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THE FORMATION OF FILAMENTARY VOIDS IN SOLIDS

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ABSTRACT

WE SHOW that solutions which create spherical voids are unstable relative to the formation of long, thin voids that point in the radial direction, for a large class of nonlinearly elastic materials. We compare sufficient conditions for the formation of such "filamentary voids" to criteria for crazing in glassy polymers.

1. INTRODUCTION

IN 1958 GENT and LINDLEY observed a striking rupture phenomenon in short rubber cylinders that were bonded at their ends to parallel steel plates and pulled in tension. At a load that was in many cases less than a fourth of the ultimate breaking load, they observed the appearance of small, approximately spherical holes in the interior of the test piece. GENT and LINDLEY (1958) measured the load at which these internal flaws appeared as a function of the dimensions of the cylinder and of the type of rubber. They found that this load varied linearly with the Young's modulus of the rubber, for eight rubber formulations with Young's modulus varying from 10 to 60 kg/cm^2, and nonlinearly with the length.

To analyze this phenomenon, they first calculated the maximum hydrostatic tension $P_{\max}$ (the maximum trace of the Cauchy stress) in a bonded cylinder pulled in tension, using an approximate stress analysis. The value $P_{\max}$ that they obtained from their calculation depended on the material parameters and the dimensions of the cylinder. They gave these parameters definite numerical values based upon the known dimensions of the test-piece and upon the independently measured energy functions of the rubbers used in the tests. [STRINGFELLOW and ABEYARATNE (1988) have recently confirmed Gent and Lindley's approximate stress analysis. They computed the stress in the interior of a bonded neo-Hookean cylinder and obtained values very close to those given by Gent and Lindley.]

They then considered a different theoretical problem. They supposed that an infinite, incompressible elastic body contained a finite hole at the origin and was loaded at infinity by a pure hydrostatic tension P. They found an equilibrated radial deformation that solved this problem and which of course depended on P. At a critical value $P = P_{\mathrm{crit}}^{\mathrm{radial}}$ they found that the deformed hole radius became infinite. Even though the cylinders in the experiments were not in a state of pure hydrostatic tension, Gent and Lindley theorized that the hydrostatic part of the stress is most important and they, in effect, proposed the criterion that macroscopic holes will form in an inhomogeneous deformation at any place where the hydrostatic part of the stress reaches a critical value obtained in the *radial* problem with pure hydrostatic stress at infinity; in short, their criterion for void formation is

$$P_{\max} = P_{\text{crit}}^{\text{radial}}. \tag{1.1}$$

Here "max" refers to a maximation over all points in the body. This criterion involves no free parameters once the energy function for the rubber and the test-piece dimensions are assigned. Gent and Lindley compared the criterion (1.1) with experiment and the agreement between theory and experiment was truly remarkable considering the range of dimensions and rubbers tested. Subsequent experiments by LINDSEY (1967) and GENT and TOMPKINS (1969a) have confirmed the above criterion, with some interesting qualifications in the work of GENT and TOMPKINS (1969b), GENT and PARK (1984) and CHO and GENT (1988).

A different point of view of this phenomena was made in a fundamental paper by BALL (1982). He considered a *finite* ball of compressible elastic material, with no hole present initially, subject to a pure radial displacement of amount $\hat{\lambda}-1$ at its boundary. He found that for appropriate materials and for $\hat{\lambda}$ sufficiently large the minimizer of the total stored energy among radial deformations contains a spherical cavity. For incompressible elastic materials, he considered the problem of given radial traction P on the boundary of a finite ball. He found that the equilibrium equations admit a weak solution with a traction-free cavity when $P > P_{\text{crit}}^{\text{radial}}$, the same critical value as given by GENT and LINDLEY (1958). The reason for this agreement, which is based upon the scaling laws for finite elasticity, was explained by BALL (1982). In addition, he analyzed the dynamic (radial) stability of the solutions with cavities and briefly considered the effect of surface energies.

A major conceptual notion in Ball's work is that holes do not have to be present initially. Hence, the idea is applicable in principle to inhomogeneous deformations since it removes the necessity of having to judge the effect of introducing (in the reference configuration) a sufficiently small hole at an arbitrary point in the body, surely a computational nightmare. Instead one can minimize the total stored energy in a sufficiently large space (e.g. an appropriate subset of the Sobolev space $W^{1,p}$, $1 \leqslant p < 3$ for a three-dimensional body) so that deformations with voids can compete for a minimum. Then one views the formation of voids as a (nonsmooth) bifurcation phenomenon. Of course, it must be determined that in such large spaces the expression for the total energy reasonably measures this energy. Questions of this nature are highly nontrivial and have been considered by BALL and MURAT (1984). We discuss this point in JAMES and SPECTOR (1991).

To our knowledge all analyses of the phenomenon of void formation in elastic materials† have been concerned with the radial problem in which all deformations $\mathbf{f}$: $\Omega \to \mathbb{R}^3$ that compete for a minimum are required to have the special form

† Analyses by CHOU-WANG and HORGAN (1989a, b), HORGAN and ABEYARATNE (1986), HORGAN and PENCE (1989a, b), PODIO-GUIDUGLI *et al.* (1986), SIVALOGANATHAN (1986a, b), and STUART (1985) have provided a rather complete picture of the radial problem for isotropic materials, while ANTMAN and NEGRON-MARRERO (1987) have obtained results for radially or azimuthally reinforced materials. PERICAK-SPECTOR and SPECTOR (1988) have shown that the equations of nonlinear elastodynamics can be used to predict the spontaneous formation of spherically symmetric holes.

$$\mathbf{f}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \quad R = |\mathbf{x}|. \tag{1.2}$$

To realize the program of actually predicting the formation of voids at a stress concentration in an elastomer two questions naturally arise: (1) Are the radial solutions with holes minimizers of the energy when nonradial deformations are allowed to compete for a minimum? (2) What information from the analyses of homogeneous boundary-value problems in which $\mathbf{f}(\mathbf{x}) = \mathbf{F}_0\mathbf{x}$ on $\partial\Omega$ can be carried over to inhomogeneous deformations that happen to take on the deformation gradient $\mathbf{F}_0$ at some point $\mathbf{x}_0 \in \Omega$? More specifically, under what conditions does GENT and LINDLEY'S (1958) criterion (1.1) follow from a minimum energy criterion?

Regarding (2) we have previously shown (JAMES and SPECTOR, 1991) that if a void of any kind, not necessarily radially symmetric, reduces the energy in the boundary-value problem with linear boundary conditions $\mathbf{f}(\mathbf{x}) = \mathbf{F}_0\mathbf{x}$, $\mathbf{x} \in \partial\Omega$, then one can find a closely related deformation that reduces the energy of an inhomogeneous deformation that assumes the deformation gradient $\mathbf{F}_0$ at some interior point.

The answer to (1) clearly depends on the stored energy function. In this paper we show that, for a large and realistic class of stored energy functions, radial solutions with holes are not minimizers of energy. Our method is based on the following ideas. Consider first the radial problem with boundary conditions $\mathbf{f}(\mathbf{x}) = \hat{\lambda}\mathbf{x}$ at $|\mathbf{x}| = 1$. The homogeneous deformation satisfying these boundary conditions, $\mathbf{f}(\mathbf{x}) = \hat{\lambda}\mathbf{x}$, $|\mathbf{x}| \leqslant 1$, induces a large volume change when $\hat{\lambda}$ is large. A material may find it energetically unfavorable to undergo such a large change in volume and will instead open a spherical hole at its center. That is, the typical radial deformation that creates a traction-free hole has principal stretches of the form

$$\alpha(R), \lambda(R), \lambda(R), R = |\mathbf{x}|, \tag{1.3}$$

with $\alpha\lambda^2$ bounded and $\lambda(R) \to +\infty$ as $R \to 0^+$. Our analysis involves the same idea carried one step further. A deformation with principal stretches of the form (1.3) might also be judged energetically unfavorable because of the large change in *area* $\lambda(R)^2$ as $R \to 0^+$. We show that for a large class of stored energy functions the material would "prefer" to have principal stretches of the form

$$\alpha, \lambda_1, \lambda_2 \tag{1.4}$$

with $\lambda_1\lambda_2 < \lambda^2(R)$. In order to realize a competitor with principal stretches (1.4), we construct, just outside of the hole, a short but very thin filamentary void, which points in the radial direction. Roughly, the filamentary void turns stretches of the form (1.3) into those of the form (1.4) and reduces the energy.

The analysis of BALL (1982), as well as the analyses of SIVALOGANATHAN (1986a) and STUART (1985), contains certain growth assumptions that may be interpreted as saying that the materials are "soft" with respect to changes of single stretches (shear), but "stiff" with respect to changes in the product of the principal stretches (dilatation). In our assumptions the growth of the energy with respect to pairwise products of the principal stretches is important. These assumptions rule out the class of compressible materials originally studied by Ball, but do not rule out the materials studied by Sivaloganathan and Stuart. To emphasize the distinction between spherical and filamentary voids, we note that our assumptions that promote the formation of filamentary voids allow any behavior whatsoever of the stored energy for pure dilatation.

As noted above, Ball's analysis (in the incompressible case) and Gent and Lindley's analysis deliver the same critical hydrostatic tension for spherical cavitation. Further support for this connection is provided by an analysis of SIVALOGANATHAN (1986a), who considers a finite ball subject to a radial displacement of amount $\hat{\lambda}-1$ at its boundary with and without a pre-existing cavity of radius R_0 at the origin. He shows that the graph of hole radius vs $\hat{\lambda}$ for radial minimizers with a pre-existing hole tends to the analogous graph with no initial cavity as $R_0 \to 0$. This may be interpreted as saying that critical conditions for void formation and sudden growth of a pre-existing void are identical. This conclusion must be tempered by results of GENT and PARK (1984) and CHO and GENT (1988). They find that the critical conditions for cavitation near a very small inclusion are not given by (1.1), and they interpret this to mean that the probability of finding a relatively large (large enough to omit surface energy) precursor void within the small region of stress concentration is small.

The filamentary void provides another mechanism for energy reduction, different from cavitation. A striking example of a rupture process that produces something resembling a filamentary void is the phenomenon of crazing in polymers. In fact, after writing the present paper, we found a striking photograph by DONALD *et al.* (1981) that shows tiny filamentary voids in thin sheets of polystyrene pulled in tension. This prompted us to write Section 7 which relates our results to crazing. A weakness of our results, which prevents a quantitative comparison, is that we do not give critical conditions for the formation of a filamentary void, but we only say that if certain combinations of principal stretches are sufficiently large, then a filamentary void will reduce the energy. Furthermore, all polymers are to some extent viscoelastic, as is true of the elastomers studied by Gent and Lindley, and our thinking relies on the notion that most viscoelasticity theories have a Lyapunov functional which has the form of a nonlinear elastic energy. In a subsequent paper we shall examine in more detail the connection between the formation of a filamentary void and crazing.

2. NOTATION

We let Lin be the space of all linear transformations (tensors) from $\mathbb{R}^3$ into $\mathbb{R}^3$ with norm

$$|\mathbf{H}| = [\operatorname{trace}(\mathbf{H}\mathbf{H}^T)]^{1/2},$$

where $\mathbf{H}^T$ denotes the transpose of $\mathbf{H}$. We write

$$\mathrm{Lin}^+ = \{\mathbf{H} \in \mathrm{Lin} : \det \mathbf{H} > 0\},$$
$$\mathrm{Lin}^{\leqslant} = \{\mathbf{H} \in \mathrm{Lin} : \det \mathbf{H} \leqslant 0\},$$

where det denotes the determinant. Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ we write $\mathbf{a} \otimes \mathbf{b}$ for the tensor product of $\mathbf{a}$ and $\mathbf{b}$; in components

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j.$$

We write ∇ for the gradient operator in $\mathbb{R}^3$: for a vector field $\mathbf{u}$, $\nabla\mathbf{u}$ is the tensor field with components $(\nabla\mathbf{u})_{ij} = \partial u_i/\partial x_j$. Given any function $\Phi(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{c})$ with vector or tensor arguments, we write e.g. $\partial\Phi/\partial\mathbf{a}$ for the partial derivative with respect to $\mathbf{a}$ holding the remaining arguments fixed.

We call a bounded open region $\Omega \subset \mathbb{R}^n$ *regular* provided that $\partial\Omega$ has measure zero. For $1 \leqslant p \leqslant \infty$ we let $\|\cdot\|_{\Omega,m,p}$ denote the $W^{m,p}$-norm on Ω and $\|\cdot\|_{\Omega,p}$ denote the L^p-norm on Ω. Thus, in particular

$$\|\mathbf{f}\|_{\Omega,p} := \|\mathbf{f}\|_{\Omega,0,p} = \left\{ \begin{array}{ll} \left[\int_\Omega |\mathbf{f}(\mathbf{x})|^p \, d\mathbf{x} \right]^{1/p}, & 1 \leqslant p < \infty \\ \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |\mathbf{f}(\mathbf{x})|, & p = \infty \end{array} \right\}$$

$$\|\mathbf{f}\|_{\Omega,1,p} = \left\{ \int_\Omega \left[|\mathbf{f}(\mathbf{x})|^p + |\nabla\mathbf{f}(\mathbf{x})|^p \right] d\mathbf{x} \right\}^{1/p}, \quad 1 \leqslant p < \infty.$$

For $1 \leqslant p < \infty$ we define

$$W^{1,p}(\Omega) \text{ to be the completion of } \{\mathbf{f} \in C^\infty(\Omega, \mathbb{R}^3) : \|\mathbf{f}\|_{\Omega,1,p} < +\infty\}$$

with respect to the $W^{m,p}$-norm and let

$$W_0^{1,p}(\Omega) \text{ be the closure of } C_0^\infty(\Omega, \mathbb{R}^3) \text{ in } W^{1,p}(\Omega).$$

3. Deformation, Stress and Stored Energy

We consider a three-dimensional homogeneous body that, for convenience, we identify with the region $\bar{\Omega}$ that it occupies in a fixed homogeneous reference configuration. Let $1 \leqslant p < 3$. We call a function $\mathbf{f}: \bar{\Omega} \to \mathbb{R}^3$ a *deformation* of the body provided that (i) $\mathbf{f}$ is one-to-one on $\bar{\Omega}$ and, (ii)† $\mathbf{f} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. We denote by Def (Ω) the set of all such deformations.

Remark 3.1. The set of deformations has been chosen so that the energy will be well-defined, voids can form in the material, and the interpenetration of matter is prohibited. For $p < 3$ the spaces $W^{1,p}$, even with the added requirement that the deformation gradient have positive determinant almost everywhere, is too large a space for nonlinear elasticity. One major difficulty is that such functions are not necessarily invertible. Furthermore, such functions can arise as the absolute minimizer of the energy in certain compression problems since the body is able to relax severe compressive strains by overlapping material (see James and Spector, 1991). The spaces $W^{1,p}$ for $p > 3$ do not suffer this pathology, but unfortunately do not permit the formation of voids.

We assume that the body is hyperelastic with continuous stored energy function W: Lin $\to \mathbb{R}^{\geqslant} \cup \{+\infty\}$. W gives the energy stored per unit volume in Ω

† More precisely, the function $\mathbf{f}$ belongs to an equivalence class that is contained in the indicated spaces.

$$W(\nabla \mathbf{f}(\mathbf{x}))$$

at any point $\mathbf{x} \in \Omega$ when the body is deformed by a smooth deformation $\mathbf{f}$.

We further assume that W restricted to $\mathrm{Lin}^{>}$ is C^2 and that $W = +\infty$ on $\mathrm{Lin}^{\leqslant}$. The derivative

$$\mathbf{S}(\mathbf{F}) := \frac{\mathrm{d}}{\mathrm{d}\mathbf{F}} W(\mathbf{F})$$

is the Piola–Kirchhoff stress while the Cauchy stress is defined by

$$\mathbf{T}(\mathbf{F}) := \mathbf{S}(\mathbf{F})\mathbf{F}^T/\det \mathbf{F}. \tag{3.1}$$

We assume that the response of the material is invariant under a change in observer and hence that

$$W(\mathbf{QF}) = W(\mathbf{F}) \tag{3.2}$$

for every $\mathbf{F} \in \mathrm{Lin}^{>}$ and $\mathbf{Q} \in \mathrm{Lin}^{>}$ with $\mathbf{QQ}^T = \mathbf{I}$. One consequence of this assumption is that $\mathbf{T}$ is symmetric. The eigenvalues of $\mathbf{T}$ are called the *principal stresses*.

In Sections 5–7 we will consider isotropic materials that is, materials for which there is a symmetric function $\Phi : (\mathbb{R}^{>})^3 \to \mathbb{R}^{\geqslant}$ with the property that for every $\mathbf{F} \in \mathrm{Lin}^{>}$

$$W(\mathbf{F}) = \Phi(\lambda_1(\mathbf{F}), \lambda_2(\mathbf{F}), \lambda_3(\mathbf{F})), \tag{3.3}$$

where $\lambda_i(\mathbf{F})$ are the *principal stretches*, i.e. the eigenvalues of $(\mathbf{FF}^T)^{1/2}$.

4. RESULTS ON THE LOCAL REDUCTION OF ENERGY

We assume that there is a potential $\beta \in C^1(\bar{\Omega} \times \mathbb{R}^3, \mathbb{R})$ such that

$$\mathbf{b}_{\mathbf{f}}(\mathbf{x}) := \frac{\partial}{\partial \mathbf{f}} \beta(\mathbf{x}, \mathbf{f}(\mathbf{x}))$$

gives the body force exerted by the environment on the material at the point $\mathbf{x}$ when the body is deformed by a smooth deformation $\mathbf{f}$. We let

$$E(\mathbf{f}, \Omega) = \int_{\Omega} [W(\nabla \mathbf{f}(\mathbf{x})) - \beta(\mathbf{x}, \mathbf{f}(\mathbf{x}))] \,\mathrm{d}\mathbf{x} \tag{4.1}$$

denote the *total energy* when the body is deformed by $\mathbf{f}$. Later, we shall introduce restrictions on W and β so that the total energy is defined for all $\mathbf{f}$ in Def (Ω). We use the term *total stored energy* for the right-hand side of (4.1) with β omitted.

Let $\mathbf{d} \in C^1(\bar{\Omega})$ be one-to-one. We are interested in deformations that are local minimizers of the total energy and that have the same boundary values and orientation as $\mathbf{d}$. We therefore let

$$\mathrm{Kin}_{\mathrm{d}}(\Omega) = \{\mathbf{f} \in \mathrm{Def}\,(\Omega) : (\mathbf{f} - \mathbf{d}) \in W_0^{1,p}(\Omega),\ \mathbf{f}(\Omega) \subset \mathbf{d}(\Omega)\};$$

the set of kinematically admissible deformations.

Remark 4.1. The constraint $\mathbf{f}(\Omega) \subset \mathbf{d}(\Omega)$ is used in the proof of theorem 4.2, where a certain deformation is altered on a sphere in Ω. It seems to us a reasonable restriction on deformations that are allowed to complete for a minimum. This constraint may be a consequence of other conditions one might impose in order to insure that a function $\mathbf{f} \in \mathrm{Def}\,(\Omega)$ corresponds to a reasonable physical notion of a deformation that has finite energy and satisfies the boundary condition $\mathbf{f} = \mathbf{d}$ on $\partial\Omega$.

Let $\mathbf{f} \in \mathrm{Kin}_{\mathbf{d}}(\Omega)$ satisfy $E(\mathbf{f}, \Omega) < +\infty$. We say that $\mathbf{f}$ is a strong local minimizer (in $W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$) of the energy E provided that there is an $\varepsilon > 0$ such that

$$E(\mathbf{f}, \Omega) \leqslant E(\mathbf{g}, \Omega)$$

for every $\mathbf{g} \in \mathrm{Kin}_{\mathbf{d}}(\Omega)$ that satisfies

$$\|\mathbf{f}-\mathbf{g}\|_{\Omega,\infty} + \|\mathbf{f}-\mathbf{g}\|_{\Omega,1,p} < \varepsilon. \tag{4.2}$$

THEOREM 4.2 (JAMES and SPECTOR, 1991). Let $\mathbf{f} \in \mathrm{Kin}_{\mathbf{d}}(\Omega)$ be a strong local minimizer of $E(\cdot, \Omega)$. Suppose that $\mathbf{f}$ is C^1 in a neighborhood of $\mathbf{x}_0 \in \Omega$ and let

$$\mathbf{F}_0 := \nabla\mathbf{f}(\mathbf{x}_0)\,; \quad \mathbf{f}_0(\mathbf{x}) := \mathbf{F}_0(\mathbf{x}-\mathbf{x}_0) + \mathbf{f}(\mathbf{x}_0),\ \mathbf{x} \in \mathbb{R}^3.$$

Assume that $W(\mathbf{F}_0) < \infty$. Then for every regular region $\mathscr{D} \subset \mathbb{R}^3$

$$\int_{\mathscr{D}} W(\mathbf{F}_0)\,\mathrm{d}\mathbf{x} \leqslant \int_{\mathscr{D}} W(\mathbf{F}_0 + \nabla\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \tag{4.3}$$

whenever $\mathbf{f}_0 + \mathbf{u} \in \mathrm{Kin}_{\mathbf{f}_0}(\mathscr{D})$.

In other words a necessary condition for $\mathbf{f}$ to be a strong local minimizer is that, at each point $\mathbf{x}_0$ of smoothness of $\mathbf{f}$, the affine deformation $\mathbf{f}_0(\mathbf{x}) = \nabla\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + \mathbf{f}(\mathbf{x}_0)$ is a global minimizer of the total energy of any body that is composed of the same material but is not subjected to body forces.

Remark 4.3. Theorem 4.2 shows that the introduction of a new hole as a method of energy reduction is a local phenomenon; that is, it is possible to introduce such a hole at any point in the body without changing the deformation outside of a small neighborhood of that point. (Of course when equilibrium solutions are considered the introduction of a new hole will cause changes in the solution at all points in the body.) We think that the introduction of new holes in incompressible bodies is global in nature since we expect that the total volume of the body should not change in an isochoric deformation (even if the deformation is only contained in $W^{1,1}$) and hence the introduction of a new hole can only be achieved by allowing the boundary of the body to move. This fact makes a rigorous justification of a failure criterion such as GENT and LINDLEY'S (1958) difficult for incompressible materials.

5. THE ENERGETICS OF FILAMENTARY VOIDS

5.1. *Constitutive assumptions*

The formation of filamentary voids is promoted by assumptions that limit the growth of the stored energy for large, nearly isochoric shears while at the same time require fairly strong growth for large equibiaxial stretches. Also, we wish to differentiate the phenomenon of the formation of filamentary voids from the formation of spherical voids by being completely noncommittal about the behavior of the energy for dilatation. We will henceforth assume that the body is isotropic and write the stored energy

$$\Phi(\alpha,\beta,\gamma) = \phi(\alpha)+\phi(\beta)+\phi(\gamma)+\psi(\alpha\beta)+\psi(\alpha\gamma)+\psi(\beta\gamma)+\chi(\alpha\beta\gamma)+\Delta(\alpha,\beta,\gamma), \tag{5.1}$$

where $\psi \in C^1(\mathbb{R}^>, \mathbb{R})$, $\phi : \mathbb{R}^> \to \mathbb{R}$, $\chi : \mathbb{R}^> \to \mathbb{R}$, and $\Delta : (\mathbb{R}^>)^3 \to \mathbb{R}$. The assumption (5.1) entails no restrictions on Φ so far because Δ is arbitrary.

Our main constitutive assumptions are:

$S1$. *Stiffness for equibiaxial stretch.*
(a) ψ is convex,
(b) $\psi(t) \to +\infty$ as $t \to +\infty$,
(c) χ is nondecreasing,
(d) ϕ, ψ, and Δ are nonnegative.

$S2$. *Softness for isochoric families of shear.* There are constants $q \in (0, 2)$ and $c > 0$ and a function $h \in C^0(\mathbb{R}^>, \mathbb{R})$ such that
(a) $\phi(t) \leqslant c[t^q + t^{-q}]$ for all $t > 0$,
(b) $\psi(t) \leqslant c[t^q + 1]$ for all $t > 0$,
(c) $\Delta(\alpha,\beta,\gamma) \leqslant h(\alpha\beta\gamma)[1+\alpha^q+\beta^q+\gamma^q+(\alpha\beta)^{q/2}+(\alpha\gamma)^{q/2}+(\beta\gamma)^{q/2}]$ for all $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.

The convexity of ψ together with the nonnegativity of ϕ, ψ, and Δ insure that the material is stiff for equibiaxial stretch; i.e. that $\Phi(\alpha, \lambda, \lambda)$ grows at least quadratically in λ, uniformly in α, for large λ. The upper bounds on ϕ, ψ, and Δ insure that the material is soft for isochoric families of shear, i.e. that $\Phi(\alpha, \beta, k/\alpha\beta)$ grows subquadratically in α and β for each $k > 0$. These upper bounds also insure that certain deformations, which open cylindrical holes in the material, have finite energy.

Remark 5.1. The subquadratic growth of $\Phi(\alpha, \beta, k/\alpha\beta)$ is necessary for the formation of essentially cylindrical holes as is evident from the work of BALL (1982) specialized to the case $n = 2$. At the expense of complicating our proofs greatly, our results remain valid when the constant c in $S2$ is different on different lines and when the value of $q \in (0, 2)$ changes from term to term.

The constitutive hypotheses $S1$ and $S2$ yield a comparison between the energy of a certain triaxial stretch and the energy of a related biaxial stretch.

LEMMA 5.2. Let $S1$ and $S2$ be satisfied. Suppose that α, λ, r', ω, and a are real numbers that satisfy

$$\left.\begin{array}{ll} 1 < a, & 0 < r' \leqslant \omega < 1 \\ 1 < \lambda, & 0 < \alpha < a \end{array}\right\}. \tag{5.2}$$

Then

$$\Phi(\lambda r', \lambda\omega/r', \alpha) - \Phi(\lambda, \lambda, \alpha) \leqslant 8a^q\lambda^q(r')^{-q}[c+h(\omega\alpha\lambda^2)] - \lambda^2(1-\omega)\psi'(\omega\lambda^2).$$

Proof. If we evaluate Φ at the indicated arguments we find, with the aid of (5.1), $(5.2)_2$, $S1$(c) and (d), and $S2$ that

$$\Phi(\lambda r', \lambda\omega/r', \alpha) - \Phi(\lambda, \lambda, \alpha) \leqslant k\lambda^q (r')^{-q} + \psi(\omega\lambda^2) - \psi(\lambda^2),$$

$$k := \left\{ \begin{array}{c} c\begin{bmatrix} (r')^{2q} + \lambda^{-2q} + \omega^q + \lambda^{-2q}(r'/\omega)^q (r')^q \\ \alpha^q\omega^q + 2\lambda^{-q}(r')^q + \alpha^q (r')^{2q} \end{bmatrix} \\ + h(\omega\alpha\lambda^2)\begin{bmatrix} \lambda^{-q}(r')^q + (r')^{2q} + \omega^q + \alpha^q\lambda^{-q}(r')^q + \omega^{q/2}(r')^q \\ + \alpha^{q/2}\lambda^{-q/2}(r')^{3q/2} + \alpha^{q/2}\lambda^{-q/2}\omega^{q/2}(r')^{q/2} \end{bmatrix} \end{array} \right\}.$$

The desired result then follows from (5.2) and the convexity of ψ, i.e.

$$\psi(\lambda^2) \geqslant \psi(\omega\lambda^2) + \lambda^2(1-\omega)\psi'(\omega\lambda^2).$$

5.2. *Formation of a cylindrical hole through a plate*

Let $\Gamma \subset \mathbb{R}^2$ be a regular region with $(0, 0) \in \Gamma$. Given $L > 0$ define

$$\mathscr{P}^L_{-L} := \Gamma \times (-L, L). \tag{5.3}$$

A *plane radial deformation* of the plate $\mathscr{P}^L_{-L}$ is a deformation $\mathbf{f} \in \mathrm{Def}(\mathscr{P}^L_{-L})$ that satisfies

$$\mathbf{f}(\mathbf{x}) = r(R)\mathbf{e}_R(\mathbf{x}) + \alpha x_3 \mathbf{e}_3, \quad \mathbf{x} \in \mathscr{P}^L_{-L}, R > 0, \tag{5.4}$$

for some $\alpha > 0$ and $r \in C^0(\mathbb{R}^{\geqslant}, \mathbb{R}^{\geqslant})$. Here $R = (x_1^2 + x_2^2)^{1/2}$ and

$$\mathbf{e}_R(\mathbf{x}) := \left\{ \begin{array}{ll} (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)/R, & R > 0 \\ \mathbf{e}_1, & R = 0 \end{array} \right\}, \tag{5.5}$$ †

the radial unit vector. The special plane radial deformation given by (5.4) with the choice

$$r(R) = \lambda R, \quad \lambda = \text{constant} > 0,$$

is called an *equibiaxial stretch.*

Let $R_0 > 0$ be sufficiently small so that

$$\{(x_1, x_2) : x_1^2 + x_2^2 \leqslant R_0^2\} \subset \Gamma.$$

Fix $\omega \in (0, 1)$ and define $r \in C^0(\mathbb{R}^{\geqslant}, \mathbb{R}^{>})$ by

$$r(R) := \left\{ \begin{array}{ll} [\omega R^2 + (1-\omega)R_0^2]^{1/2}, & 0 \leqslant R < R_0 \\ R, & R_0 \leqslant R \end{array} \right\}. \tag{5.6}$$

Then, for each $\alpha > 0$ and $\lambda > 0$,

$$\mathbf{p}_\lambda^\alpha(\mathbf{x}) = \lambda r(R)\mathbf{e}_R(\mathbf{x}) + \alpha x_3 \mathbf{e}_3, \quad \mathbf{x} \in \mathscr{P}^L_{-L}, \tag{5.7}$$

† We have put $\mathbf{e}_R(0) = \mathbf{e}_1$ in (5.5) so that $\mathbf{f}$ is well defined at $R = 0$.

is a plane radial deformation of $\mathscr{P}^L{}_L$ (cf. proposition 5.4) that satisfies the boundary condition

$$\mathbf{p}_\lambda^\alpha = \mathbf{b}_\lambda^\alpha \quad \text{on} \quad \partial\Gamma \times [-L, L],$$

where

$$\mathbf{b}_\lambda^\alpha(\mathbf{x}) := \lambda R \mathbf{e}_R(\mathbf{x}) + \alpha x_3 \mathbf{e}_3 \tag{5.8}$$

is the equibiaxial stretch associated with $\mathbf{p}_\lambda^\alpha$. The deformation $\mathbf{p}_\lambda^\alpha$ opens a circular cylindrical hole of radius $\lambda(1-\omega)^{1/2}R_0$ in the plate. The principal stretches of $\nabla\mathbf{p}_\lambda^\alpha$ are (in no special order)

$$\lambda_1 = \lambda r'(R), \qquad \lambda_2 = \lambda r(R)/R, \qquad \lambda_3 = \alpha, \tag{5.9}$$

where prime denotes differentiation with respect to R.

We now show that, for λ sufficiently large, the deformation (of a material governed by $S1$ and $S2$) that introduces a hole in the plate has less energy than the associated equibiaxial stretch.

PROPOSITION 5.3. Let the stored energy satisfy constitutive hypotheses $S1$ and $S2$. Let $a > 1$ and suppose that $\hat{\alpha}: (1, \infty) \to (0, a)$ is such that both

$$0 < \inf_{\lambda > 1} [\hat{\alpha}(\lambda)\lambda^2] \quad \text{and} \quad \sup_{\lambda > 1} [\hat{\alpha}(\lambda)\lambda^2] < +\infty, \tag{5.10}$$

or

$$\sup_{\lambda > 1} h(\hat{\alpha}(\lambda)\lambda^2) < +\infty \tag{5.11}$$

[cf. $S2$(c)]. Then there is a $\lambda_0 > 1$ and a $\kappa > 0$, which are independent of λ and L, such that

$$\int_{\mathscr{P}^L{}_L} [W(\nabla\mathbf{p}_\lambda^\alpha(\mathbf{x})) - W(\nabla\mathbf{b}_\lambda^\alpha(\mathbf{x}))]\, \mathrm{d}\mathbf{x} \leqslant -2\kappa L \lambda^q$$

for all $\lambda \geqslant \lambda_0$. Here $\mathbf{p}_\lambda^\alpha$ is the plane radial deformation given by (5.7), with $\alpha = \hat{\alpha}(\lambda)$, and $\mathbf{b}_\lambda^\alpha$ is the associated equibiaxial stretch given by (5.8).

Proof. We first note that (5.10) and the continuity of h imply (5.11). Next, by (5.6), $0 < r'(R) < \omega$ and $r'(R) = \omega R/r(R)$ provided $0 < R < R_0$. Thus (5.9) and lemma 5.2 imply that for $0 < R < R_0$

$$W(\nabla\mathbf{p}_\lambda^\alpha) - W(\nabla\mathbf{b}_\lambda^\alpha) \leqslant 8a^q\lambda^q\omega^q(r/R)^q[c + h(\omega\alpha\lambda^2)] - \lambda^2(1-\omega)\psi'(\omega\lambda^2). \tag{5.12}$$

The definition (5.6) of $r(\cdot)$ shows that

$$\int_0^{R_0} \left|\frac{r(R)}{R}\right|^q R\, \mathrm{d}R \leqslant \int_0^{R_0} \left|\frac{R_0}{R}\right|^q R\, \mathrm{d}R < +\infty, \tag{5.13}$$

for $q < 2$, and hence that the function $|r(R)/R|^q$ is integrable on $\mathscr{P}^L{}_L$. Also, $S1$(a) and $S2$(b) imply that $\psi'(t)$ is positive and nondecreasing for large t. Thus, if we integrate (5.12) over $\mathscr{P}^L_{-L}$ we conclude, with the aid of (5.11), that there are $c_1 > 0$ and $c_2 > 0$, which are independent of L and λ, such that

$$\int_{\mathscr{P}^L_{-L}} [W(\nabla \mathbf{p}^\alpha_\lambda) - W(\nabla \mathbf{b}^\alpha_\lambda)]\, d\mathbf{x} \leqslant 2L(c_1\lambda^q - c_2\lambda^2).$$

The desired result is now immediate since $q < 2$.

5.3. *Kinematic description of the caps*

The deformation $\mathbf{p}^\alpha_\lambda$ of the preceding section opens a hole through a plate and reduces the total energy relative to the associated equibiaxial stretch. We intend to use a deformation like $\mathbf{p}^\alpha_\lambda$ to show that the conclusion of theorem 4.2 is violated and therefore that any deformation that at some point has principal stretches λ, λ, and α [with suitable assumptions on $\alpha = \hat{\alpha}(\lambda)$ and with λ sufficiently large] is unstable. In particular this will sometimes occur outside a spherical cavity. At this point we cannot use $\mathbf{p}^\alpha_\lambda$ for this purpose because it does not satisfy linear boundary conditions at $x_3 = \pm L$. The linear boundary conditions are essential for applying theorem 4.2. The problem is that the hole pierces the top and bottom of the plate. To circumvent this difficulty we shall, in this section, place *caps* on the ends of the holes. Eventually, we will find it necessary to make the void very long and thin ($R_0 \ll L$) in order to take advantage of the energy reducing nature of the cylindrical void relative to the caps.

Fix $\omega \in (0, 1)$ and let $\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ be a bridging function with

$$\theta(t) = \begin{cases} \omega, & t \leqslant 1 \\ 1, & t \geqslant 2 \end{cases} \tag{5.14}$$

$$\dot{\theta}(t) > 0, \quad 1 < t < 2.$$

PROPOSITION 5.4. Fix $\alpha > 0$, $\lambda > 0$ and $L > 0$. Define

$$\mathbf{c}^\alpha_\lambda(\mathbf{x}) := \lambda r(R, x_3)\mathbf{e}_R(\mathbf{x}) + \alpha x_3 \mathbf{e}_3, \quad \mathbf{x} \in \mathscr{P}^{L+3}_L, \tag{5.15}$$

where

$$r(R, z) := \begin{cases} [\theta(z-L)R^2 + (1-\theta(z-L))R_0^2]^{1/2}, & 0 \leqslant R < R_0, \\ R, & R_0 \geqslant R. \end{cases} \tag{5.16}$$

Then, $\mathbf{c}^\alpha_\lambda \in \mathrm{Def}\,(\mathscr{P}^{L+3}_L)$, i.e. $\mathbf{c}^\alpha_\lambda \in W^{1,p}(\mathscr{P}^{L+3}_L) \cap L^\infty(\mathscr{P}^{L+3}_L)$ for $1 \leqslant p < 2$ and $\mathbf{c}^\alpha_\lambda$ is one-to-one. Moreover, if the stored energy function satisfies $S2$ then there is a $K_1 > 0$, which does not depend on L, such that

$$\int_{\mathscr{P}^{L+3}_L} W(\nabla \mathbf{c}^\alpha_\lambda)\, d\mathbf{x} \leqslant K_1. \tag{5.17}$$

Proof. It is clear from (5.15) and (5.16) that $\mathbf{c}^\alpha_\lambda$ is bounded and one-to-one. We claim that $\mathbf{c}^\alpha_\lambda \in W^{1,p}(\mathscr{P}^{L+3}_L)$ with weak derivative

$$\mathbf{C}^\alpha_\lambda := \lambda r_R \mathbf{e}_R \otimes \mathbf{e}_R + \lambda(r/R)\mathbf{e}_T \otimes \mathbf{e}_T + \alpha \mathbf{e}_3 \otimes \mathbf{e}_3 + \lambda r_z \mathbf{e}_R \otimes \mathbf{e}_3 \quad \text{a.e.}, \tag{5.18}$$

where r_R and r_z are the classical partial derivatives of r, and $\mathbf{e}_T(\mathbf{x}) := \mathbf{e}_3 \times \mathbf{e}_R(\mathbf{x})$ is the tangential unit vector. To prove the above claim we first show that $\mathbf{C}^\alpha_\lambda \in L^p(\mathscr{P}^{L+3}_L)$.

If we square both sides of (5.16), take the partial derivatives, and square the results we find that for $0 < R < R_0$ and $L+1 < z < L+2$

$$r_R^2 = \frac{\theta^2 R^2}{\theta R^2 + (1-\theta)R_0^2} \leqslant \frac{\theta^2 R^2}{\theta R^2} = \theta,$$

$$r_z^2 = \frac{\dot\theta^2 (R^2 - R_0^2)^2}{4[\theta R^2 + (1-\theta)R_0^2]} \leqslant \frac{\dot\theta^2 R_0^4}{4(1-\theta)R_0^2} = \frac{\dot\theta^2 R_0^2}{4(1-\theta)}. \tag{5.19}$$

L'Hospital's rule shows that

$$\lim_{t\to 2} \frac{[\dot\theta(t)]^2}{1-\theta(t)} = \lim_{t\to 2} \frac{2\dot\theta(t)\ddot\theta(t)}{-\dot\theta(t)} = 0.$$

Using the definition (5.14) of θ, we conclude that r_R and r_z are bounded on $\mathscr{P}^{L+3}_L$. The argument leading to (5.13) can be repeated with ω replaced by θ and q replaced by p to show that $|r(R)/R|^p$ is integrable on $\mathscr{P}^{L+3}_L$. It follows that $\mathbf{C}^\alpha_\lambda \in L^p(\mathscr{P}^{L+3}_L)$.

To show that $\mathbf{C}^\alpha_\lambda$ is the weak derivative of $\mathbf{c}^\alpha_\lambda$, we let $\psi \in C_0^\infty(\mathscr{P}^{L+3}_L)$ and let $\mathscr{C}_\varepsilon = \{\mathbf{x} \in \mathscr{P}^{L+3}_L : x_1^2 + x_2^2 \leqslant \varepsilon\}$. Then, by (5.14)–(5.16) and the divergence theorem

$$\begin{aligned}\int_{\mathscr{P}^{L+3}_L \setminus \mathscr{C}_\varepsilon} [\psi \mathbf{C}^\alpha_\lambda + \mathbf{c}^\alpha_\lambda \otimes \nabla\psi]\, \mathrm{d}\mathbf{x} &= \int_{\mathscr{P}^{L+3}_L \setminus \mathscr{C}_\varepsilon} \operatorname{div}[\psi \mathbf{c}^\alpha_\lambda]\, \mathrm{d}\mathbf{x} \\ &= \int_{\partial(\mathscr{P}^{L+3}_L \setminus \mathscr{C}_\varepsilon)} \psi \mathbf{c}^\alpha_\lambda \cdot \mathbf{n}\, \mathrm{d}a \\ &= \varepsilon \int_L^{L+3} \int_0^{2\pi} \psi \mathbf{c}^\alpha_\lambda \cdot \mathbf{e}_R \, \mathrm{d}\Theta\, \mathrm{d}x_3,\end{aligned} \tag{5.20}$$

where (R, Θ, x_3) are the cylindrical coordinates of the point $\mathbf{x}$. The functions ψ, $\mathbf{c}^\alpha_\lambda$, and $\mathbf{e}_R$ are bounded on $\mathscr{P}^{L+3}_L$. Thus, if we let $\varepsilon \to 0^+$ in (5.20) and use the dominated convergence theorem, we get

$$\int_{\mathscr{P}^{L+3}_L} [\psi \mathbf{C}^\alpha_\lambda + \mathbf{c}^\alpha_\lambda \otimes \nabla\psi]\, \mathrm{d}\mathbf{x} = \lim_{\varepsilon\to 0} \int_{\mathscr{P}^{L+3}_L \setminus \mathscr{C}_\varepsilon} [\psi \mathbf{C}^\alpha_\lambda + \mathbf{c}^\alpha_\lambda \otimes \nabla\psi]\, \mathrm{d}\mathbf{x} = 0,$$

which proves that $\nabla \mathbf{c}^\alpha_\lambda = \mathbf{C}^\alpha_\lambda$.

We now assume that the stored energy satisfies $S2$ and we show that $\mathbf{c}^\alpha_\lambda$ has finite total stored energy. The definition (5.18) of $\mathbf{C}^\alpha_\lambda$ yields

$$\mathbf{C}^\alpha_\lambda(\mathbf{C}^\alpha_\lambda)^T = \lambda^2(r_R^2 + r_z^2)\mathbf{e}_R \otimes \mathbf{e}_R + \lambda^2\left(\frac{r}{R}\right)^2 \mathbf{e}_T \otimes \mathbf{e}_T + \alpha^2 \mathbf{e}_3 \otimes \mathbf{e}_3 + \alpha\lambda r_z(\mathbf{e}_3 \otimes \mathbf{e}_R + \mathbf{e}_R \otimes \mathbf{e}_3),$$

so the eigenvalues $\{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$ of $\mathbf{C}^\alpha_\lambda(\mathbf{C}^\alpha_\lambda)^T$ satisfy

$$\lambda_1^2 = \lambda^2\left(\frac{r}{R}\right)^2, \quad \lambda_{2,3}^4 - \lambda_{2,3}^2(\alpha^2+\lambda^2 r_R^2+\lambda^2 r_z^2)+\alpha^2\lambda^2 r_R^2 = 0.$$

Hence

$$\lambda_1 = \lambda\frac{r}{R}, \quad \lambda_2^2+\lambda_3^2 = \alpha^2+\lambda^2 r_R^2+\lambda^2 r_z^2, \quad \lambda_2\lambda_3 = \alpha\lambda r_R, \tag{5.21}$$

and therefore

$$\begin{aligned}\lambda_1 &= \lambda(r/R),\\ \lambda_2 &= \alpha\lambda r_R/\lambda_3,\\ \lambda_3 &= \tfrac{1}{2}\{[(\alpha+\lambda r_R)^2+\lambda^2 r_z^2]^{1/2}+[(\alpha+\lambda r_R)^2+\lambda^2 r_z^2]^{1/2}\}.\end{aligned} \tag{5.22}$$

We also note that

$$\lambda_1\lambda_2\lambda_3 = \alpha\lambda^2 r_R\left(\frac{r}{R}\right) = \alpha\lambda^2\theta \in (\alpha\lambda^2\omega, \alpha\lambda^2).$$

By (5.1) and $S2$, there is a constant $\bar{c} > 0$ such that

$$\Phi(\lambda_1,\lambda_2,\lambda_3) \leqslant \bar{c}\left\{1+\sum_i(\lambda_i^q+\lambda_i^{-q})+\sum_{j\neq k}\lambda_j^q\lambda_k^q\right\}.$$

Therefore, in order to prove that the deformation $\mathbf{c}_\lambda^\alpha$ has finite total stored energy, we shall show that $R\lambda_i(R,z)^q$, $R\lambda_i(R,z)^{-q}$ and $R(\lambda_j(R,z)\lambda_k(R,z))^q$, are integrable on $(0,R_0)\times(L,L+3)$.

By $(5.22)_3$ $\lambda_3 \geqslant \alpha$. Since also r_R and r_z are bounded on $\mathscr{P}_L^{L+3}$ we find from (5.21) that λ_2 and λ_3 are bounded on $\mathscr{P}_L^{L+3}$. The argument above, which shows that $\mathbf{C}_\lambda^\alpha \in L^p(\mathscr{P}_L^{L+3})$, also shows that λ_1^q is integrable on $\mathscr{P}_L^{L+3}$. Putting these facts together, we have established the integrability of

$$\begin{aligned}&\lambda_i^q, \quad i=1,2,3,\\ &\lambda_3^{-q},\\ &(\lambda_j\lambda_k)^q, \quad j,k\in\{1,2,3\}, j\neq k.\end{aligned}$$

It remains to examine the integrability of λ_1^{-q} and λ_2^{-q}. We note that (5.14) and (5.16) imply that $r(R,z) \geqslant \omega R$ which yields the integrability of λ_1^{-q}. The integrability of λ_2^{-q} follows from the identity

$$\lambda_2^{-q} = \frac{\lambda_3^q}{\alpha^q\lambda^q\theta^q}\left(\frac{r}{R}\right)^q,$$

and (5.13). Finally, it is clear from (5.15) and (5.16) that the total stored energy $E(\mathbf{c}_\lambda^\alpha, \mathscr{P}_L^{L+3})$ is independent of L.

5.4. *Main result*

We now use theorem 4.2 and propositions 5.3 and 5.4 to show that by introducing a filamentary void at a point $\mathbf{x}_0 \in \Omega$ of large equibiaxial stretch, we can reduce the energy.

THEOREM 5.5. Let the stored energy function Φ satisfy constitutive hypotheses *S*1 and *S*2. Let $a > 1$ and suppose that $\hat{\alpha}: (1, \infty) \to (0, a)$ is such that both

$$0 < \inf_{\lambda > 1} [\hat{\alpha}(\lambda)\lambda^2] \quad \text{and} \quad \sup_{\lambda > 1} [\hat{\alpha}(\lambda)\lambda^2] < \infty$$

or

$$\sup_{\lambda > 1} h(\hat{\alpha}(\lambda)\lambda^2) < \infty$$

[cf. *S*2(c)]. Let λ_0 be as in proposition 5.3 and let $\mathbf{f} \in \mathrm{Kin_d}(\Omega)$ be C^1 in a neighborhood of $\mathbf{x}_0 \in \Omega$. Suppose that $\mathbf{f}$ is a sufficiently large equibiaxial stretch at $\mathbf{x}_0$ up to a rigid rotation: for some unit vector $\mathbf{e}$, some proper orthogonal tensor $\mathbf{Q}$, and some $\lambda \geqslant \lambda_0$,

$$\mathbf{Q}\nabla\mathbf{f}(\mathbf{x}_0) = \lambda\mathbf{I} + (\hat{\alpha}(\lambda) - \lambda)\mathbf{e} \otimes \mathbf{e}.$$

Then $\mathbf{f}$ is not a strong local minimizer of the total energy $E(\cdot, \Omega)$.

Proof. Suppose on the contrary that $\mathbf{f}$ is a strong local minimizer of $E(\cdot, \Omega)$. Then by theorem 4.2, we have that for every regular region $\mathscr{D} \subset \mathbb{R}^3$,

$$\int_{\mathscr{D}} W(\nabla\mathbf{f}(\mathbf{x}_0))\, d\mathbf{x} \leqslant \int_{\mathscr{D}} W(\nabla\mathbf{f}(\mathbf{x}_0) + \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x} \tag{5.23}$$

whenever $\mathbf{f}_0 + \mathbf{u} \in \mathrm{Kin}_{\mathbf{f}_0}(\mathscr{D})$. Here

$$\mathbf{f}_0(\mathbf{x}) := \mathbf{f}(\mathbf{x}_0) + \nabla\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0), \quad \mathbf{x} \in \mathscr{D}.$$

However, we can apply proposition 5.3 to show that for any $L > 0$ and any $\lambda \geqslant \lambda_0$

$$\int_{\mathscr{P}^L_{-L}} [W(\nabla\mathbf{p}^{\alpha}_{\lambda}) - W(\mathbf{Q}\nabla\mathbf{f}(\mathbf{x}_0))]\, d\mathbf{x} \leqslant -2\kappa L\lambda^q \tag{5.24}$$

where $\mathbf{p}^{\alpha}_{\lambda}$ is defined by (5.7) with R_0 fixed and κ is independent of λ and L. Note that $\mathbf{Q}^T\mathbf{p}^{\alpha}_{\lambda}$ cannot be used as a competitor in (5.23) because it does not satisfy the linear boundary conditions. We put "caps" on $\mathbf{Q}^T\mathbf{p}^{\alpha}_{\lambda}$ by defining for each $L > 0$ the function $\mathbf{f}^L : \mathscr{P}^{L+3}_{-L-3} \to \mathbb{R}^3$ by

$$\mathbf{f}^L(\mathbf{x}) = \begin{cases} \mathbf{Q}^T\mathbf{c}^{\alpha}_{\lambda}(x_1, x_2, x_3), & L \leqslant x_3 < L+3, \\ \mathbf{Q}^T\mathbf{p}^{\alpha}_{\lambda}(x_1, x_2, x_3), & -L < x_3 < L, \\ \mathbf{Q}^T\mathbf{c}^{\alpha}_{\lambda}(x_1, x_2, -x_3), & -L-3 < x_3 \leqslant -L, \end{cases}$$

[cf. (5.3), (5.5), (5.7), (5.14)–(5.16)]. Here, x_1, x_2, x_3 are components relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3 = \mathbf{e}$. It is clear from propositions 5.3 and 5.4 that $\mathbf{f}^L \in \mathrm{Def}\,(\mathscr{P}^{L+3}_{-L-3})$ and $\mathbf{f}^L(\mathbf{x}) = \nabla\mathbf{f}(\mathbf{x}_0)\mathbf{x}$, $\mathbf{x} \in \partial\mathscr{P}^{L+3}_{-L-3}$. Therefore, if we define $\mathbf{u}^L(\mathbf{x}) := \mathbf{f}^L(\mathbf{x}) - \nabla\mathbf{f}(\mathbf{x}_0)\mathbf{x}$, $\mathbf{x} \in \mathscr{P}^{L+3}_{-L-3}$, then $\mathbf{f}_0 + \mathbf{u}^L \in \mathrm{Kin}_{\mathbf{f}_0}(\mathscr{P}^{L+3}_{-L-3})$.

We now combine (3.1), (5.24), and proposition 5.4 to conclude that

$$\int_{\mathscr{P}^{L+3}_{-L-3}} [W(\nabla \mathbf{f}(\mathbf{x}_0)+\nabla \mathbf{u}^L(\mathbf{x}))-W(\nabla \mathbf{f}(\mathbf{x}_0))]\,\mathrm{d}\mathbf{x} \leqslant 2K_1-2\kappa\lambda^q L, \tag{5.25}$$

with K_1 and κ independent of L. For L sufficiently large (5.25) contradicts (5.23).

Remark 5.6. In the proof of theorem 5.5 we have fixed R_0 and we have let $L \to \infty$ in order to take advantage of the energy reducing nature of the cylindrical deformation relative to the caps. At the expense of a slightly more complicated argument, we could have fixed L and let $R_0 \to 0$. The important point is that the void must be very long and thin to force it to be energy reducing, which motivates the term "filamentary void".

Remark 5.7. Our construction of the filamentary void shows that the conclusions of theorem 5.5 are valid for certain anisotropic materials. The crucial assumptions are: that the material at the point $\mathbf{x}_0$ is transversely isotropic in the plane perpendicular to $\mathbf{e}$, and that the stored energy function W satisfies both

$$\text{(i)}\quad W(\mathbf{F}) \leqslant c(\det \mathbf{F})\left[\sum_i (\lambda_i^q+\lambda_i^{-q})+\sum_{i\neq k} (\lambda_i\lambda_k)^q\right]$$

for every $\mathbf{F}\in \mathrm{Lin}^>$ and some $q\in(0, 2)$ and $c: \mathbb{R}^> \to \mathbb{R}$, and

$$\text{(ii)}\quad W(\mathbf{F}) = \phi(\alpha)+\phi(\beta)+\phi^*(\gamma)+\psi(\alpha\beta)+\psi^*(\alpha\gamma)+\psi^*(\beta\gamma)+\chi(\alpha\beta\gamma)+\Delta(\alpha,\beta,\gamma)$$

whenever α, β, γ are the eigenvalues of $\mathbf{F}\mathbf{F}^T$, $\mathbf{F}\mathbf{F}^T\mathbf{e} = \gamma^2\mathbf{e}$, and ϕ, ψ, χ, and Δ satisfy $S1$ and $S2$.

Our description of the filamentary void as well as our proof of its energy reducing property appears to require that the underlying smooth deformation be an equibiaxial stretch at the point where the void is to be inserted. This requirement is not essential, but was made to simplify both the analysis and the conceptual description of our results. In a more general deformation it is possible to achieve a reduction of energy by introducing a filamentary void with elliptical cross section. More precisely, given $0<\alpha<\lambda_1 \leqslant \lambda_2$, let

$$\begin{aligned} \mathbf{g}(\mathbf{x}) &:= \lambda_1 x_1\mathbf{e}_1+\lambda_2 x_2\mathbf{e}_2+\alpha x_3\mathbf{e}_3,\\ \mathbf{f}(\mathbf{x}) &= r(R)\mathbf{e}_R(\mathbf{x})+x_3\mathbf{e}_3,\ R = (x_1^2+x_2^2)^{1/2}, \end{aligned} \tag{5.26}$$

be deformations of the plate $\mathscr{P}^L_{-L}$ [cf. (5.3)], where r is given by (5.6) and $\mathbf{e}_R$ is the radial unit vector [cf. (5.5)]. Then $\mathbf{g}\in \mathrm{Def}\,(\mathscr{P}^L_{-L})$, $\mathbf{g}\circ\mathbf{f}\in \mathrm{Def}\,(\mathscr{P}^L_{-L})$, and $(\mathbf{g}\circ\mathbf{f})(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ for $x_1^2+x_2^2 \geqslant R_0^2$.

Let $\mathbf{x}_0\in\mathscr{P}^L_{-L}$, $\mathbf{x}_0\times\mathbf{e}_3 \neq \mathbf{0}$, and denote the principal stretches of $\nabla(\mathbf{g}\circ\mathbf{f})(\mathbf{x}_0)$ by v_1, v_2, and v_3. Then it is not difficult to show that

$$\begin{aligned} &\lambda_1 r' \leqslant v_1 \leqslant \lambda_2 r/R, \quad v_3 = \alpha,\\ &\lambda_1 r' \leqslant v_2 \leqslant \lambda_2 r/R, \quad v_1 v_2 = \omega\lambda_1\lambda_2, \end{aligned} \tag{5.27}$$

and hence that (cf. lemma 5.2)

$$W(\nabla(\mathbf{g}\circ\mathbf{f})(\mathbf{x}_0)) - W(\nabla\mathbf{g}(\mathbf{x}_0)) \leqslant \begin{bmatrix} 8a^q\lambda_2^q(r')^{-q}(c+h(\omega\alpha\lambda_1\lambda_2)) \\ -\lambda_1\lambda_2(1-\omega)\psi'(\omega\lambda_1\lambda_2) \end{bmatrix}, \tag{5.28}$$

provided $1 < a$, $\alpha \in (0, a)$, and $\lambda_1 \in [a^{-1}, \lambda_2]$.

The energy reduction achieved by the introduction of the filamentary void $\mathbf{g}\circ\mathbf{f}$ is now clear from (5.28). For example if we let $\lambda_2 = \lambda$, $\lambda_1, = \varepsilon\lambda$, for some $\varepsilon \in (0, 1]$ and suppose that $h(\omega\alpha\varepsilon\lambda^2)$ is bounded uniformly in λ for $\lambda \geqslant 1$ then the right-hand side of (5.28) will be negative for sufficiently large λ.

Finally, a computation similar to that done in the proof of proposition 5.4 shows that the "cap" $\mathbf{g}\circ\mathbf{c}_1^L$ is contained in Def $(\mathscr{P}_L^{L+3})$ and has finite total stored energy that is independent of L. Thus we arrive at the following result.

THEOREM 5.8. Let the stored energy function Φ satisfy constitutive hypotheses $S1$ and $S2$. Let $a > 1$, $\varepsilon_0 \in (0, 1]$, and suppose that $\hat{\alpha}: (1, \infty) \to (0, a)$ is such that both

$$0 < \inf_{\lambda > 1} [\hat{\alpha}(\lambda)\lambda^2] \quad \text{and} \quad \sup_{\lambda > 1} [\hat{\alpha}(\lambda)\lambda^2] < +\infty$$

or

$$\sup_{\substack{\lambda > 1 \\ \varepsilon \in [\varepsilon_0, 1]}} h(\varepsilon\hat{\alpha}(\lambda)\lambda^2) < +\infty. \tag{5.29}$$

Then there is a $\lambda^* > 1$ such that any $\mathbf{f} \in \mathrm{Kin}_d(\Omega)$ that satisfies property LS given below is not a strong local minimizer of the total energy $E(\cdot, \Omega)$.

(LS) f is C^1 in a neighborhood of a point $\mathbf{x}_0 \in \Omega$ where the principal stretches λ_1, λ_2, and λ_3 of $\nabla\mathbf{f}(\mathbf{x}_0)$ satisfy

$$\lambda_1 \geqslant \lambda^*, \qquad \lambda_2 \in [\varepsilon_0\lambda_1, \lambda_1], \qquad \lambda_3 = \hat{\alpha}(\lambda).$$

Remark 5.9. If h is convex then hypothesis (5.29) is equivalent to (5.11).

6. AN APPLICATION. INSTABILITY OF RADIAL MINIMIZERS

BALL (1982) considered the problem of minimizing the total energy of a ball composed of a homogeneous, isotropic, compressible hyperelastic material among those deformations that are radial, orientation preserving, and satisfy given displacement or traction conditions on the boundary. In particular Ball showed that when the load on the boundary is sufficiently large, the global radial minimizer of the energy will exhibit a spherical hole at its center for a large class of stored energy functions.

Motivated by Ball's work others have shown that both radial minimizers and radial solutions of the equilibrium equations can exhibit spherical cavities under a wide range of constitutive hypotheses. However, it is still not known whether the minimizer of the energy among radial deformations is a global minimizer of the energy.

Although it may be true that the radial minimizer is indeed a global minimizer for many stored energy functions, we show in this section that a radial deformation that exhibits cavitation cannot be a global minimizer of the energy for materials that satisfy constitutive hypotheses $S1$ and $S2$ of Section 5. We also show that the constitutive hypotheses of SIVALOGANATHAN (1986a) and STUART (1985) include some polyconvex materials satisfying $S1$ and $S2$. Hence, for these materials the radial solutions that exhibit cavitation are not global minimizers of the total energy.

We now assume that the body in its reference configuration occupies the ball of radius P_0

$$\mathscr{B} := \{\mathbf{x} \in \mathbb{R}^3 ;\ |\mathbf{x}| < P_0\}.$$

A *radial deformation* of $\mathscr{B}$ is a deformation $\mathbf{f} \in \mathrm{Def}\,(\mathscr{B})$ that satisfies

$$\mathbf{f}(\mathbf{x}) = \frac{\rho(P)}{P}\mathbf{x}, \quad P = |\mathbf{x}| \neq 0, \tag{6.1}$$

for some ρ: $[0, P_0] \to \mathbb{R}$. We say that a radial deformation is *smooth* provided that $\rho \in C^0([0, P_0], \mathbb{R}) \cap C^1((0, P_0), \mathbb{R})$. A radial deformation will *exhibit cavitation* provided that

$$0 < \rho(0) = \lim_{P \to 0^+} \rho(P). \tag{6.2}$$

LEMMA 6.1 (BALL, 1982, p. 566). Let $\mathbf{f}$: $\mathscr{B} \to \mathbb{R}^3$ satisfy (6.1). Then $\mathbf{f} \in W^{1,q}(\mathscr{B})$ if and only if ρ is absolutely continuous on every closed subinterval of $(0, P_0)$ and

$$\int_0^{P_0} \left[|\dot\rho(P)|^q + \left|\frac{\rho(P)}{P}\right|^q \right] P^2 \,\mathrm{d}P < +\infty. \tag{6.3}$$

In this case the weak derivatives of $\mathbf{f}$ are given by

$$\nabla\mathbf{f}(\mathbf{x}) = \frac{\rho(P)}{P}\mathbf{I} + \left(\dot\rho(P) - \frac{\rho(P)}{P}\right)\frac{\mathbf{x}}{P} \otimes \frac{\mathbf{x}}{P} \quad \text{a.e. } \mathbf{x} \in \mathscr{B} \tag{6.4}$$

and hence the principal stretches are given by

$$|\dot\rho(P)|, \left|\frac{\rho(P)}{P}\right|, \left|\frac{\rho(P)}{P}\right|, \quad \text{a.e. } \mathbf{x} \in \mathscr{B}.$$

It follows from (6.4) that $\det \nabla f(\mathbf{x}) = \dot\rho(P)\rho^2(P)/P^2$ a.e. $\mathbf{x} \in \mathscr{B}$. Thus we will call a smooth radial deformation *orientation preserving* provided that

$$\dot\rho(P) > 0 \quad \text{for every} \quad P > 0. \tag{6.5}$$

We next note that by (6.4) almost every $\mathbf{x} \in \mathscr{B}$ in a radial deformation experiences an equibiaxial stretch. In addition, in radial deformations that exhibit cavitation this stretch becomes infinite at the center of the ball. The following result is therefore a consequence of theorem 5.5.

THEOREM 6.2. Let the stored energy satisfy constitutive hypotheses $S1$ and $S2$. Suppose that $\mathbf{f} \in \mathrm{Def}\,(\mathscr{B})$ is a smooth, orientation preserving,† radial deformation that exhibits cavitation. If there is a sequence of radii $P_i \to 0^+$ such that

$$0 < \lim_{i\to\infty} \dot{\rho}(P_i)\frac{\rho(P_i)^2}{P_i^2} < +\infty, \tag{6.6}$$

or

$$\lim_{i\to\infty} \dot{\rho}(P_i) < +\infty \quad \text{and} \quad \lim_{i\to\infty} h\left(\dot{\rho}(P_i)\frac{\rho(P_i)^2}{P_i^2}\right) < +\infty, \tag{6.7}$$

then $\mathbf{f}$ is not a strong local minimizer of the total energy $E(\cdot, \mathscr{B})$.

Proof. Equations (6.2) and (6.5) show that (6.6) implies $\dot{\rho}(P_i) \to 0$ as $i \to \infty$. Therefore (6.6) implies (6.7) and so we consider assumption (6.7) only. By (6.2) $\rho(P)/P \to +\infty$ as $P \to 0^+$ and hence there is a subsequence $P_i' \to 0^+$ of $\{P_i\}$ such that

$$\begin{gathered} P'_{i+1} < P'_i, \\ \rho(P'_{i+1})/P'_{i+1} > \rho(P'_i)/P'_i. \end{gathered} \tag{6.8}$$

Let

$$a := 1 + \sup_i \dot{\rho}(P'_i), \quad v_1 := \dot{\rho}(P'_1)\rho(P'_1)^2/(P'_1)^2$$

and define $\hat{\alpha}\colon (1, \infty) \to (0, a)$ by

$$\hat{\alpha}(\lambda) := \begin{cases} \dot{\rho}(P'_i) & \text{if} \quad \lambda = \rho(P'_i)/P'_i \\ v_1/\lambda^2 & \text{otherwise} \end{cases}. \tag{6.9}$$

Then by (6.8) $\hat{\alpha}$ is well defined and using (6.9) we find that

$$\sup_{\lambda>1} h(\hat{\alpha}(\lambda)\lambda^2) \leqslant h(v_1) + \sup_i h\left(\dot{\rho}(P_i)\frac{\rho(P_i)^2}{P_i^2}\right) < +\infty.$$

Therefore, by theorem 5.5 $\mathbf{f}$ is not a strong local minimizer.

We will next replace (6.6) or (6.7) by an alternative physical assumption, which can be more easily adapted to the analyses found in the literature. In particular, in order for a radial deformation that creates a hole to be at equilibrium we would expect, in the case of a vacuous hole, that there there are no radial forces on the surface of the hole. More precisely, let $\mathbf{f} \in \mathrm{Def}\,(\mathscr{B})$ be a smooth radial deformation that exhibits cavitation. If $\mathbf{f}$ is the global minimizer of the energy then BALL (1982) and SIVALOGANATHAN (1986a) have shown, under various constitutive hypotheses, that the radial component of the Cauchy stress is zero on the surface of the cavity. In order

† A smooth radial deformation that is not orientation preserving has infinite total energy.

to allow for cavities with contents† we will make the slightly more general assumption that the radial component of the Cauchy stress is bounded as one approaches the surface of the cavity.

We now consider a class of stored energy functions for which we are able to deduce the conditions (6.7) from the aforementioned bound on the radial component of the Cauchy stress at the surface of the cavity. This class of stored energy functions satisfies our $S1$ and $S2$ and at the same time can be further restricted to satisfy the constitutive hypotheses of other authors. We assume

$$\Phi(\alpha,\beta,\gamma) = \phi(\alpha)+\phi(\beta)+\phi(\gamma)+\psi(\alpha\beta)+\psi(\alpha\gamma)+\psi(\beta\gamma)+\delta(\alpha\beta\gamma)$$

where ϕ, ψ, and δ are C^1, nonnegative and satisfy

$G1$. ϕ is convex and $\phi(t) \leqslant c[t^q+t^{-r}]$;
$G2$. ψ is convex, $\lim_{t\to\infty}\psi(t) = +\infty$, and $\psi(t) \leqslant c[t^s+1]$;
$G3$. $\delta'(t) \to +\infty$ as $t \to +\infty$;

for some $c > 0$, $q\in[1, 2)$, $r\in(0, 2)$, and $s\in[1, 2)$. For such a material the radial component of the Cauchy stress in a radial deformation is given by

$$T(P) := P^{-2}\mathbf{x}\cdot\mathbf{T}\mathbf{x} = \frac{P^2}{\rho^2}\phi'(\dot\rho)+\frac{2P}{\rho}\psi'\left(\dot\rho\frac{\rho}{P}\right)+\delta'\left(\dot\rho\frac{\rho^2}{P^2}\right), \tag{6.11}$$

for $P\in(0, P_0)$.

THEOREM 6.3. Let the stored energy satisfy constitutive hypotheses $G1$–$G3$. Suppose that $\mathbf{f}\in \mathrm{Def}\,(\mathscr{B})$ is a smooth, orientation preserving radial deformation that exhibits cavitation and for which

$$\limsup_{P\to 0^+} |T(P)| < +\infty. \tag{6.12}$$

Then $\mathbf{f}$ is not a strong local minimizer of the total energy $E(\cdot, \mathscr{B})$.

Proof. By $G3$ there is a $\tau > 0$ such that $\delta'(t) \geqslant 0$ for all $t > \tau$. Define

$$\chi(t) := \begin{Bmatrix} 0, & 0 < t < \tau \\ \delta(t)-\delta(\tau), & \tau \leqslant t < \infty \end{Bmatrix}, \qquad h(t) := \begin{Bmatrix} \delta(t), & 0 < t < \tau \\ \delta(\tau), & \tau \leqslant t < \infty \end{Bmatrix}.$$

Then it is clear that $G1$–$G3$ imply that $S1$ and $S2$ are satisfied. We show that (6.12) together with $G1$–$G3$ imply that (6.7) is also satisfied so that the desired result follows from theorem 6.2.

Let $\mathbf{f}\in \mathrm{Def}\,(\mathscr{B})$ be a smooth orientation preserving radial deformation that exhibits cavitation and for which (6.12) is satisfied. Then it is clear from (6.2), (6.11), (6.12), and $G1$–$G3$ that

† We note that GENT and TOMPKINS (1969a, b) have studied the creation of holes in an unloaded material that has been subject to high pressure over a long period of time. They modified criterion (1.1) to account for a pressure in the hole equal to the pressure that was imposed.

$$\dot{\rho}(P) \to 0^+ \quad \text{as} \quad P \to 0^+. \tag{6.13}$$

If there is a sequence of radii $P_i \to 0^+$ such that

$$v(P_i) = \dot{\rho}(P_i)\frac{\rho(P_i)^2}{P_i^2} \geqslant \varepsilon > 0$$

then we are done since $h(v(P_i))$ is uniformly bounded. Thus we only need to consider the case

$$v(P) \to 0 \quad \text{as} \quad P \to 0^+. \tag{6.14}$$

We claim that, given (6.14), equations (6.12), (6.13), $G1$ and $G2$ imply that the constitutive function δ must satisfy

$$\liminf_{t \to 0^+} \delta'(t) > -\infty. \tag{6.15}$$

If not, (6.14) and the continuity of v yield a sequence $P_i' \to 0^+$ such that

$$\delta'(v(P_i')) \to -\infty. \tag{6.16}$$

However, (6.14) and the continuity of ρ and $\dot{\rho}$ imply that

$$a(P) := \dot{\rho}(P)\frac{\rho(P)}{P} \to 0^+ \quad \text{as} \quad P \to 0^+$$

and hence we conclude, with the aid of (6.13), $G1$ and $G2$, that $\phi'(\dot{\rho}(P_i'))$ and $\psi'(a(P_i'))$ are uniformly bounded. These uniform bounds and (6.16) contradict (6.12), i.e. for this sequence $T(P_i') \to -\infty$. Therefore (6.15) must hold.

Finally, we note that (6.15) yields for $0 < s < \sigma$ and σ sufficiently small

$$0 < \delta(s) = \delta(\sigma) + \int_s^\sigma -\delta'(t)\, \mathrm{d}t \leqslant \delta(\sigma) + k(\sigma - s),$$

which shows that $(6.7)_2$ is satisfied.

Before stating our final theorem, we quote a result of SIVALOGANATHAN (1986a) which establishes the existence of radial minimizers that open a hole.

PROPOSITION 6.4 (SIVALOGANANTHAN, 1986a). Suppose the stored energy function $\Phi \in C^3((\mathbb{R}^>)^3)$ satisfies $\Phi_{,i}(1,1,1) = 0$ as well as the following constitutive hypotheses [in the notation of SIVALOGANATHAN (1986a)]

$H1$. $\Phi_{,11}(\lambda_1, \lambda_2, \lambda_3) > 0$.
$H2$. $(\lambda_i\Phi_{,i}(\lambda_1, \lambda_2, \lambda_3) - \lambda_j\Phi_{,j}(\lambda_1, \lambda_2, \lambda_3))/(\lambda_i - \lambda_j) \geqslant 0$, $\lambda_i \neq \lambda_j$.
$H5$. $\Phi_{,1}(\lambda, a, a) \to \{+\infty; -\infty\}$ as $\lambda \to \{+\infty; 0\}$ for fixed $a \in (0, \infty)$.
$H7$. $(\Phi_{,i}(\lambda_1, \lambda_2, \lambda_3) - \Phi_{,j}(\lambda_1, \lambda_2, \lambda_3))/(\lambda_i - \lambda_j) + \Phi_{,ij}(\lambda_1, \lambda_2, \lambda_3) \geqslant 0$ for $\lambda_i \neq \lambda_j$.
$H9$. $(\lambda^2/(\lambda^3-1)^2)\Phi(\lambda^{-2}, \lambda, \lambda) \in L^1(\delta, \infty)$ for $\delta \in (1, \infty)$.
$H10$. There are constants k, $M > 0$ such that $(\Phi(\lambda, \lambda, \lambda)/\lambda^3) \geqslant M$ for $\lambda \geqslant k$.

$E1$. $\Phi(\lambda_1, \lambda_2, \lambda_3) \geqslant \Sigma_{i=1}^3 \Psi(\lambda_i)$ where
(i) $\Psi \in C^0((0, \infty), (0, \infty))$,

(ii) $\Psi(\lambda)/\lambda \to +\infty$ as $\lambda \to +\infty$,
(iii) $\Psi(\lambda) \to +\infty$ as $\lambda \to 0^+$.

*E*2. There exist constants M, $\varepsilon_0 \in (0, \infty)$ such that

$$|\Phi_{,i}(\lambda_1, \alpha_2\lambda_2, \alpha_3\lambda_3)\lambda_i| < M(\Phi(\lambda_1, \lambda_2, \lambda_3)+1) \quad \text{if} \quad |\alpha_i - 1| < \varepsilon_0, \quad i = 2, 3.$$

Then

$$I(\rho) := \int_0^1 \Phi\left(\dot{\rho}(P), \frac{\rho(P)}{P}, \frac{\rho(P)}{P}\right) P^2 \, \mathrm{d}P$$

attains its infimum on the set

$$A_\lambda := \{\rho \in W^{1,1}(0,1) : \rho(1) = \lambda, \dot{\rho} > 0 \text{ a.e.}, \rho(0) \geqslant 0\}.$$

If $\rho_\lambda \in A_\lambda$ is an absolute minimizer of I on A_λ then $\rho_\lambda \in C^0([0,1]) \cap C^2((0,1])$, and ρ_λ satisfies the corresponding Euler–Lagrange equation. Moreover, there is a $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$, ρ_λ exhibits cavitation and

$$\lim_{P \to 0^+} T(P) = 0,$$

where T is the radial component of the Cauchy stress in the radial deformation ρ_λ.

Remark 6.5. For $P_0 = 1$ lemma 6.1 implies that for any radial deformation $\mathbf{f}(\mathbf{x}) = [\rho(P)/P]\mathbf{x}$, $P = |\mathbf{x}|$, with finite stored energy,

$$\int_{\mathscr{B}} W(\nabla \mathbf{f}(\mathbf{x})) \, \mathrm{d}\mathbf{x} = 4\pi I(\rho).$$

Therefore proposition 6.4 establishes the existence of a radial minimizer.

Remark 6.6. A difficulty with the application of proposition 6.4 to our constitutive hypotheses is that the natural choice of Ψ in *E*1 is our ϕ. However, if we put $\Psi = \phi$, the condition *E*1(iii) would violate simple known sufficient conditions for polyconvexity (cf. BALL, 1977), these conditions being that ϕ and ψ are convex and increasing and δ is convex. We would clearly like to include many polyconvex and therefore elliptic stored energy functions in our hypotheses. For this purpose we note that proposition 6.4 remains valid when *E*1 is replaced by the following hypotheses:

*E*1′. $\Phi(\alpha, \beta, \gamma) \geqslant \phi(\alpha)+\phi(\beta)+\phi(\gamma)+\psi(\alpha\beta)+\psi(\beta\gamma)+\psi(\alpha\gamma)+\delta(\alpha\beta\gamma)$, where ϕ, ψ and δ belong to $C^0(\mathbb{R}^>, \mathbb{R}^\geqslant)$ and satisfy
(i) $\lim_{v \to 0^+} \delta(v) = \lim_{v \to +\infty} [\delta(v)/v] = +\infty$;
(ii) $\lim_{\lambda \to +\infty} \phi(\lambda) = +\infty$ or $\lim_{\xi \to +\infty} \psi(\xi) = +\infty$.

To see that this is true, we note that *E*1 is only used in the proof of proposition 6.4 at one place, i.e. in SIVALOGANATHAN's (1986a) proof of his proposition 4.1 which establishes the existence of minimizers of $I(\rho)$ in A_λ. An alternate proof of existence is given by BALL (1982, theorem 7.1) which can be easily adapted to the hypotheses *E*1′ and *H*1.

Finally, we use proposition 6.4 and theorem 6.3 to obtain a class of stored energy functions for which the radial minimizer is not a local minimizer among general deformations.

THEOREM 6.7. Suppose that the stored energy function is given by

$$\Phi(\alpha,\beta,\gamma) = \phi(\alpha)+\phi(\beta)+\phi(\gamma)+\psi(\alpha\beta)+\psi(\beta\gamma)+\psi(\alpha\gamma)+\delta(\alpha\beta\gamma), \tag{6.19}$$

where ϕ, ψ, and δ are C^3, nonnegative and satisfy $G1$, $G2$, and $G3$ with $c > 0$, $q\in[1,2)$, $r\in(0,3/2)$, $s\in[1,3/2)$. In addition, assume that $\Phi_{,i}(1,1,1) = 0$ and

$$G4.\quad \left.\begin{aligned} &[t\phi'(t)]' \geqslant 0,\\ &|t\phi'(\alpha t)| \leqslant c[\phi(t)+1] \text{ whenever } |\alpha-1| < c^{-1} \end{aligned}\right\};$$

$$G5.\quad \left.\begin{aligned} &[t\psi'(t)]' \geqslant 0,\\ &|t\psi'(\alpha t)| \leqslant c[\psi(t)+1] \text{ whenever } |\alpha-1| < c^{-1} \end{aligned}\right\};$$

$$G6.\quad \left.\begin{aligned} &\delta''(t) > 0, \lim_{t\to 0^+} \delta(t) = +\infty,\\ &|t\delta'(\alpha t)| \leqslant c[\delta(t)+1] \text{ whenever } |\alpha-1| < c^{-1} \end{aligned}\right\}.$$

Then there is a $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ the infimum of

$$I(\rho) := \int_0^1 \Phi\left(\dot\rho(P), \frac{\rho(P)}{P}, \frac{\rho(P)}{P}\right) P^2\, \mathrm{d}P$$

on the set $A_\lambda := \{\rho\in W^{1,1}(0,1) : \rho(1) = \lambda,\ \dot\rho > 0 \text{ a.e.},\ \rho(0) \geqslant 0\}$ is attained by a function $\rho_\lambda \in A_\lambda$ and ρ_λ exhibits cavitation. However, the radial deformation $\mathbf{f}_\lambda \in \mathrm{Def}(\mathscr{B})$, $\mathscr{B} := \{\mathbf{x} : |\mathbf{x}| < 1\}$, defined by

$$\mathbf{f}_\lambda(\mathbf{x}) := \frac{\rho_\lambda(P)}{P}\mathbf{x}, \quad P = |\mathbf{x}| < 1, \tag{6.20}$$

is not a strong local minimizer of the total energy $E(\cdot, \mathscr{B})$ on the set of $\mathrm{Kin}_{\mathbf{d}}(\mathscr{B})$ where $\mathbf{d}(\mathbf{x}) = \lambda\mathbf{x}$, $\mathbf{x}\in\bar{\mathscr{B}}$.

Remark 6.8. A simple class of stored energy functions that satisfies the hypotheses of theorem 6.7 is given by

$$\Phi(\alpha,\beta,\gamma) = c_1(\alpha^q+\beta^q+\gamma^q)+c_2((\alpha\beta)^s+(\beta\gamma)^s+(\gamma\alpha)^s)$$
$$+c_3(\alpha\beta\gamma)^l+c_4(\alpha\beta\gamma)^{-m}+c_5(\alpha\beta\gamma), \tag{6.21}$$

where $c_1 \geqslant 0$, $c_i > 0$, $i = 2, 3, 4$, $q\in[1,2)$, $s\in[1,3/2)$, $l\in(1,\infty)$, $m\in(0,\infty)$ and $c_5 = mc_4 - lc_3 - sc_2 - qc_1$. [The term $c_5\alpha\beta\gamma$ is a null Lagrangian and is included so that Φ satisfies the condition $\Phi_{,i}(1,1,1) = 0$.] The function (6.21) with the restrictions listed above is also polyconvex, quasiconvex [but not $W^{1,p}$-quasiconvex for $1 \leqslant p < 3$ (see JAMES and SPECTOR, 1991)] and rank-one convex (cf. remark 6.6).

Proof of Theorem 6.7. To prove this theorem it is sufficient to show that a function $\Phi\in C^3((\mathbb{R}^>)^3)$ of the form (6.19) satisfying $G1$–$G6$ with $q\in[1,2)$, $r\in(0,3/2)$ and

$s \in [1, 3/2)$ necessarily satisfies Sivaloganathan's hypotheses $H1$, $H2$, $H5$, $H7$, $H9$, $H10$, $E2$, and either $E1$ or $E1'$. If so, then by theorem 6.3 $\mathbf{f}_\lambda$ is not a strong local minimizer, whereas by proposition 6.4 there is a $\rho_\lambda \in A_\lambda$ which minimizes $I(\cdot)$ on A_λ, and for $\lambda > \lambda_0 \rho_\lambda$ exhibits cavitation.

Assume $G1$–$G6$ are satisfied by Φ having the form (6.19) with ϕ, ψ, and δ in $C^3(\mathbb{R}^>, \mathbb{R}^\geq)$. By $G1$, $G2$, and $G6$ we have $\phi'' \geq 0$, $\psi'' \geq 0$, $\delta'' > 0$. Hence, $H1$ is satisfied. The condition $H2$ applied to a function of the form (6.19) becomes

$$\frac{\lambda_i \phi'(\lambda_i) - \lambda_j \phi'(\lambda_j)}{\lambda_i - \lambda_j} + \frac{\lambda_k^2 [\lambda_i \psi'(\lambda_i \lambda_k) - \lambda_j \psi'(\lambda_j \lambda_k)]}{\lambda_k \lambda_i - \lambda_k \lambda_j} \geq 0, \tag{6.22}$$

for $\lambda_i \neq \lambda_j$ and $k \neq i$, $k \neq j$. Clearly, $G4$ and $G5$ imply (6.22) and therefore $H2$. By (6.19)

$$\Phi_{,1}(\lambda, a, a) = \phi'(\lambda) + 2a\psi'(\lambda a) + a^2 \delta'(\lambda a^2).$$

Thus since ϕ and ψ are convex and nonnegative $G3$ implies that the first part of $H5$ holds. The second part of $H5$ follows from the convexity of ϕ, ψ, and δ together with $G6_2$. The condition $H7$ applied to (6.19) becomes

$$\frac{\phi'(\lambda_i) - \phi'(\lambda_j)}{\lambda_i - \lambda_j} + \frac{\lambda_k^2 [\psi'(\lambda_i \lambda_k) - \psi'(\lambda_j \lambda_k)]}{\lambda_i \lambda_k - \lambda_j \lambda_k} + \lambda_i \lambda_j \psi''(\lambda_i \lambda_j) + \lambda_i \lambda_j \lambda_k^2 \delta''(\lambda_1 \lambda_2 \lambda_3) \geq 0 \tag{6.23}$$

for$\lambda_i \neq \lambda_j$ and $k \neq i$, $k \neq j$. Thus the conditions $\phi'' \geq 0$, $\psi'' \geq 0$, and $\delta'' \geq 0$ imply (6.23) and therefore $H7$. Sufficient conditions for $H9$ are

$$\begin{aligned} &\frac{\lambda^2}{(\lambda^3 - 1)^2} \phi(\lambda) \in L^1(2, \infty), \\ &\frac{\lambda^2}{(\lambda^3 - 1)^2} \phi(\lambda^{-2}) \in L^1(2, \infty), \\ &\frac{\lambda^2}{(\lambda^3 - 1)^2} \psi(\lambda) \in L^1(2, \infty), \\ &\frac{\lambda^2}{(\lambda^3 - 1)^2} \psi(\lambda^{-1}) \in L^1(2, \infty). \end{aligned} \tag{6.24}$$

However $G1$ with $q \in [1, 2)$ and $r \in (0, 3/2)$ implies $(6.24)_{1,2}$, while $G2$ with $s \in [1, 3/2)$ implies $(6.24)_{3,4}$. The conditions $\delta'' > 0$ (from $G7$) and $\delta'(t) \to +\infty$ as $t \to +\infty$ (from $G3$) and the nonnegativity of ϕ and ψ imply $H10$. Recalling remark 6.6, we verify $E1'$ rather than $E1$. Clearly, $G2$ implies $E1_{(ii)}$, while the convexity of δ yields

$$\delta(t) \geq \delta\left(\frac{t}{2}\right) + \frac{t}{2} \delta'\left(\frac{t}{2}\right). \tag{6.25}$$

We divide (6.25) by t and let $t \to +\infty$; it then follows from $G3$ that $E1'_{(i)}$ is satisfied, completing the verification of $E1'$. Finally, $E2$ follows from $G4_2$, $G5_2$, $G6_3$, the triangle inequality, and the fact that ϕ and ψ are nonnegative.

Remark 6.9. STUART (1985) gives existence theorems for radial equilibrium solutions with cavities under constitutive hypotheses that differ from those of Sivaloganathan. Stuart does not examine the stability of his solutions but he does compare the energy of the homogeneous deformation $\mathbf{h}(\mathbf{x}) = \lambda\mathbf{x}$, $\mathbf{x} \in \mathscr{B}$, with the energy of solutions $\mathbf{f}_\lambda$ that exhibit cavitation and satisfy $\mathbf{f}_\lambda(\mathbf{x}) = \lambda\mathbf{x}$, $\mathbf{x} \in \partial\mathscr{B}$. He also gives similar results for boundary conditions of pressure and dead loading, which we do not discuss here. In our terminology his existence theorem states that under his constitutive hypotheses A1–A7 (discussed below) there is a radial deformation $\mathbf{f}_\lambda(\mathbf{x}) = [\rho_\lambda(P)/P]\mathbf{x}$ that satisfies

$$\begin{gathered}\rho_\lambda \in C^0([0,1]) \cap C^2((0,1)),\\ \rho_\lambda(1) = \lambda > 0, \quad \rho_\lambda' > 0 \quad \text{on} \quad (0,1), \\ [P^2\Phi_{,1}(\rho_\lambda(P), \rho_\lambda(P)/P, \rho_\lambda(P)/P)]' = 2P\Phi_{,2}(\rho_\lambda'(P), \rho_\lambda(P)/P, \rho_\lambda(P)/P).\end{gathered} \tag{6.26}$$

Furthermore, he shows that there is $\lambda_0 > 0$ such that for $0 < \lambda \leqslant \lambda_0$

$$\rho_\lambda(P) = \lambda P, \tag{6.27}$$

while for $\lambda > \lambda_0$ there are exactly two solutions of (6.26), one given by (6.27) and the other of which exhibits cavitation and satisfies $T(P) \to 0$ as $P \to 0$. Here, $T(P)$ is the radial component of the Cauchy stress defined by (6.11).

Thus, according to theorem 6.3 Stuart's equilibrium solutions will be unstable relative to the formation of filamentary voids in the case where the stored energy function satisfies both his A1–A7 and our $S1$ and $S2$. A relatively easy special case to work out is the case of a function $\Phi(\alpha, \beta, \gamma)$ of the form (6.19) with ϕ, ψ, δ in $C^3((0,\infty))$ and satisfying $G1$, $G2$, and $G3$ with $c > 0$, $r = 0$, $q \in [1,2)$, and $s \in (1, 3/2)$. A lengthy computation shows that this stored energy function also satisfies Stuart's A1–A7 if

$J1$. $\phi' \geqslant 0$, $\phi'' \geqslant 0$,
$\phi'(t) \leqslant A + Bt^v$,
$t\phi''(t) \leqslant A + Bt^v$,

$J2$. $\psi' \geqslant 0$, $\psi'' > 0$
$t\psi''(t) \leqslant A + Bt^w$,

$J3$. $\delta''(t) > \delta_0 > 0$,
$\delta'(t) < -A/t$, $0 < t < t_0$,

for some constants $A > 0$, $B > 0$, $v \in (0, 2)$, $w \in (0, 1/2)$, and $t_0 > 0$. A simple function satisfying all the hypotheses is given by (6.21) with $s > 1$, $l \geqslant 2$, and the qualifications listed there. Note that for the function (6.21) Sivaloganathan's constitutive hypotheses are also satisfied so that in this case the cavitating equilibrium solutions found by Stuart are also radial minimizers.

7. THE FILAMENTARY VOID AND CRAZING IN GLASSY POLYMERS

We have shown that under appropriate constitutive hypotheses, a deformation of a material that at some point has principal stretches λ, λ, $\alpha(\lambda)$ with $0 < c_1 < \lambda^2\alpha(\lambda) < c_2$ and λ sufficiently large is unstable relative to the formation of a filamentary void. These constitutive hypotheses embody the idea that the material is stiff for large equibiaxial stretches and is soft for a certain isochoric family of shear deformations. The generalization of our results given at the end of Section 5 shows that a similar instability involving the formation of a filamentary void with elliptical cross-section is possible with principal stretches

$$\left.\begin{aligned} \lambda_1 &= k_1\lambda \\ \lambda_2 &= k_2\lambda \\ \lambda_3 &= \alpha(\lambda) \end{aligned}\right\}, \quad k_1 > 0, k_2 > 0, 0 < c_1 < \lambda^2\alpha(\lambda) < c_2$$

and with λ sufficiently large. The void is termed a filamentary void because our methods require that it be made long and thin in order that the energy reducing nature of the essentially cylindrical deformation compensates for the energy excess in the tip region. Our hypotheses that promote the formation of filamentary voids as $\lambda \to +\infty$ place no restriction on the dilatational behavior of the stored energy; h in $S2$ is an assignable function.

The geometry of the filamentary void suggests a possible connection with the phenomenon of crazing in polymers such as polystyrene (low molecular weight), polycarbonate, and polymethyl methacrylate. In this section we present some qualitative evidence both for and against this connection.

Many authors who have studied crazing have interpreted it in terms of a material instability. That is, it is found to be an instability that occurs when certain conditions on stress, strain, and temperature are met locally and is independent of the nature of the loading device or the boundary conditions, unlike a buckling instability or the Treloar instability (see, for example, CHEN, 1987). This is especially clear from work by DEKKER and HEIKENS (1983) who compare a variety of crazing criteria to tensile experiments on polystyrene containing either well or poorly adhering glass beads. The adherence of the beads does not directly affect their criteria although it drastically changes the local stress distribution. Our treatment of filamentary voids similarly treats them as material instabilities, this idea being directly embodied in theorem 4.2.

An early and influential treatment of crazing was given by STERNSTEIN *et al.* (1968) and later modified by STERNSTEIN and ONGCHIN (1969). They drilled 1/16″ holes in 1/2″ × 2″ × 1/31″ strips of polymethyl methacrylate and pulled them in tension. Crazes appeared in the specimen in regions adjacent to the hole; the crazes assumed curvilinear trajectories beginning at the surface of the hole. Most of the crazes occurred in two symmetric regions that were near poles of the hole that were connected by a diameter that was perpendicular to the tensile axis. STERNSTEIN *et al.* (1968) observed both the direction of the crazes and the shape of the region of predominantly crazed material. To arrive at a criterion for crazing, they calculated a linear elastic solution for the stress field near a hole in an infinite plate pulled in tension. They found that the individual crazes pointed in the direction of the principal axis of stress corresponding to the *minimum* principal stress, while the first formation of a craze was governed by the *value* of the maximum principal stress σ_1. They also inferred from the absence of crazing in compression that conditions of dilatation [tr $\mathbf{T} > 0$, cf. (3.1)] were necessary for craze formation. It is clear from their principal stress contours that various other criteria besides $\sigma_1 = \sigma_{\text{crit}}$ would also fit the shape of the crazed zone; later STERNSTEIN and ONGCHIN (1969) formulated a craze criterion based on $\operatorname{tr} \mathbf{T} := \sigma_1 + \sigma_2 + \sigma_3$ and the stress bias $\sigma_1 - \sigma_2$. Here $\sigma_3 \leqslant \sigma_2 \leqslant \sigma_1$ represent the principal Cauchy stresses. STERNSTEIN *et al.* (1968) also state "... [T]he craze always propagates transverse to the major principal stress vector, thereby maximizing the spreading stress on the tip of the craze....The requirement that the formation of a craze necessitates an anisotropic driving force is stated by eq. (15) $[\sigma_1 \neq \sigma_2, \sigma_1 \neq \sigma_3]$." These remarks and our calculation seem to reflect a similar point of view.

DEKKER and HEIKENS (1983) also compare various crazing criteria to results of their experiments on 10^{-5} m glass beads embedded in polystyrene. They find that either a criterion for the maximum principal stress or the dilatation $(\lambda_1 + \lambda_2 + \lambda_3)$ can match their experimental results; they do not examine the stress-bias criterion of STERNSTEIN and ONGCHIN, nor do they compare principal stress axes with the trajectories of the crazes. The idea that dilatation alone is a criterion for crazing seems to us to be inconsistent with the ideas of STERNSTEIN *et al.* (1969) quoted above, and also does not relate very well to our calculation, as discussed below.

In recent years a great deal of information on the microstructure of crazes has emerged, as presented for example in the review article by KRAMER (1983). It is found that the crazes observed for example by STERNSTEIN *et al.* (1969) and DEKKER and HEIKENS (1983) consist of thin bands of fibrous material with the fibers lying in a direction perpendicular to the band. As pictured by KRAMER (1983, Fig. 2c), a craze in its early stages of formation consists of many small filamentary voids. A striking photograph of these voids is shown by DONALD *et al.* (1981, Fig. 1c). Once a craze band is established, a meniscus instability similar to the Saffman–Taylor instability is regarded as a mechanism for the advance of the craze tip. This mechanism is based on surface energy considerations at the craze tip and contains no bulk energy contribution of the polymer; in the analysis the polymer is modelled as a non-Newtonian fluid. It would be interesting to investigate whether this theory has a Lyapunov function bearing some relation to our nonlinear elastic energy.

GENT (1970) proposes a rather different criterion for crazing. Noting that crazing only occurs in glassy polymers that are capable of transformation to the rubbery state, he proposes a mechanism whereby the material at a stress concentration first experiences a stress-induced phase transformation and then undergoes cavitation. Using his criterion for (spherical) cavitation [Eq. (1.1)] in the rubbery region, he then is able to account for the effect of pressure, preorientation and the presence of craze inducing liquids and vapors on crazing. The agreement between his theory and experiment, covering several very different physical phenomena, cannot be ignored. However, his cavitation criterion for the rubbery region could be replaced by a criterion for the formation of a filamentary void, rather than a spherical one.

The idea that the material undergoes a stress-induced phase transformation at a flaw could easily be built into a stored energy function. For example, stored energy functions that exhibit martensitic and other structural phase transformations have been studied by ERICKSEN (1988), JAMES (1986) and many others. In these studies, calculations of the effect of stress on transformation temperature are in excellent agreement with experiment. The hypotheses on the stored energy function $\Phi(\mathbf{F}, T)$ needed to cause a stress-induced phase transformation do not contradict our hypotheses $S1$ and $S2$, which promote the formation of a filamentary void. Thus, a single stored energy function could be defined that necessarily gives a stress-induced phase transformation at an appropriate flaw and that also promotes void formation in the transformed zone, although at this time we would be more inclined to search for the possibility of a filamentary void in this zone.

Indirect support for both Gent's hypothesis and the use of thermoelastic energy functions comes from recent work of PLUMMER and DONALD 1989a, b). They show that in some polymers that exhibit high entanglement density and low molecular weight (such as polycarbonate) crazes will heal. That is, upon heating the crazed material to above the glass transition, the crazes diappear in the optical microscope and the material recovers its original material properties. It should be mentioned that high molecular weight polystyrene, for example, heals only after very long heat treatments and may undergo chain scission, suggesting that rather different models may be appropriate. For this reason we chose to discuss PMMA, PC and low molecular weight PS in this section.

Several studies (DONALD and KRAMER, 1982; DEKKERS and HEIKENS, 1985; DONALD, 1985; PLUMMER and DONALD, 1989a, b) have focussed on the competition between crazing and shear yielding in polycarbonate, polystyrene, and polyphenylene oxide. Presumably a stored energy function could be defined that exhibited both the formation of filamentary voids and shear bands, the latter being a material instability associated with a loss of ellipticity of the stored energy function.† Such a competition does not seem to be inconsistent with Gent's hypotheses, since many of the energy function appropriate for materials that undergo stress-induced phase transformations in fact lose ellipticity at some deformation gradient.

Although the filamentary void forming deformation is not in general an energy minimizer, it is interesting to determine the energy reduction per unit volume achieved by different distributions of filamentary voids. In order to obtain more energy reduction it is clear that one should use as little cap (see Section 5.3) as possible, since there is an energy penalty proportional to the volume of the cap for each one used. More precisely, suppose that the unit cube is subject to a pure stretch with principal stretches $[\lambda, \lambda, \hat{\alpha}(\lambda)]$ with $\lambda \geqslant \lambda_0$ (see proposition 5.3). The deformation $\mathbf{f}^L$ used in theorem 5.5 creates a filamentary void with domain

$$0 < R < R_0, \quad -(L+3) < x_3 < L+3,$$

and achieves an energy reduction

$$\Delta E_1 \leqslant 2(K_1 - \kappa\lambda^q L),$$

where K_1 and κ are independent of L. If we scale $\mathbf{f}^L$ using the usual scaling laws of nonlinear elasticity, we find that the deformation $\mathbf{f}^L_\mu(\mathbf{x}) := \mu\mathbf{f}^L(\mu^{-1}\mathbf{x})$ occupies the domain

$$0 < R < \mu R_0, \quad -\mu(L+3) < x_3 < \mu(L+3)$$

and achieves an energy reduction

† Analyses of the formation of shear bands in nonlinear elasticity due to a loss of ellipticity are given by many authors, see e.g. KNOWLES and STERNBERG (1978), ABEYARATNE (1980).

$$\Delta E_\mu \leq 2\mu^3(K_1 - \kappa\lambda^q L).$$

Thus, in order to insert a periodic array of $n \times n$ filamentary voids in the unit cube we set $\mu := (2nR_0)^{-1}$ and $L := nR_0 - 3$ to achieve the energy reduction

$$\begin{aligned} n^2 \Delta E_{(2nR_0)^{-1}} &\leq 2n^2(2nR_0)^{-3}(K_1 - \kappa\lambda^q(nR_0 - 3)) \\ &= \frac{1}{4R_0^3}\left(\frac{K_1 + 3\kappa\lambda^q}{n} - \kappa\lambda^q R_0\right), \end{aligned} \tag{7.1}$$

which shows that many filamentary voids lowers the energy more than a few voids occupying the same volume. This should be contrasted with a similar calculation due to BALL (private communication; see also BALL and MURAT, 1984, proposition 2.3) which shows that a single spherical void at the center of a ball gives the same energy reduction as a ball filled with small spherical voids. The reason for this is due to the caps which scale differently than the cylindrical holes and which are regions of excess energy. The inclusion of surface energy proportional to the area of the new surface formed would contribute a positive term proportional to n to the right-hand side of (7.1) and therefore would mediate against the formation of infinitely many voids. It is interesting (but perhaps misleading) to note that crazes often consist of bands containing many fine parallel voids.

An unsatisfactory aspect of the present analysis is that the behavior of the material appears to be crucially dependent on the growth of the stored energy function for arbitrarily large stretches and this growth can never be directly examined experimentally. (This point is also discussed BALL, 1982.) This feature is really a defect of the analysis. That is, it is extremely likely that all materials satisfy "slow" growth hypotheses for large stretches and therefore permit the formation of various kind of spherical, elliptical, and filamentary voids. For example, Cauchy's molecular theory of elasticity gives the stored energy function

$$\Phi(\mathbf{F}) = \frac{m}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3} \sum_{n^1, n^2, n^3} \phi(|n^i \mathbf{F} \mathbf{e}_i|), \tag{7.2}$$

where $\mathbf{F}$ is the deformation gradient $(\mathbf{e}_i)$ are reference lattice vectors, m is the mass of an atom, and the sum is taken over all triples of integers. A popular choice of the central force potential ϕ gives $\phi(r) \sim r^{-6}$, for large r, which implies that for every fixed $\mathbf{F} \in \mathrm{Lin}^>$

$$\Phi(t\mathbf{F}) \to 0 \quad \text{as} \quad t \to \infty. \tag{7.3}$$

It is then realized that at sufficiently large stretches voids may occur (as well as the possibility of shear instability)† and the other features of the stored energy become important in deciding which instability comes first. For example, GENT and LINDLEY'S (1958) criterion for spherical cavitation, $P_{\text{crit}} = 5E/6$, only depends on the single easily accessible constant E, the Young's modulus, but behind this criterion is the assumption

† Cauchy's energy (7.2) fails the condition of strong ellipticity, as discussed, for example, by ERICKSEN (1977).

that the stored energy has a simple form (verified experimentally for moderate stretches) that satisfies the crucial growth hypothesis

$$\int_1^\infty \left[\frac{d\hat{\Phi}/dv}{v^3 - 1}\right] dv < \infty,$$

where $\hat{\Phi}(v) = \Phi(v^{-2}, v, v)$ and the material is assumed to be incompressible.

A final objection against the present approach could be the failure to account for viscoelastic properties of the polymer, particularly for T near T_g, the glass transition temperature, although many of the above mentioned criteria for crazing have relied on interpreting the experiments in terms of pure elasticity. This objection does not appear particularly difficult in that most viscoelasticity theories have a Lyapunov function consisting of a nonlinear elastic energy. If it is possible to reduce this energy by inserting a void, then it is likely that the associated viscoelasticity theory will have a dynamic solution† involving the creation and evolution of a similar void. Of course, verifying this property for general theories of viscoelasticity involves great mathematical difficulties. In special cases, however, this hope is often borne out, for example, in the analysis by CALDERER (1986) of the radial motion of a viscoelastic sphere containing a void. Her principal hypothesis is a slow growth assumption on the underlying elastic energy, and under sufficiently large imposed hydrostatic tension a pre-existing hole grows suddenly to infinite size. The physical interpretation of the nonlinear elastic/viscoelastic relation described here is that under appropriate time independent boundary conditions, a void or filamentary void eventually appears.

In summary, the formation of a filamentary void warrants further study as a possible explanation for crazing in glassy polymers. To test the idea, critical conditions for the formation of a filamentary void must be determined and meaningful stored energy functions for polymers must be found. Ideally, these functions would also model the glass transition (including the observed effect of stress on transition temperature) so a direct connection with GENT'S (1970) proposal could be established. Measurements of mechanical properties under carefully controlled conditions of multiaxial stress are particularly important. The difficulty with obtaining critical conditions for void formation from the theory is that the filamentary void has a complex geometry, particularly near the tip of the void. With a leap of faith, we would guess that in the equibiaxial case such critical conditions might turn out to be

$$\lambda^\alpha_{\text{crit}}, \lambda^\alpha_{\text{crit}}, \alpha,$$

where $\lambda^\alpha_{\text{crit}}$ is the critical condition for *circular* void formation in the radial deformation of a two-dimensional elastic material defined by

$$\Phi^\alpha(\lambda_1, \lambda_2) = \Phi(\lambda_1, \lambda_2, \alpha).$$

This guess is mainly based upon the observation in theorem 5.5 that the energy contribution of the caps scales away for long thin voids. The formation of a filamentary void also seems to be possible (under different constitutive hypotheses) when λ_1 and λ_2 are fixed and $\alpha > \alpha_{\text{crit}}$. A further speculation is that essentially the same critical

† PERICAK-SPECTOR and SPECTOR (1988) show that the equations of dynamic nonlinear elasticity have a time-dependent solution involving the creation and evolution of a spherical void.

conditions would be obtained for a sufficiently small preexisting filamentary void. This speculation is based on the fact that a similar connection between the creation of spherical voids and the growth of a preexisting spherical void has been established by SIVALOGANATHAN (1986a). A criterion such as the one given above would seem to indicate that commonly quoted criteria such as a "dilatational stress field" or "adequate free volume" are not sufficient conditions for crazing. Finally, a potential advantage of the present approach is the same advantage enjoyed by GENT and LINDLEY's (1958) criterion for spherical voids—that conditions for instability are only dependent on measureable nonlinear elastic properties of the solid.†

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† Another advantage is the accessibility of the present approach to comprehensive parameter studies which can be directly compared with experiment. For example, the effect of superimposed hydrostatic or nonhydrostatic pressure, and of changes in the linearized moduli (Young's modulus and Poisson's ratio) on the critical conditions for the formation of a filamentary void can be studied.

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