Processing STIM-LAB's experimental data on slot flow of proppants

Part I

D.D. Joseph, and Jimmy Wang
University of Minnesota

Part II

Neelesh Patankar
Northwestern University

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Part I

STIM-LAB carried out two types of experiments looking at the transport of proppants in thin fluids. The first experiment can be described as a lift-off experiment. A somewhat simplified description of the experiment is that a bed of proppant is eroded by the flow of water. Proppant is not injected. The faster the flow of water the deeper is the channel above. We are seeking to predict the height above the channel. The experiment is described in Appendix 1. Of particular relevance is their data Table 4.9.4.9-1. This data table described the condition called "channelized flow" in section 4.9.2.2. The channel width is 0.8cm in all experiments. The density and size of proppants and the water temperature is given in Table 4.9.4.1-1. The water temperature controls the viscosity.

The second experiment (Appendix 2) is different than the first because proppant plus fluid is continuously injected. If the inflow rates are fixed the system reaches a steady state as is described in "Modeling 4-6" in which proppant is transported as a moving, mildly inflated porous bed, called a "mobile bed," below a fully fluidized suspension called a traction carpet. If we look away from the ends of the eight-foot long by one-foot tall by 3/16" wide channel, the flow of proppants and liquid is fully developed as in the diagram shown in our Figure 1 below.
Traction carpet moves forward
Fluidized bed: the height of 1-2 grains; rolls and bounces
Mobile bed, moving at a very low velocity

Figure 1. Proppant flowing in a channel.

**Nomenclature**
- $\rho_p$: density of the particles, gm/cm³
- $d$: mean diameter of the particles, cm
- $\rho_f$: density of the fluid, gm/cm³
- $\eta$: dynamic viscosity of the fluid, gm/cm-s
- $\bar{p}$: average pressure gradient applied in the flow direction (if available), gm/(cm-s)²
- $Q_f$: volumetric flow rate of the fluid, cm³/s
- $M_p$: mass flow rate of the proppants, gm/s
- $Q_p$: volumetric flow rate of the proppants = $M_p/\rho_p$, cm³/s
- $Q_T$: total volumetric flow rate (fluid + proppant) = $Q_f + Q_p$, cm³/s
- $h$: open channel height cm, ($H_2$)
- $H_1$: see Figure 1, cm
- $H_2$: see Figure 1, cm
- $W$: channel width 0.79375 cm
- $A$: area cm², ($A = W \times H_2$)
- $\nu$: kinematic viscosity cm²/s, $\nu = \frac{\eta}{\rho_f}$
- $V$: fluid velocity cm/s, $V = \frac{Q}{A} = \frac{Q}{WH_2}$
- $\bar{V}$: fluid velocity cm/s, $\bar{V} = \frac{Q}{W^2}$
- $R$: Reynolds number (based on $V$); $R = \frac{Vd}{\nu}$
- $\bar{R}$: Reynolds number (based on $\bar{V}$); $\bar{R} = \frac{\bar{V}d}{\nu} = \frac{RH_2}{W}$
- $G$: gravity parameter; $G = \frac{(\rho_p - \rho_f)gd^2}{\eta V}$
- $R_G$: gravity Reynolds number; $R_G = \frac{\rho_f(\rho_p - \rho_f)gd^3}{\eta^3}$
- $S$: Shields parameter is defined as: $S = \frac{\tau}{(\rho_p - \rho_f)gd}$
- If we take $\tau = \eta \dot{\gamma}_w$ and $V = \dot{\gamma}_w d$, then $S = \frac{\eta V}{(\rho_p - \rho_f)gd^2} = \frac{1}{G}$. 

\[2\]
**Scaling parameters**

In the paper *Lift Correlations from Direct Numerical Simulation of Solid-Liquid Flows* (Joseph 2001) we found scaling parameters for the flow of particles in fluids for simulations in two dimensions. The two most important were a Reynolds number for the forward flow

\[ R = \frac{Vd}{\eta} \]  

(1)

and a sedimentation “Reynolds number” based on the settling velocity \( \frac{\rho_f (\rho_p - \rho_f)gd^2}{\eta} \) in Stokes flow.

\[ R_G = \frac{\rho_f (\rho_p - \rho_f)gd^3}{\eta^2} \]  

(2)

The fluid velocity \( V \) is related to the pressure drop across the flow cell.

It is important to do correlations in terms of dimensionless parameters; this leads to maximum generality. To see this consider power laws, which we found, that are of the form

\[ R_G = aR^n \]  

(3)

for lift-off. For \( R \) larger than \( (R_G)_{1/a} \), the particle of radius \( d \) and density \( \rho_p \) will levitate. This equation (3) may be written as

\[ \rho_f \frac{(\rho_p - \rho_f)gd^3}{\eta^2} = d\left( \frac{Vd}{\eta} \right)^n \]  

(4)

Suppose we did experiments for lift-off in a certain fluid with a given proppant of different size. We would find a correlation of the form

\[ V = bd^m \]  

(5)

We wouldn't know that \( m = \frac{3}{n} - 1 \) or how all the other parameters enter into the correlation.

**Erosion experiments; for these experiments \( H_1 = H_2 \) and only water is moving.**

STIM-LAB did experiments on bed erosion. These are essentially lift-off experiments since the flow rate is dropped to a critical lift-off value below which particles are not eroded from the bed.

In Table 1 we reformulate the data from tables 4.9.4.9-1 and 4.9.4.1-1 in Appendix 1 for processing in terms of \( R_G \) and

\[ \bar{R} = \frac{\bar{V}d}{V} \]  

(6)

where

\[ \bar{V} = \frac{Q}{W^2} \]  

(7)

and \( Q \) is the volume flow rate. We use \( \bar{V} \) because \( Q \) and \( W \) are prescribed data.

There are seven groups in the data shown in Table 1; each one corresponds to a value of \( R_G \). The data corresponding to a given \( R_G \) is ordered by increasing \( H_2 \); the larger \( R \) corresponds to a larger \( H_2 \) more or less, but there are exceptions.
Table 1: Data set for lift-off experiments from Table 4.9.4.9-1 in Appendix 2

<table>
<thead>
<tr>
<th>Proppants</th>
<th>d (cm)</th>
<th>( H_2 ) (cm)</th>
<th>( \eta ) (g/cm s)</th>
<th>( \rho_f ) (g/cm cc)</th>
<th>( U ) (cm/s)</th>
<th>( \rho_p ) (g/cm cc)</th>
<th>( R_G )</th>
<th>( \bar{V} ) (cm/s)</th>
<th>( \bar{R} )</th>
<th>( H_2/W )</th>
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<tbody>
<tr>
<td>60/40 Brady</td>
<td>0.034212</td>
<td>0.01115</td>
<td>0.999</td>
<td>0.011161</td>
<td>36.778</td>
<td>2.65</td>
<td>521.1645</td>
<td>50.37416</td>
<td>178.9337</td>
<td>2.141732</td>
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<td>648.512</td>
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<td>369.1554</td>
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<td>9.826772</td>
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</tr>
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<td>0.011161</td>
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<td>73.89383</td>
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<td>2.897638</td>
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<td>10.33071</td>
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<td>0.011161</td>
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<td>521.1645</td>
<td>369.1644</td>
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<td>2.65</td>
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<td>2.65</td>
<td>521.1645</td>
<td>369.1644</td>
<td>1131.596</td>
<td>9.826772</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Data set for lift-off experiments from Table 4.9.4.9-1 in Appendix 2

Power fit: \( H_2/W \) vs. \( \bar{R} \) in a log-log plot

It is rather obvious that the height \( H_2 \) will increase with \( Q \) at a fixed \( R_G \) (for fixed proppant and fluid). With \( R_G \) fixed we can hope for a two-parameter correlation. In Figure 2 we show seven more or less straight lines for the seven values of \( R_G \). The power law for these is given by

\[
\frac{H_2^2}{W} = \alpha(R_G) R_G^{m(R_G)}
\]

where \( \alpha(R_G) \) and \( m(R_G) \) are listed in Table 2. The value of the exponent \( m(R_G) \approx 0.87 \) for all cases except \( R_G = 86.84 \) corresponding to nearly neutrally buoyant particles (\( \rho_p = 1.05 \) gm/cc).
\[ H/W = a(RG) * R^m(RG) \]

Table 2. The coefficients (as functions of \( RG \)) used in power fit for \( H/W \).

<table>
<thead>
<tr>
<th>( RG )</th>
<th>( a(RG) )</th>
<th>( m(RG) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>86.84</td>
<td>1.479E-1</td>
<td>0.6140</td>
</tr>
<tr>
<td>521.16</td>
<td>2.326E-2</td>
<td>0.8672</td>
</tr>
<tr>
<td>2290.82</td>
<td>1.700E-2</td>
<td>0.8600</td>
</tr>
<tr>
<td>13363.76</td>
<td>6.393E-3</td>
<td>0.9174</td>
</tr>
<tr>
<td>14513.72</td>
<td>6.40E-3</td>
<td>0.9170</td>
</tr>
<tr>
<td>20342.90</td>
<td>8.476E-3</td>
<td>0.8508</td>
</tr>
<tr>
<td>100415.72</td>
<td>3.847E-3</td>
<td>0.8672</td>
</tr>
</tbody>
</table>

Inspection of Table 2 shows that when \( RG \geq 521 \)
\[ m(RG) \approx 0.87 \quad (9) \]

whereas when \( RG = 86.84 \) (\( \rho_p = 1.05 \)) we get
\[ m(RG) = 0.61 \quad (10) \]

This shows that the exponent \( m \) does depend on and we may hope to describe this dependence in intervals (as is true for the Richardson-Zaki \( n(R) \), which depends in intervals on the Reynolds number \( R \)). For example, we could suppose that there is a certain value of \( RG = RG_C \) for which the \( m(RG) \) take on the two values, (9) if \( RG > RG_C \) and (10) is \( RG < RG_C \) where 86.84 \( \leq RG_C \leq 521.16 \).
Summarizing we may propose

\[ \frac{H_2}{W} = a(R_G) R_m^{m(R_G)} \]  (11)

where \( m(R_G) \) are given approximately by (9) and (10).

**The final correlation \( a(R_G) \) as \( R_G \) in a log-log plot**

We plotted \( a(R_G) \) as given in Table 2 against \( R_G \) in a log-log plot. This plot is shown in Figure 3 and gives rise to the formula

\[ a = 0.8007 \ R_G^{-0.4854} \]  (12)

For the coefficient \( a(R_G) \)

\[ a(R_G) = b \times R_G^k \]

\[ a = 0.8007 \ R_G^{-0.4854} \]

**Figure 3. Power fit for \( a(R_G) \) vs. \( R_G \).**

Combining now (11) and (12)

\[ \frac{H_2}{W} = 0.8007 \times R_G^{-0.4854} \times R_m^{m(R_G)} \]  (13)

where \( m(R_G) = 0.87 \) for \( R_G > R_{GC} \), where \( 86 < R_{GC} < 500 \) and \( m = 0.614 \) for \( R_G < R_{GC} \).

We are proposing (13) as a widely applicable correlation for lift-off valid beyond where data has already been taken. More experiments, testing (13) validity for different values of \( W \) under more extreme conditions ought to be undertaken; of particular interest are light particles for which \( R_G < R_{GC} \).
When $R_G > R_G^{oc}$ we can write (13) as

$$ \frac{H_2}{W} = 0.8007 \left[ \frac{\rho_f (\rho_f - \rho_r)}{\eta^3} \right]^{0.4854} \left( \frac{\rho_f Q d}{W^2 \eta} \right)^{0.07} $$

(14)

Equation (14) gives the fracture height in terms of given quantities. Formula (13) can be expressed in terms of the Shield's parameter $1/G$ by writing $\tilde{R} = \frac{R_G H_2}{GW}$.

**Correlations for “channelized flow” (see Figure 1) with continuous slurry injection**

Tables 3 and 4 prepare the data taken from Table 4.9.4.1-1 for processing in log-log plots. In Figure 4 we plotted $H_2$ (the height of the open channel) against the proppant rate $Q_p$ in a log-log plot for Ottawa sand and in Figure 5 we do the same for Carbolite. These plots give rise to straight lines corresponding to power laws of the form

$$ H_2 = a Q_p^n $$

(15)

where $Q_f/Q$ is nearly the same on each straight line but differs from line to line.

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$Q_p$</th>
<th>$Q_r$</th>
<th>$Q_f$</th>
<th>$Q_p/Q$</th>
<th>$Q_f/Q$</th>
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<td>284.1</td>
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<td>314.8</td>
<td>326.2</td>
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Table 3. Parameters associated with the channeling experiment for Ottawa in Figure 1.
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Table 4. Parameters associated with the channeling experiment for Carbolite in Figure 1.
Figure 4. $H_2$ vs. $Q_p$ for Ottawa (see table 3). This correlation is not framed between dimensionless variables, but the lines are pretty straight.

Figure 5. $H_2$ vs. $Q_p$ for Carbolite (see table 4). Straight lines describe the data well.
Correlations for 

\[ a \left( \frac{Q_f}{Q} \right) \quad \text{and} \quad n \left( \frac{Q_f}{Q} \right) \]

are obtained from the 6 mean values of \( \frac{Q_f}{Q} \) listed in Table 5. These correlations are shown in Figures 6 and 7.

<table>
<thead>
<tr>
<th>( H_1 )</th>
<th>( H_2 )</th>
<th>( Q_p )</th>
<th>( Q_r )</th>
<th>( Q )</th>
<th>( Q_r/Q )</th>
<th>( Q_f/Q )</th>
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Table 5. Mean values of \( H_1, H_2, Q_p, Q_r, Q, Q_r/Q, Q_f/Q \). These mean values are used for the "a" and "n" correlations in Figures 6 and 7.

Power fit for the coefficient \( a \) were tried by using:

\[ a = b \left( \frac{Q_f}{Q} \right)^m \]  

(16)

![Figure 6. Fit for coefficient a with Q_f/Q. The “a” correlation for Carbolite is not perfect.](image-url)
Power fit for the coefficient $n$ were tried by using:

$$-n = c(Q_f/Q)^k$$

(17)

Combining now (15), (16) and (17) we get the final dimensional correlation:

$$H_2 = b(Q_f/Q)^m(Q_p)^{-c(Q_f/Q)^k}$$

(18)

where

For Ottawa, $b = 4.315$, $m = -2.0914$, $c = 0.2105$, $k = -7.6864$;

For Carbolite, $b = 2.1$, $m = -19.251$, $c = 0.2113$, $k = -12.62$

**Dimensionless correlation**

A dimensionless correlation

$$\frac{H_2}{W} = a\left(\frac{Q_p}{Q}\right)^n$$

(19)

is presented in Figures 8 through 11. The correlations for Ottawa and Carbolite are satisfactory.
Figure 8. $H_2/W$ vs. $Q_p/Q$ for Ottawa (see table 3).

Figure 9. $H_2$ vs. $Q_p/Q$ for Carbolite (see table 4).
Logically \( a \) and \( n \) in the correlation should depend on the fluid rate fraction \( Q_f/Q \). A power fit

\[
a = b(Q_f/Q)^m
\]  

(20)
is given in Figure 10.

\[a=b(Q_f/Q)^m\]

**Figure 10. Fit for coefficient \( a \) with \( Q_f/Q \).**

A power fit for the exponent \( n \)

\[-n = c(Q_f/Q)^k\]

(21)
is given in Figure 11.

**Figure 11. Fit for coefficient \( n \) with \( Q_f/Q \).**
The fit for "a" and "n" in Figures 10 and 11 are not very good. Putting together (19), (20) and (21) we get

\[ H_2/W = b(Q_f/Q)^m (Q_p/Q)^{-c(Q_f/Q)^k} \]  \hspace{1cm} (22)

where

For Ottawa, \( b = 1.92, m = 16.71; c = 0.2015, k = -8.5 \).

For Carbolite, \( b = 1.303, m = 26.36; c = 0.2085, k = -13.54 \).

The bedload correlations just developed lack generality since they are not expressed in terms of the basic parameters, the Reynolds numbers \( R \) and \( R_d \). In the next section Neelesh Patankar will look at this more general problem.
Part II

Power law correlation for bed load transport

Neelesh A. Patankar
Northwestern University

Variables:

- \( \rho_p \) density of the particles, gm/cm\(^3\)
- \( d \) mean diameter of the particles, cm
- \( \rho_f \) density of the fluid, gm/cm\(^3\)
- \( \eta \) dynamic viscosity of the fluid, gm/cm-s
- \( \bar{P} \) average pressure gradient applied in the flow direction (if available), gm/(cm-s)\(^3\)
- \( Q_f \) volumetric flow rate of the fluid, cm\(^3\)/s
- \( M_p \) mass flow rate of the proppants, gm/s
- \( Q_p \) volumetric flow rate of the proppants = \( M_p/\rho_p \), cm\(^3\)/s
- \( Q_T \) total volumetric flow rate (fluid + proppant) = \( Q_f + Q_p \), cm\(^3\)/s
- \( H_1 \) see Figure 1, cm
- \( H_2 \) see Figure 1, cm
- \( W \) width of the channel, cm

**Figure 1. Proppant flowing in a channel**

**Data set 1: \( H_1 = H_2 = H \)**

Define a Reynolds number \( R_q = \frac{\rho_f Q_f}{W \eta} \). Figure 2 shows a plot of \( R_q \) vs. \( H/W \) at different values of \( R_g \)

where \( R_g = \frac{981 \rho_f (\rho_p - \rho_f) d^3}{\eta^2} \).
We get

$$R_q = a \left(\frac{H}{W}\right)^n,$$  \hspace{1cm} (1)

where $a$ and $n$ are functions of $R_g$. We see that $a$ and $n$ may be regarded to be constant for $521.37 \leq R_g \leq 20342.9$. More experimental data will be required to obtain a smooth equations for $a$ and $n$ as functions of $R_g$. The values we have are plotted in Figures 3(a) and 3(b).

Figure 2. Plot (log-log) of $R_q$ vs. $H/W$ at different values of $R_g$.

Figure 3(a). Plot (log-log) of $a$ vs. $R_g$. 
The data above represents the critical condition for the bed to start moving. We define an effective Reynolds number at the fluid bed interface as

$$R_{\text{eff}} = R_q(W/H)^n,$$

where the values of $n$ are given in Figure 2 and plotted in Figure 3.

From the above plots we can write an expression for the critical effective Reynolds number $R_{\text{eff}}^{cr}$ for bed motion as

$$R_{\text{eff}}^{cr} = a(R_G)$$

Data set 2: $H_1 \neq H_2$

**Correlation for the fluid flow rate**

We have two cases: 1. Carbolite: $R_G = 12229$ and 2. Ottawa: $R_G = 3496$. From the above correlation we see that

$$R_{\text{eff}}^{cr} = 2053.4$$

Define the effective Reynolds number at the fluid bed-load interface as $R_{\text{eff}} = R_q(W/H_2)^{1.0937}$ for the Carbolite and Ottawa data.

Figure 4 shows a plot of $R_{\text{eff}}$ vs. $\ln(H_1/H_2)$ for the combined Carbolite and Ottawa data.
Two regimes are observed for the given values of $R_G$

1. $R_{\text{eff}} - R_{\text{eff}}^c = 26838 \ln(H_1/H_2)$, $\ln(H_1/H_2) \leq 1.3$.  
2. $R_{\text{eff}} = 8177(H_1/H_2)^{1.1648}$, $\ln(H_1/H_2) > 1.3$.

Note that the y-intercept is taken to be equal to $R_{\text{eff}}^c$. Regime 1 shows logarithmic behavior whereas regime 2 shows power law behavior. When $H_1/H_2 = 1$, we recover the correlation in (3). Thus (5) represents a combined correlation from the data set 1 and 2. Lastly, regimes 1 and 2 have an overlap region. Hence $\ln(H_1/H_2) = 1.3$ is only an approximate value for transition.

The correlation can be rewritten as:

$$\left( \frac{\rho_f Q_f}{H_2 \eta} \right) (W/H_2)^{n-1} - a(R_G) = 26838 \ln(H_1/H_2); \ln(H_1/H_2) \leq 1.3$$

and

$$\left( \frac{\rho_f Q_f}{H_2 \eta} \right) (W/H_2)^{n-1} = 8177(H_1/H_2)^{1.1648}; \ln(H_1/H_2) > 1.3.$$
\[ R_{\text{eff}} = \frac{8177 \left( \frac{H_1}{H_2} \right)^{1.1648}}{1 + \left( \frac{H_1}{0.9} \right)^9} \]  \hfill (9)

**Model for bed load transport** The above correlation suggests the following model for bed-load transport.

Consider a channel with a given height \( H_2 \) and width \( W \). Begin the fluid flow in the channel. After a critical value \( Q_f \), given by (2), the fluid begins to move the proppant in the bed. The height \( H_2 \) remains the same but the depth \( H_2 - H_1 \) increases as the fluid flow rate is increased. The depth \( H_1 \) to which the bed moves can be found from (7). This is regime 1 or the bed erosion regime.

The correlation suggests a logarithmic behavior in regime 1 given by
\[ dR_{\text{eff}} = -m \frac{dH}{H} \]  \hfill (10)

Integrating we get
\[ R_{\text{eff}} = -m \frac{H_1}{H} dH \]
\[ R_{\text{eff}} = -m \frac{H_1}{H} \ln \left( \frac{H_1}{H_2} \right) \]  \hfill (11)

A theoretical model to explain this equation may be constructed, details of which are not included here.

The depth \( H_1 \) increases up to a certain value (in this case corresponding to \( \ln(H_1/H_2) \sim 1.3 \)). Further increase in the fluid flow rate does not increase \( H_1 \), hence the bed begins to 'inflate' or 'expand'. This situation is identical to our numerical simulations. Bed expansion now causes the value of \( H_2 \) to decrease. This is regime 2 or the bed expansion regime. As expected from our numerical simulation results, we observe power law behavior in this regime.

**Correlation for proppant flow rate**

We need another correlation for the proppant flow rate. It is shown in Figure 5. Define a non-dimensional variable \( R_p \) for proppant flow rate as 
\[ R_p = \frac{\rho_f Q_p}{H_1 \eta} \]
Once again we observe two regimes. The correlations can be written as

\[ R_{\text{eff}} - R_{\text{eff}}^\sigma = 1057.7R^0.397 \]  \hspace{1cm} (12)  

\[ R_{\text{eff}} - R_{\text{eff}}^\sigma = 0.0356R^{1.8522} \]  \hspace{1cm} (13)  

Equations (12) and (13) can be represented by a single equation given by (Bob Baree, private communication)

\[ R_{\text{eff}} - R_{\text{eff}}^\sigma = \frac{1057.7R^0.397}{1 + \left( \frac{R}{1185.07} \right)^{-0.391}} \]  \hspace{1cm} (14)  

**Conclusion:** Correlations obtained are given by (3), (5)-(6) (or (9)) and (12)-(13) (or (14)).