$$e^{\sigma(\alpha,\beta)t} p_j(\alpha, \beta, 0),$$

since

$$\lim_{k=\sqrt{\alpha^2+\beta^2}\to\infty} \sigma(\alpha, \beta) = +\infty.$$

Therefore the integral defining $q_j(x, y, t)$ can exist only if $p_j(\alpha, \beta, 0)$ decays exponentially as $k \to \infty$ with an exponent dominating $\sigma(\alpha, \beta, t)$. By the Paley–Wiener theorem, this amounts to saying that the initial data $q_j(x, y, 0)$ can be extended as an analytic function in a (complex) strip containing the plane (x, y). We give below a direct proof that the Cauchy problem cannot be solved for C^k initial data.

Suppose the second x derivative of $q_l(x, y, 0)$ is discontinuous at $x = x^*$, that $q_l(x, y, 0)$ tends to zero at large x, and y is as required for functions in the Fourier transform class. Then, after integrating by parts, we find

$$2\pi p_j(\alpha, \beta, 0) = -\int_{-\infty}^{\infty} dy\, e^{-i\beta y} \int_{-\infty}^{\infty} \frac{1}{i\alpha^3} e^{-i\alpha x} \frac{\partial^3 q_j(x, y, 0)}{\partial x^3} + \frac{1}{i\alpha^3} \int_{-\infty}^{\infty} dy\, e^{-i\beta y} e^{-i\alpha x^*} \left[\frac{\partial^2 q_j}{\partial x^2}\right](y), \tag{21.6}$$

where

$$\left[\frac{\partial^2 q_j(x^*, y)}{\partial x^2}\right](y) \tag{21.7}$$

is the jump in $q_j(x, y)$ at $x = x^*$. It follows that, in general, $p_j(\alpha, \beta, 0)$ decays like $1/\alpha^3$ for large α when (20.7) holds. In this case (20.5) is unbounded as $\alpha \to \infty$ whenever $\sigma(\alpha, \beta) > 0$ is irregular. It is obvious how we could proceed with this proof when derivatives higher than the second are discontinuous.

The results just given might be taken to mean that there is no difficulty in solving ill-posed initial-value problems with analytic initial values by inverting the Fourier transform. Appearances are deceiving. The difference between an analytic function with a large derivative tending to a discontinuity and an actual discontinuity could not be important. There is surely a sense in which ill-posed problems are "overly sensitive" to changes in analytic initial data. In fact, it is just this fact which is implied by saying that ill-posed Cauchy problems do not depend continuously on the data.

22. Some Further Comments About Frozen Coefficients

Analysis of short waves on frozen coefficients has the following useful properties:

1. It leads us to linear equations. Richtmeyer and Morton (1967) note that, "Indeed, it is in checking the 'local' stability of linearized equations obtained from truly nonlinear equations that the constant coefficient theory is mainly of use."

2. The short waves allow as to freeze the coefficients; the coefficients do not vary on a sufficiently small region, so the linearized problem has constant coefficients.
3. Since derivatives of the quasi-linear system on short waves become unboundedly large, the lower-order terms are increasingly unimportant and they may be neglected. This leads to a homogeneous system, to a set of linear homogeneous equations which can be solved only if a determinant of constants vanish. What could be easier?
4. The reduced homogeneous system is now in $\mathbb{R}^n$ and boundary conditions can be neglected. The short-wave instabilities start as a local phenomenon.

The foregoing analysis of short waves on frozen coefficients was formal and not rigorous. The connection to ill-posedness can be framed as in Section 20, but more needs to be done. Kreiss (1978) gives an example of an equation where the problem with variable coefficients is properly posed, yet all corresponding constant coefficient problems are improperly posed; on the other hand, he gives an example in which the constant coefficient problems are all properly posed, yet the variable coefficient problem is not. So Kreiss' examples show that, in general, local stability is neither necessary nor sufficient for the overall stability of the variable coefficient problem. However, as Strang (1966) points out, if a quasilinear system of first-order equations is properly posed, then all frozen coefficient problems are properly posed. So first-order systems, like (18.8), must be properly posed locally, in the sense of frozen coefficients.

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