Slip Velocity and Lift

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Abstract

The lift force on a circular particle in plane Poiseuille flow is studied by direct numerical simulation and by explicit mathematical analysis of a long particle model. The model leads to a formula for the particle velocity that is in excellent agreement with simulation values for the circular cylinder. The value of the Poiseuille flow velocity at the point at the particle’s center when the particle is absent is always larger than the particle velocity; the slip velocity is positive at equilibrium (at steady flow). The angular slip velocity \( \Omega_s = \dot{\gamma}/2 - \Omega_p \), where \( \dot{\gamma}/2 \) is the angular velocity of the fluid at a point where the shear rate is \( \dot{\gamma} \) and \( \Omega_p \) is the angular velocity of the particle, is always positive at equilibrium.

The particle migrates to its equilibrium position and adjusts \( \Omega_p \) so that \( \Omega_s > 0 \) is nearly zero because of \( \Omega_p = \dot{\gamma}/2 \). No matter where the particle is placed it drifts to an equilibrium position with a unique, slightly positive equilibrium angular slip velocity. The slip angular velocity discrepancy defined as the difference between the slip angular velocity of a migrating particle and the slip angular velocity at its equilibrium position is positive below the position of equilibrium and negative above it. This discrepancy is the quantity that changes sign above and below the equilibrium position for neutrally buoyant particles, and also above and below the lower equilibrium position for heavy particles.

1. Introduction

The experiments of Segré and Silberberg (1961, 1962) have had a big influence on fluid mechanics studies of migration and lift. They studied the migration of dilute suspensions of neutrally buoyant spheres in pipe flows at Reynolds numbers between 2 and 700. The particles migrate away from the wall and centerline and accumulate at 0.6 of a pipe radius.

The lift on heavier than liquid particles is also influenced by the factors that determine the equilibrium position of neutrally buoyant particles. The heavy particles must reach an equilibrium that balances the hydrodynamic lift and buoyant weight. If the buoyant weight is very small, the equilibrium position of the particles will be close to the value for the neutrally buoyant case. The effect of increasing the weight is to lower the equilibrium position whose zero is established for the case of zero buoyant weight.

Most attempts to explain the Segré-Silberberg effects have been based on linearized low Reynolds number hydrodynamics. Possibly the most famous of these attempts is due to Saffman (1965), who found that the lift on a sphere in a linear shear flow is given by

\[
L = 6.46\eta a^2 U_o \left(\frac{\gamma}{\nu}\right)^{1/2} + \text{lower order terms}
\]  

(1.1)
where $U_s = U_f - U_p$ is the slip velocity, $U_f$ is the fluid velocity and $U_p$ is the particle velocity, $a$ is the sphere radius, $v = \eta/\rho_f$, $\eta$ is the fluid viscosity, $\rho_f$ is the fluid density and $\dot{\gamma}$ is the shear rate. The lower order terms are

$$-U_s a^2 \rho_f + \pi \Omega_s - \left( \pi - \frac{22}{8} \right) \dot{\gamma}$$

(1.2)

where

$$\Omega_s = \frac{\gamma}{2} - \Omega_p$$

(1.3)

is the angular slip velocity and $\Omega_p$ is the particle angular velocity. There are a number of formulas like Saffman’s that are in the form of $U_s$ times a factor, which can be identified as a density times a circulation as in the famous formula $\rho U \Gamma$ for aerodynamic lift. A relatively recent review of such formulas can be found in McLaughlin (1991).

Formulas like Saffman’s cannot explain Segré-Silberberg's observations, which require migration away from both the wall and the center. There is nothing in these formulas to account for the migration reversal near 0.6 of a radius. Moreover the slip velocity $U_s$, the angular slip velocity $\Omega_s$, the particle velocity and the particle angular velocity, which are functionals of the solution are prescribed quantities in these formulas.

The fluid motion drives the lift on a free body in shear flow; no external forces or torques are applied. If there is no shear there is no lift. In Poiseuille flow there is not only a shear but a shear gradient. Gradients of shear (curvature) produce lateral forces. At the centerline of a Poiseuille flow the shear vanishes, but the shear gradient does not. To understand the Segré-Silberberg effect it is necessary to know that the curvature of the velocity profile at the center of Poiseuille flow makes the center of the channel an unstable position of equilibrium. A particle at the center of the channel or pipe will be driven by shear gradients toward the wall; a particle near the wall will lag the fluid and be driven away from the wall. An equilibrium radius away from the center and wall must exist.

Ho and Leal (1974) were the first to combine these effects in an analysis of the motion of a neutrally buoyant sphere rotating freely between plane walls so closely spaced that the inertial lift can be obtained by perturbing Stokes flow with inertia. They treated wall effects by a method of reflection and found an equilibrium position at 0.69 from the center. Vasseur and Cox (1976) used another method to treat wall effects and their results are close to Ho and Leal’s near the center line but rather different than those of Ho and Leal near the wall. Feng, Hu and Joseph (1994) studied the motion of solid circles in plane Poiseuille flow by DNS. The circle migrates to the 0.6 of a radius equilibrium position. They compared their 2D results with those of Ho & Leal and Vasseur & Cox. Schonberg and Hinch (1989) analyzed the lift on a neutrally buoyant small sphere in a plane Poiseuille flow using matched asymptotic methods. The same problem for neutrally buoyant and non-neutrally buoyant small sphere has been studied using asymptotic method by Asmolov (1999). The linearized analysis given so far takes the effect of inertia $(u \cdot \nabla)u$ into account only in an Oseen linear system; the comparison of the results of these analyses with experiments is far from perfect. The analysis is heavy and explicit formulae for lift are not obtained.
This paper approaches the problem of migration and lift in a different way. Basically we have used direct numerical simulation (DNS) to formulate and validate a long particle model that gives a very good, completely explicit analytical approximation to the velocity and slip velocity of circular particles. DNS is used here as a diagnostic tool to analyze the role of the slip velocity, and the angular slip velocity on migration and lift. We are able in this way to establish a rather simple picture of lift and migration that in particular clarifies the role of the angular slip velocity, and is not restricted to low Reynolds numbers. Our analysis is carried out in two dimensions but should apply in principle to 3D, which is at present under study.

2. Mechanism for lift

First we can look on formulas in a fluid without viscosity for which viscous drag is impossible. The most famous formula for lift on a body of arbitrary shape moving forward with velocity \( U \) in a potential flow with circulation \( \Gamma \) was given by

\[
L' = \rho U \Gamma
\]

where \( \rho \) is the fluid velocity and \( L' \) is the lift per unit length.

The lift on circular cylinder of radius \( a \) is of special interest. A viscous potential flow solution for a stationary cylinder rotating with angular velocity \( \Omega \) which satisfies the no slip boundary condition is given by

\[
u(r) = e_\theta \Omega a^2 / r, \quad \mathbf{u}(a) = e_\theta \Omega a.
\]

The circulation for this viscous potential flow is

\[
\Gamma = \oint \mathbf{u} \cdot d\mathbf{x} = 2\pi \Omega a^2.
\]

When this rotating cylinder moves forward it generates a lift \( L' \) per unit length

\[
L' = 2\pi \rho a^2 U \Omega.
\]

The direction of the lift can be determined by noting that the velocity due to rotation adds to the forward motion of the cylinder at the top or bottom of the rotating cylinder according to the directions of \( \Omega \) and \( U \). In figure 2.1 the velocity is smaller on the bottom of the cylinder; by the Bernoulli equation, the pressure is greater there and it pushes the cylinder up.

![Figure 2.1. The lift per unit length \( L' = 2\pi \rho a^2 U \Omega \) on a cylinder of radius \( a \) moving forward at speed \( U \) and rotating with angular velocity \( \Omega \) in such a way as to reduce the velocity at the bottom and add at the top.](image)
Another formula for the lift off on a particle in an inviscid fluid in which uniform motion is perturbed by a weak shear was derived by Auton (1987) and a more recent satisfying derivation of the same result was given by Drew and Passman (1999). They find that in a plane flow

\[ L = \frac{4}{3} \pi a^3 \rho \Omega_f U_s e_z \]  

(2.5)

where

\[ 2 \Omega_f = \frac{\gamma}{dy} = \frac{du}{dy}. \]

If \( du/dy > 0 \) the sphere is lifted against gravity when the slip velocity \( U_s \) is positive; if \( U_s \) is negative the sphere will fall. Particles which lag the fluid migrate to streamlines with faster flow, particles which lead the fluid migrate to streamlines with slower flow.

There are rather striking differences between (2.5) and (2.4); first (2.4) depends on the angular velocity of the particle but (2.5) depends on the angular velocity of the fluid. Both formulas leave the slip velocity undetermined, \( U_s \) appears in (2.5) because of the shear, in (2.4), \( U_f = 0 \). The slip velocities have to be prescribed in these theories because the particle velocity is not determined by viscous drag. Similarly the angular velocity of the particle cannot arise from torques arising from viscous shears. The effects of particle rotation cannot be obtained by the method of Auton (1987).

The lift formula \( \rho U \Gamma \) captures the essence of the mechanism in which the motion of the particle relative to the fluid is such as to increase the pressure on the side of the particle as it moves forward.

The lift on a spherical or circular particle in a shear flow is different; there is no exterior agent to move and rotate the freely moving particle. Instead the particle is impelled forward and rotated by the shear flow. Previous theoretical studies and our simulations show that the relevant velocity is the slip velocity and the relevant circulation is proportional to an angular slip velocity discrepancy

\[ \Gamma \propto \Omega_s - \Omega_{se} \]  

(2.6)

where \( \Omega_{se} \) is the slip angular velocity in steady flow (equilibrium). This conclusion will be established in the sequel. For now we simply note that in our simulations the angular slip velocity discrepancy \( \Omega_s - \Omega_{se} < 0 \) when the cylinder is above the equilibrium (Segrè-Silberberg) position and \( \Omega_s - \Omega_{se} > 0 \) when it is below the equilibrium (figure 5.2).

3. Numerical simulation of migration and lift

The simulation method used in this paper is described in the papers by Patankar, Huang, Ko and Joseph (2000), Choi and Joseph (2000), and Joseph (2000). A circular particle of diameter \( d \) is placed in a plane Poiseuille flow in a channel of height \( h \) where it moves forward in the direction \( x \) and migrates up or down in the direction \( y \) under the action of hydrodynamic forces. The fluid satisfies the Navier-Stokes equations

\[ \rho_f \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \eta \nabla^2 \mathbf{u} + \rho \mathbf{e}_x \]  

(3.1)
where $\bar{p}$ is the constant pressure gradient. The particle satisfies Newton’s equations of motion. The motion $\mathbf{U}_p$ of the mass center is governed by

$$
\rho_p \frac{d \mathbf{U}_p}{dt} = - (\rho_p - \rho_f) g e_y + \bar{p} e_x + \frac{1}{V_p} \left\{ - p \mathbf{1} + 2 \eta \mathbf{D}[\mathbf{u}] \right\} : \mathbf{n} d \Gamma \quad (3.2)
$$

where $V_p = \pi a^2$ is the volume of the particle and $(- p \mathbf{1} + 2 \eta \mathbf{D}[\mathbf{u}]) \cdot \mathbf{n}$ is the stress on the surface $\Gamma$ of the particle that is a circle of radius $a = d/2$ and

$$
\left\{ (\cdot) d\Gamma = a \int^{2\pi}_0 (\cdot) d\theta . \right. \quad (3.3)
$$

The angular velocity around the mass center at $\mathbf{x} = \mathbf{X}$ is governed by

$$
I \frac{d \Omega}{dt} = \left\{ (\mathbf{x} - \mathbf{X}) \wedge \left[ (- p \mathbf{1} + 2 \eta \mathbf{D}[\mathbf{u}]) \cdot \mathbf{n} \right] d \Gamma \right. \quad (3.4)
$$

where $I = \rho_p \pi a^4$ is the moment of inertia.

In steady (equilibrium) flow the forward motion of the particle is given by

$$
\bar{p} + \frac{1}{\pi a} \int^{2\pi}_0 \mathbf{e}_x \cdot \left\{ - p \mathbf{1} + 2 \eta \mathbf{D}[\mathbf{u}] \right\} : \mathbf{e}_y \ d\theta = 0 \quad (3.5)
$$

The pressure gradient force on the particle is balanced by the resultant of stress traction vector. The long particle model discussed by Joseph (2000), Choi and Joseph (2000) and here in section 4 is a realization of (3.5).

In numerical experiments of solid-liquid flows we can examine physical effects one at a time; this cannot be done in real experiments. For the present application we look first at the effect on particle migration of controlling the angular velocity of the particle. In figure 3.1 we plotted the rise to equilibrium of a neutrally buoyant particle for three different values of the slip angular velocity

$$
\Omega_s = \frac{\gamma}{2} - \Omega_p = \left\{ \frac{\gamma}{2}, \Omega_{se}, 0 \right\} . \quad (3.6)
$$

The rise is the greatest when the particle angular velocity $\Omega_p = 0$ and the least when the particle angular velocity is equal to the local rate of rotation $\Omega_p = \gamma/2$. The rise of a heavier than liquid $\rho_p/\rho_f = 1.01$ circular particle is plotted in figure 3.1 for Reynolds number $R_w = \gamma_w d^2 / \nu = 5.4$ and for $R_w = 16.2$ in figure 3.2. The angular slip velocity $\Omega_{se} > 0$ is the equilibrium value that a free circular particle takes in torque-free motion when the angular acceleration vanishes. We call attention to the fact that $\Omega_{se} > 0$ is very small, and at equilibrium

$$
\frac{\gamma}{2} > \Omega_{pe} , \quad \frac{\gamma}{2} \approx \Omega_{pe} . \quad (3.7)
$$

In another constrained motion we fix the y position of the particle and compute the slip velocities and lift (figure 3.3). A fixed particle with non-zero lift forces will migrate if the constraint is relaxed.
Figure 3.1. Rise vs. time for $R_w = 5.4$. Compare rise of freely rotating and nonrotating particles. Nonrotating ones rise more. A neutrally buoyant, freely rotating particle rises closer to the center line than the “Segré-Silberberg” experiment; the nonrotating one rises even more. Models which ignore particle rotation overestimate lift. A yet smaller lift is obtained when the slip velocity is entirely suppressed ($\Omega_s = 0$), but the particle does rise. The greater the slip angular velocity, the higher the particle will rise.

Figure 3.2. Rise vs. time for a circular particle $\rho_p/\rho_f = 1.01$ when $R_w = 16.2$. The rise of the particle is an increasing function of the slip angular velocity in the range $0 \leq \Omega_s \leq \gamma/2$ and is a maximum when the particle angular velocity is suppressed. The freely rotating particle has a small positive slip angular velocity.
**Figure 3.3.** FIXED PARTICLE: $R_w = 20, \rho_s/\rho_f = 1$. Steady state relative values for the lift force and the slip velocities. In the region close to the wall, the lift force and the horizontal slip velocity have a similar non-linear behavior. In the region close to the centerline, the lift force appears to be proportional to the angular slip velocity. Therefore, the lift force may be expressed as a function of the slip velocity product $L = L(U_s, \Omega_s)$.

### 4. Long particle model

Joseph (2000) proposed a model problem for the velocity of a long particle in Poiseuille flow (also see Choi and Joseph, 2000). We replaced the circular particle of diameter $d$ with a long rectangle whose short side is $d$. The rectangle is so long that we may neglect the effects of the ends of the rectangle at sections near the rectangle center. In that model the long particle was assumed to be rigid but it was noted that a more realistic model could be obtained by letting the long particle shear. Ocando has noted that we could choose this shear to be the same as the shear rate of the circular particle in the approximation in which $\Omega_p = \dot{\gamma}/2$ (figure 4.1). We call this long particle model a shear slip model; the diagram for this model is shown in figure 4.1.

- **Shear slip model**

The forces acting on the long particle are the force due to pressure acting on the sides perpendicular to the flow, and the force due to shear stress acting on the sides parallel to the flow (figure 4.1). The former force is always positive, while the latter may be positive or negative depending if the fluid is faster than the particle or vice versa,

$$ (\tau_A + \tau_B)l + (p_1 - p_2)ld = 0 $$

(4.1)
\[ \tau_A + \tau_B + \bar{\rho}d = 0 \quad \bar{\rho} = \frac{(p_1 - p_2)}{l} \]  

(4.2)

where the shear stresses are defined by

\[ \tau_A = -\eta \frac{du_A}{dy} (h_A) \quad \tau_B = -\eta \frac{du_B}{dy} (h_B) \]  

(4.3)

The velocity profiles above and below the long particle are given by

\[ u_A(y') = \frac{\bar{p}}{2\eta} y'(h_A - y') + \frac{U_A y'}{h_A} \]  

(4.4)

\[ u_B(y) = \frac{\bar{p}}{2\eta} y(h_B - y) + \frac{U_B y}{h_B} \]  

(4.5)

where different velocities \((U_A, U_B)\) were assumed for the top and bottom walls to take into account the angular speed of the circular particle. The relation between them is given by

\[ U_A - U_B = \frac{\dot{\gamma}(h_B + d/2)}{2} d \]  

(4.6)

where \(\dot{\gamma}(y)\) is the shear rate for the undisturbed flow (without the particle), given by

\[ \dot{\gamma}(y) = \frac{du}{dy} = \frac{\bar{p}}{2\eta} (h - 2y) \]  

(4.7)

The shear rate on the particle’s sides parallel to the flow may be evaluated from (4.4), (4.5) and (4.6),

\[ \frac{du_A}{dy'} (h_A) = -\frac{\bar{p}}{2\eta} h_A + \frac{U_B}{h_A} + \frac{\dot{\gamma}(h_B + d/2)}{2} \frac{d}{h_A} \]  

(4.8)

\[ \frac{du_B}{dy} (h_B) = -\frac{\bar{p}}{2\eta} h_B + \frac{U_B}{h_B} \]  

(4.9)
Substituting, recursively, (4.8) and (4.9) in (4.3), and then the resultant equation in (4.2), we find that at the top and bottom of the long particle (diameter $d$):

$$
U_A = \frac{\bar{p}}{\eta} \frac{(2d + h_A + h_B)h_A h_B + \bar{\gamma}(h_B + d/2)h_A d}{2(h_A + h_B)}
$$  \hspace{1cm} (4.10)

$$
U_B = \frac{\bar{p}}{\eta} \frac{(2d + h_A + h_B)h_A h_B - \bar{\gamma}(h_B + d/2)h_B d}{2(h_A + h_B)}
$$  \hspace{1cm} (4.11)

The average particle velocity is:

$$
U_p = \frac{U_A + U_B}{2} = \frac{\bar{p}}{2\eta} \frac{(2d + h_A + h_B)h_A h_B - \bar{\gamma}(h_B + d/2)(h_B - h_A)\frac{d}{2}}{2(h_A + h_B)}
$$  \hspace{1cm} (4.12)

The undisturbed flow field (without the particle) can be written as:

$$
u(y) = \frac{\bar{p}}{2\eta} \gamma(h - y)
$$  \hspace{1cm} (4.13)

At the position where the center of the particle is located ($y_c = h_B + \frac{d}{2}$), the undisturbed fluid velocity is:

$$
u(h_B + \frac{d}{2}) = \frac{\bar{p}}{2\eta} (h_B + \frac{d}{2})(h_A + \frac{d}{2})
$$  \hspace{1cm} (4.14)

The particle slip velocity can be defined as:

$$
U_s = u(h_B + \frac{d}{2}) - U_p
$$  \hspace{1cm} (4.15)

which can be written as:

$$
U_s = \frac{\bar{p}}{\eta} \frac{[(h_A + h_B)(h_B + \frac{d}{2})(h_A + \frac{d}{2}) - (2d + h_A + h_B)h_A h_B] + \bar{\gamma}(h_B + d/2)(h_B - h_A)\frac{d}{2}}{2(h_A + h_B)}
$$  \hspace{1cm} (4.16)

The channel width $h$ and the position $h_A$ and $h_B$ satisfy the following conditions:

$$
\begin{align*}
    h_A &= h - (h_B + d) \\
    h_A + h_B &= h - d
\end{align*}
$$

and the shear rate at the particle center is

$$
\bar{\gamma}(h_B + d/2) = \frac{\bar{p}}{2\eta}(h - d - 2h_B)
$$  \hspace{1cm} (4.17)

Then the slip velocity can be simplified:
\[
U_S = \frac{\bar{p}}{2\eta(h_A + h_B)} \left[ (h_A + h_B)(h_A + \frac{d}{2})(h_A + \frac{d}{2}) - (2d + h_A + h_B)h_Ah_B + \frac{1}{2}(h - d - 2h_B)(h_B - h_A) \frac{d}{2} \right] \\
= \frac{\bar{p}}{2\eta(h - d)} \left[ 2d(h - y_C)^2 + hd^2 \right] \geq 0
\]

(4.18)

when \( d \to 0 \), we can get \( U_S \to 0 \).

A comparison of the shear slip long particle with direct numerical simulation of a circular particle is given in figures 4.2-4.5. The agreement is rather better than might have been anticipated given the severe assumptions required in the model. The agreement is quite good away from the centerline, even close to the wall. Equations (4.2-4.6) can be recommended for an analytical approximation for the velocity of a circular particle in Poiseuille flow.

**Figure 4.2.** FIXED PARTICLE AT THE CENTERLINE: \( y_p = 6.0 \) [cm], \( R_w = 20 \). Axial velocity profiles at steady state. There is no rotation when the particle is on the centerline. The fluid leads the particle \( U_S > 0 \). The long particle model disturbs the flow less than the cylinder.
**Figure 4.3.** FIXED PARTICLE AT $y_p' = 3.0$ [cm]: $R_e = 20$. Axial velocity profiles at steady state. At the middle point between the centerline and the wall, the shear slip model gives a close approximation to the DNS result. The particle’s angular rotation is approximated as half of the shear rate at the particle position on the undisturbed fluid velocity profile.

**Figure 4.4.** FIXED PARTICLE AT $y_p' = 0.75$ [cm]: $R_e = 20$. Axial velocity profiles at steady state. Close to the wall, the shear slip model gives an excellent approximation to the DNS result.
Figure 4.5. SMALL PARTICLE AT $y_p' = 0.75$ [cm]: $d = 0.5$ [cm], $R_w = 20$. Axial velocity profiles at steady state. As the particle is smaller, the difference between disturbed and undisturbed velocity profiles is smaller.

5. Slip angular velocity discrepancy

Here we shall show that the quantity that changes sign when the circular particle is on one or the other side of the equilibrium (Segré-Silberberg) position is the slip angular velocity discrepancy

$$\Omega_s - \Omega_{se}$$

where $\Omega_{se}$ is the slip angular velocity at equilibrium. We shall establish this result for a neutrally buoyant particle at a Reynolds number

$$R_w = \frac{\gamma_w d^2}{\nu} = 10$$

We call this the benchmark case. Identical results may be obtained for other values of $R_w$ below the first bifurcation.

- Unconstrained dynamic simulation

In figure 5.1 we show the trajectory of a neutrally buoyant circular particle that is released from the centerline and the wall of the channel. No matter where the particle is released it will migrate to a unique position of equilibrium at $y_e = 4.18$ cm.
Figure 5.1.  BENCHMARK CASE: $R_w = 10$, $\rho_s/\rho_f = 1$. Evolution of the particle’s vertical position. The particle moves from the starting position to the final position without crossing the equilibrium position.

In figure 5.2 we show that the slip angular velocity is negative when the particle is above the equilibrium position and is positive when it is below the equilibrium position.

Figure 5.2.  BENCHMARK CASE: $R_w = 10$, $\rho_s/\rho_f = 1$. Evolution of the particle’s ANGULAR slip velocity. The angular slip velocity function evolves without crossing the equilibrium value. When the angular slip velocity is BELOW the equilibrium value, the particle moves DOWNWARD. When the angular slip velocity is ABOVE the equilibrium value, the particle moves UPWARD.
In figure 5.3 we show the evolution of the axial slip velocity to equilibrium. The axial slip velocity is positive and of course the greatest for a particle released from rest at the centerline.

**Figure 5.3.** BENCHMARK CASE: $R_w = 10$, $\rho_s/\rho_f = 1$. Evolution of the particle’s AXIAL slip velocity. There is no clear relation between the particle’s vertical position and the axial slip velocity.

**Figure 5.4.** FIXED PARTICLE AROUND EQUILIBRIUM: $R_w = 10$, $y_p = 4.16$ [cm], $y_p = 4.19$ [cm]. Evolution of the vertical force on the particle. The vertical force pulls the particle towards the equilibrium position. This result shows that this position is a STABLE equilibrium position.
**Constrained simulations**

In these simulations we fix the position $y_p$ of the particle’s center. The lift and particle’s forward and angular velocities are allowed to develop in an otherwise unconstrained motion. At a fixed position $y_p$ is given by the undisturbed Poiseuille flow.

In figure 5.4 and 5.5 we fix the particle’s center at $y_p = 4.16$ cm and 4.19 cm, below and above the equilibrium position $y_e = 4.18$. The lift force is such as to push the particle to $y_e$ (figure 5.4) and the slip angular velocity $\Omega_s - \Omega_{se}$ is positive when $y_p > y_e$, and negative when $y_p < y_e$.

![Graph](image)

**Figure 5.5.** FIXED PARTICLE AROUND EQUILIBRIUM: $R_w = 10$, $y_p = 4.16$ [cm], $y_p = 4.19$ [cm]. Evolution of the particle’s angular slip velocity. At STEADY STATE: When the angular slip velocity is BELOW the equilibrium value, the particle moves DOWNWARD (negative force). When the angular slip velocity is ABOVE the equilibrium value, the particle moves UPWARD (positive force).

In figure 5.6 we fix the particle below and above the center line at $y = 6$ cm. The centerline is an unstable equilibrium. In figure 5.7 we look at the slip angular velocity near the centerline. The magnitude of the slip angular velocity decreases as the particle angular velocity increases. The angular velocity $\Omega_s$ asymptotes at large $t$ to its equilibrium value $\Omega_{se}$.
Figure 5.6. FIXED PARTICLE AROUND CENTERLINE: $R_w = 10$, $y_p = 5.95$ [cm], $y_p = 6.05$ [cm]. Evolution of the vertical force on the particle. The vertical force pushes the particle away from the centerline. This result shows that the centerline is an UNSTABLE equilibrium position. If the particle starts at the centerline, it will stay there, but the slightest perturbation will force the particle away.
Figure 5.7. FIXED PARTICLE AROUND CENTERLINE: $R_w = 10, y_p = 5.95$ [cm], $y_p = 6.05$ [cm]. Evolution of the particle’s angular slip velocity. At STEADY STATE: When the angular slip velocity is BELOW the equilibrium value, the particle moves UPWARD. When the angular slip velocity is ABOVE the equilibrium value, the particle moves DOWNWARD. This behavior is the OPPOSITE of the previous cases, because the previous cases were STABLE equilibrium positions, and therefore THE FORCE FIELD around them is the OPPOSITE.

Figures 5.8 and 5.9 give the velocity and angular velocity of the long body from the shear slip model and the velocity and angular velocity of the circular cylinder from the DNS results. We can see that the difference between the shear slip model and the DNS is very small. From figure 5.10 we compare the relative values of the slip velocity $U_s$ with respect to the fluid velocity on the undisturbed flow at the particle’s center $u(y_c)$. The slip velocity from DNS is quite flat for most of the region while the slip velocity from shear slip model increases when it is closer to the wall, but the magnitude of the difference is so small that we believe the shear slip model is perfectly in agreement with the results of numerical computation of the motion of a free circular particle.
**Figure 5.8.** FIXED PARTICLE: $R_w = 20, \rho_s/\rho_f = 1$. Particle’s angular velocity $\Omega_p$ is approximated in the shear slip model as half the value of the shear rate on the undisturbed flow evaluated at particle’s center position $\dot{\gamma}(y_c)/2$.

**Figure 5.9.** FIXED PARTICLE: $R_w = 20, \rho_s/\rho_f = 1$. Particle’s velocity in the shear slip model vs. DNS particle velocity. The slip model overpredicts in the region close to the centerline and underpredicts in the region close to the wall.
Figure 5.10. FIXED PARTICLE: \( R_e = 20 \), \( \rho_p / \rho_f = 1 \). Slip velocity evaluated using DNS results vs. slip velocity in shear slip model. These are relative values of the slip velocity, \( U_s \), with respect to the fluid velocity on the undisturbed flow at particle’s center \( u(y_c) \).

- **Heavy particles**

A position of equilibrium is a steady state of a free particle in which the lift is balanced by the buoyant weight. For neutrally buoyant particles we may say that the position of equilibrium is a "Segré-Silberberg" position. In figure 5.11 we plotted the trajectory of a heavy particle with \( \rho_p = 1.01 \rho_f \); no matter where it is released it will migrate to a unique equilibrium \( y_e = 1.02 \) cm, which is of course smaller than the value \( y_e = 4.18 \) cm for a neutrally buoyant particle. In figure 5.12e show that the slip angular velocity changes sign across the position of equilibrium even when the particle is heavy.
Figure 5.11  EAVY PARTICLE: $R_w = 10$, $\rho_s/\rho_f = 1.01$. Evolution of the particle’s vertical position. The particle starting at the centerline crosses the equilibrium position and then moves upward.

Figure 5.12  HEAVY PARTICLE: $R_w = 10$, $\rho_s/\rho_f = 1.01$. Evolution of the particle’s angular slip velocity. For the particle starting at the centerline, the angular slip velocity function crosses the equilibrium value. When the angular slip velocity is BELOW the equilibrium value, the particle moves DOWNWARD. When the angular slip velocity is ABOVE the equilibrium value, the particle moves UPWARD.
6. Forces and torques on a neutrally buoyant accelerating free particle

We calculated the forces on a freely moving neutrally buoyant particle by computing the integrals of the traction vectors in (3.2) and the moment of theee vectors in (3.4), and displayed their values as a function of time for the benchmark case in figure 6.1, 6.2 and 6.3. These graphs can be compared with the particle trajectory and slip velocities given in section 5. There is no obvious relation between the forces and torques on the one hand and the slip velocities on the other.

![Graph showing the evolution of vertical force and torque](Figure 6.1)

**Figure 6.1.** BENCHMARK CASE: $R_w = 10, \rho_s/\rho_f = 1$. Evolution of the vertical force and torque acting on a particle starting close to the CENTERLINE. There is no clear relation between the slip parameters and the vertical force.
Figure 6.2. **BENCHMARK CASE:** $R_w = 10$, $\rho_s/\rho_f = 1$. Evolution of the vertical force and torque acting on a particle starting close to the WALL. There is no clear relation between the slip parameters and the vertical force.

Figure 6.3. **BENCHMARK CASE:** $R_w = 10$, $\rho_s/\rho_f = 1$. Evolution of the horizontal force acting the particle. The horizontal force acting on a particle starting close to the WALL is always positive, or in other words the particle is dragged from the wall to the equilibrium position. The force for the particle starting from the CENTERLINE is first positive and then negative, because the particle is moving from a higher fluid velocity region to a lower-fluid-velocity region.
7. Summary and conclusion

The lift and migration of neutrally buoyant and heavier-than-liquid circular particles in a plane Poiseuille flow was studied using direct numerical simulation. The study looks at the relation of slip velocity and angular slip velocity to lift and migration. No matter where the particle is released, it will migrate to a unique equilibrium height and move forward with a unique steady particle velocity and rotate with unique steady angular velocity. Neutrally buoyant particles migrate to a radius which can be called the "Segré Silberberg" radius. This radius is a reference; heavier-than-liquid particles also migrate to an equilibrium radius that is close to the Segré-Silberberg radius if the particle density is close to the fluid density. The particles migrate to an equilibrium position \( y_e \) with shear rate \( \dot{\gamma}_s \) such that the local fluid rotation \( \dot{\gamma}_s/2 \) is slightly greater than the particle angular velocity \( \Omega_p \). The angular slip velocity, \( \Omega_p = \dot{\gamma}_s/2 - \Omega_p \) is always positive but at equilibrium it is very small; \( \Omega_p \approx \dot{\gamma}_s/2 \) can be proposed as an approximation. The slip velocity at equilibrium \( U_s = \gamma_s y_e - U_p \) is always positive and slowly varying.

Since the shear rate and slip velocities are one signed they do not explain why the lift changes sign across the equilibrium radius. We found that the quantity which does change sign at \( y_e \) is the angular velocity discrepancy; the angular velocity minus the equilibrium angular velocity \( \Omega_s - \Omega_{se} \). \( \Omega_s - \Omega_{se} > 0 \) when \( y < y_e \) and \( \Omega_s - \Omega_{se} < 0 \) when \( y > y_e \). The adjustment of the angular velocity of a free particle is very critical to lift. One might think of the angular velocity discrepancy as a shear flow analogue to the circulation in aerodynamic lift.

We derived a shear slip version of our long particle model. The long particle model arises when the circular particle is replaced with a long rectangle of the same diameter as the circle, but so long that we may neglect end effects. In the shear slip model we allow the rectangle to shear at the rate \( \dot{\gamma}/2 \) of the local rotation. Using this model we can find explicit expressions for the fluid rotation in which the velocity on either side of the long particle is matched by the fluid velocity; then we satisfy the particles equation of motion in which the shear stress force balances the pressure drop force. This leads to explicit expression for the velocity of the particle (4.12) and the slip velocity (4.16) that is always positive. The shear slip model is in astonishing agreement with the results of numerical computation of the motion of a free circular particle, both with respect to the particle velocity and the fluid velocity on the cross section containing the center of the circular particle.

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List of symbols

d : distance between particle’s top and bottom walls.

$F_L$ : lift force.

$F_x$ : horizontal force acting on the particle.

$F_y$ : vertical force acting on the particle.

$h_A$ : distance between channel’s top wall and particle’s top wall.

$h_B$ : distance between channel’s top wall and particle’s top wall.

$p$ : absolute value of the pressure gradient.

$t$ : time.

$u_A$ : fluid velocity above particle’s top wall.

$u_B$ : fluid velocity below particle’s bottom wall.

$u_c$ : cylinder’s axial velocity.

$u_f$ : cylinder’s axial velocity.

$u_s$ : axial slip velocity.

$U$ : free stream velocity.

$U_A$ : particle’s top wall velocity.

$U_B$ : particle’s bottom wall velocity.

$y$ : vertical coordinate pointing upward.

$y'$ : vertical coordinate pointing downward.

$y_p$ : particle’s vertical coordinate.

- Greek letters

$\eta$ : dynamic viscosity.

$\dot{\gamma}$ : shear rate.

$\Omega_s$ : angular slip velocity.

$\Gamma$ : circulation around cylinder.

$\Omega_c$ : cylinder’s angular velocity.

$\Omega_s$ : angular slip velocity.

$\Omega_{sf}$ : angular slip velocity at equilibrium.

$\tau_A$ : shear stress on particle’s top wall.

$\tau_B$ : shear stress on particle’s bottom wall.
References


