Analysis of Large Scale Parameter-Varying Systems by Using Scaled Diagonal Dominance

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***Abstract*— It has been recognized recently that semidefinite optimization problems (SDP) can be cast into second order cone programs (SOCP) by replacing the positive definiteness constraints with stronger, scaled diagonal dominance (SDD) conditions. Since the scalability of SOCP solvers is much better than that of the SDPs, the new formulation allows solving large dimensional problems more efficiently. However, scaled diagonal dominant matrices form only a subset of the positive definite matrices. Hence, the new problem formulation results in more conservative solutions. This paper analyses the efficiency and conservativeness of the SDD formulation on two particular problems: the stability analysis and induced $\mathcal{L}_2$ gain computation for linear parameter-varying systems. In the paper some important features of the SDD formulation are revealed and numerical examples are provided to demonstrate the efficiency of the approach.**

I. INTRODUCTION

This paper considers the analysis of large scale linear parameter-varying (LPV) systems. LPV systems are a class of linear systems where the state matrices depend on (measurable) time-varying parameters. By large scale, both a high number of states and/or a high number scheduling parameters is meant. Such large scale system naturally arise in many applications, for instance aeroelastic vehicles [1] or wind turbines [2].

Many analysis and controller synthesis problems for LPV systems can be formulated as convex optimization problems involving linear objective functions and positive definiteness constraints in form of linear matrix inequalities (LMIs). These problems can be readily solved using semidefinite programming (SDP) solvers. While SDPs are powerful tools with many appealing features that have been successfully applied to many problems in the past, they do in general not scale well with the problem size. Recently, an approach to approximate large scale SDPs by solving linear (LP) or second order cone programs (SOCP) has been proposed in [3], [4], [5]. The key idea behind the approximation is that any SDP can be converted into an LP (or SOCP) by replacing every positive definiteness constraint by a more restrictive diagonally dominant (or scaled diagonally dominant) condition. By converting the problem into an LP or SOCP specialized solvers can be used which scale much better with the problem dimension than SDP solvers. A brief review of this approach is given in Section II. Note that any (scaled) diagonally dominant matrix is also positive definite so that the solution of the LP or SOCP respectively is a feasible solution of the original SDP. The reverse is in general not true, so that the approximation leads to conservative results.

In this paper, we focus on the problems of stability analysis and induced $\mathcal{L}_2$-gain computation for LPV systems, see Section III. These problems can lead to computationally demanding optimization tasks if considering large scale systems. The paper has two main contributions. First, theoretical results are presented in Section IV that show that for certain problems and structures of linear time invariant (LTI) systems there is no conservatism between the SDP and the SOCP. The results for LTI systems are used to gain some intuition for LPV systems and an iterative algorithm is proposed to reduce the gap. Second, numerical examples both for LTI and LPV systems are given in Section V to show the efficiency of the proposed approach. Focusing on the stability analysis and induced gain computation problems, important features of the scaled diagonally dominant formulation are pointed out, which help understanding the relation between the two formulations.

II. BACKGROUND

A. Notation

The notations used in the paper are fairly standard. The set of $n\times k$ dimensional real matrices is denoted by $\mathbb{R}^{n\times k}$. If $M\in\mathbb{R}^{n\times k}$ then we refer to its entries and sub-blocks by m_{ij} and M_{ij}, respectively. The notation $M=[m_{ij}]=[M_{ij}]$ is used to emphasize that M is built up from entries m_{ij} and sub-matrices M_{ij}. The sets of symmetric positive definite and semi-definite matrices are denoted by $\mathcal{PD}$ and $\mathcal{PSD}$, respectively. If the dimension is also important then it is indicated in the top-right index, e.g. $\mathcal{PD}^n$.

B. Scaled Diagonally Dominance

This section provides formal definition of diagonally dominant and scaled diagonally dominant matrices. In addition, an approach to convert a SDP into a SOCP as proposed in [4] is presented. The following definitions were taken from [4] and [6]. First the notion of diagonally dominance is defined in the following way:

Definition 1. A matrix $M\in\mathbb{R}^{n\times n}$ is called diagonally dominant, denoted by $M\in\mathcal{DD}$, if $m_{ii}>\sum_{j,j\neq i}^{n}|m_{ij}|$ holds[1] for all $i=1,\dots,n$.

Based on the previous definition, scaled diagonally dominance can be defined.

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[1]In this paper we define the diagonal dominance by strict inequality, because we would like to exclude the semidefinite solutions in the forthcoming problems.

Definition 2. A matrix M is scaled diagonally dominant, denoted by $M \in \mathcal{SDD}$, if there exists a diagonal scaling matrix $S = \mathrm{diag}(s_{ii}, \ldots, s_{nn})$, such that $s_{ii} > 0$, $\forall i$ and $MS \in \mathcal{DD}$.

It is clear from the definition that $\mathcal{DD}^n \subseteq \mathcal{SDD}^n$. Specifically if a matrix $M \in \mathcal{DD}^n$, it is trivial to show that $M \in \mathcal{DD}^n$ by setting $S = I$ in Definition 2. By using the theorems on eigenvalue localization and Gershgorin disks, it can also be proved that $\mathcal{SDD}^n \subseteq \mathcal{PD}^n$, i.e. any scaled diagonal dominant matrix is also positive definite. To simplify the derivations we introduce the following notation: for a matrix M, the fact that $-M \in \Omega$ with $\Omega \in \{\mathcal{DD}, \mathcal{SDD}, \mathcal{PD}, \mathcal{PSD}\}$ will be denoted by $M \in -\Omega$.

Using the relations above, it was shown in Theorem 3.1 in [3] that by relaxing the PD constraints by SDD constraints, an SDP can be cast into a SOCP problem. A feasible solution of the SOCP will still be a feasible solution of the original SDP. This follows from the following Theorem that provides a definition for *symmetric* SDD matrices.

Theorem 1 ([3]). Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, i.e. $M = M^T$. M is scaled diagonally dominant if it can be decomposed in the following way:

$$M = \sum_{i=1,j>i}^{i=n,j=n} M^{(ij)}, M_{2\times 2}^{(ij)} := \begin{bmatrix} m_{ii}^{(ij)} & m_{ij}^{(ij)} \\ m_{ji}^{(ij)} & m_{jj}^{(ij)} \end{bmatrix} \succ 0 \quad (1)$$

where for all (i, j), $M^{(ij)} \in \mathbb{R}^{n\times n}$ is a symmetric matrix with all entries zero, except the elements $m_{ii}^{(ij)}$, $m_{ij}^{(ij)}$, $m_{ji}^{(ij)}$, $m_{jj}^{(ij)}$ and $M_{2\times 2}^{(ij)}$ is a 2×2 matrix that contains the nonzero entries of $M^{(ij)}$.

The following example shall illustrate the decomposition.

Example 1. Let $M \in \mathbb{R}^{3\times 3}$ be a symmetric SDD matrix, i.e. $M = M^T$. Then the following decomposition exists:

$$M = \underbrace{\begin{bmatrix} m_{11}^{(12)} & m_{12}^{(12)} & 0 \\ m_{21}^{(12)} & m_{22}^{(12)} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{M^{(12)}} + \underbrace{\begin{bmatrix} m_{11}^{(13)} & 0 & m_{13}^{(13)} \\ 0 & 0 & 0 \\ m_{31}^{(13)} & 0 & m_{33}^{(13)} \end{bmatrix}}_{M^{(13)}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & m_{22}^{(23)} & m_{23}^{(23)} \\ 0 & m_{32}^{(23)} & m_{33}^{(23)} \end{bmatrix}}_{M^{(23)}} \quad (2)$$

For this decomposition the 2×2 matrices have the form

$$M_{2\times 2}^{(12)} = \begin{bmatrix} m_{11}^{(12)} & m_{12}^{(12)} \\ m_{21}^{(12)} & m_{22}^{(12)} \end{bmatrix} \qquad M_{2\times 2}^{(13)} = \begin{bmatrix} m_{11}^{(13)} & m_{13}^{(13)} \\ m_{31}^{(13)} & m_{33}^{(13)} \end{bmatrix}$$
$$M_{2\times 2}^{(23)} = \begin{bmatrix} m_{22}^{(23)} & m_{23}^{(23)} \\ m_{32}^{(23)} & m_{33}^{(23)} \end{bmatrix}. \quad (3)$$

Theorem 1 states that checking whether a symmetric matrix is SDD amounts to checking whether a set of 2×2 matrices is positive definite. Since the positive definiteness of a 2 by 2 symmetric matrix is equivalent to a rotated quadratic cone constraint, (e.g. in (1), $M_{2\times 2}^{(ij)} \succ 0$ is equivalent to $m_{ii}^{(ij)} > 0$, $m_{jj}^{(ij)} > 0$, $m_{ii}^{(ij)} m_{jj}^{(ij)} - (m_{ij}^{(ij)})^2 > 0$). Thus the condition $M = M^T \in \mathcal{SDD}^n$ can be checked by a second order cone program (SOCP). If in any optimization problem, every LMI constraint $M \in \mathcal{PD}$ is replaced by the stronger, scaled diagonally dominance condition $M \in \mathcal{SDD}$ then the problem can be solved more efficiently by a SOCP solver. As this reformulation restricts the search space to a subset of positive definite matrices, more conservative solutions can be expected. The efficiency vs. conservativeness has to be always analyzed before recommending this approach for a specific problem class.

C. Linear Parameter Varying Systems

Linear parameter varying (LPV) systems are a class of systems whose state space matrices depend on a time-varying parameter vector $\rho : \mathbb{R}^+ \to \mathbb{R}^m$. The parameter is assumed to be a continuously differentiable function of time and admissible trajectories are restricted both in magnitude and rates such as

$$\underline{\rho}_i \leq \rho_i(t) \leq \overline{\rho}_i, \quad \underline{\mu}_i \leq \dot{\rho}_i(t) \leq \overline{\mu}_i, \ \forall t, \ i = 1, \ldots, m, \quad (4)$$

where $\underline{\rho}, \overline{\rho}, \underline{\mu}, \overline{\mu} \in \mathbb{R}^m$ are known, constant vectors. The set of all parameter vectors $p \in \mathbb{R}^m$ satisfying the magnitude bounds in (4) will be denoted by $\mathcal{P}$. The set of admissible parameter trajectories $\rho : \mathbb{R}^+ \to \mathcal{P}$ satisfying the rate limits in (4) will be denoted by $\mathcal{A}$.

The system matrices of an LPV system $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ $D(\cdot)$ are assumed to be continuous functions of the parameter: $A : \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$, $B : \mathbb{R}^m \mapsto \mathbb{R}^{n\times p}$, $C : \mathbb{R}^m \mapsto \mathbb{R}^{q\times n}$, $D : \mathbb{R}^m \to \mathbb{R}^{q\times p}$. An n^{th} order LPV system G_ρ is given by

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))w(t) \\ z(t) &= C(\rho(t))x(t) + D(\rho(t))w(t) \end{aligned} \quad (5)$$

Throughout the paper the explicit dependence on t is occasionally suppressed to shorten the notation.

Many analysis and control synthesis problems related to LPV system can be formulated as semidefinite programs [7], [8]. There are several special classes of LPV systems that are categorized based on how the state matrices depend on the scheduling parameters. One special class assumes the state matrices of the LPV system have a rational dependence on the parameters. In this case finite dimensional semidefinite programs (SDPs) can be formulated to synthesize/analyze LPV controllers [9], [10], [11]. Another class of LPV systems assumes the state matrices have an arbitrary dependence on the parameters. The controller synthesis/analysis problem for this class of systems leads to an infinite collection of parameter dependent linear matrix inequalities (LMIs) [8].

This paper focuses on the latter class of LPV system. Two particular problems are considered in the paper. Specifically these are the stability analysis and the computation of the induced $\mathcal{L}_2$ norm. Note that while the papers restricts itself to a specific class of LPV systems and only two particular problems, the results can be easily extended to handle more problems and different classes.

First, quadratic stability of (5) is considered. This problem requires to find a Lyapunov function $V(x, \rho) > 0$ such that $\dot{V}(x, \rho) < 0$. If $V(x, \rho)$ is chosen to be quadratic, i.e. $V(x, \rho) := x^T P(\rho)x$, where $P(\rho) = P_0 + \sum_{i=1}^{F} f_i(\rho)P_i$ and $f_i(\cdot)$ are a priori fixed, differentiable basis functions, then the stability analysis problems can be formulated by an SDP, see [12], [8]. The following Lemma gives a sufficient condition for stability of an LPV system.

Lemma 1. An LPV system G_ρ is exponentially stable if there exists $P_0, P_1, \dots, P_F$ such that for all $\rho \in \mathcal{A}$

$$P(\rho) = P_0 + \sum_{i=1}^{F} f_i(\rho) P_i \in \mathcal{PD}$$
$$N(\rho,\dot\rho) := \frac{\partial P(\rho)}{\partial \rho}\dot\rho + A^T(\rho)P(\rho) + P(\rho)A(\rho) \in -\mathcal{PD} \tag{6}$$

The induced $\mathcal{L}_2$ norm of an exponentially stable LPV system is defined as follows:

$$\|G_\rho\| := \sup_{\rho(\cdot)\in\mathcal{A}} \sup_{w\neq 0, w\in\mathcal{L}_2} \frac{\|z\|}{\|w\|}$$

where $\|\cdot\|$ on the right side denotes the $\mathcal{L}_2$ signal norm, i.e. $\|w\|^2 = \int_0^\infty w(t)^T w(t)\ dt$. Similar to Lemma 1, sufficient conditions can be formulated to upper bound the induced $\mathcal{L}_2$ gain of an LPV system. This corresponds to a generalization of the LTI Bounded Real Lemma. The next Lemma states the condition provided in [8] to bound the $\mathcal{L}_2$ gain of an LPV system.

Lemma 2. An LPV system G_ρ is exponentially stable and $\|G_\rho\| < \gamma$ if there exists $P_0, P_1, \dots, P_F$ such that for all $\rho \in \mathcal{A}$

$$P(\rho) \in \mathcal{PD}$$
$$M(\rho,\dot\rho) := \begin{bmatrix} N(\rho,\dot\rho) & P(\rho)B(\rho) & C(\rho)^T \\ B(\rho)^T P(\rho) & -\gamma I & D(\rho)^T \\ C(\rho) & D(\rho) & -\gamma I \end{bmatrix} \in -\mathcal{PD} \tag{7}$$

III. PROBLEM FORMULATION

Lemma 1 and Lemma 2 do not provide numerically tractable conditions in this form, because they involve infinite number of constraints. Therefore, let Γ be a suitable fine grid over the $\mathcal{P}$, i.e. let $\Gamma := \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_m$ where $\Gamma_i := \{\rho_{i,1} = \underline{\rho}_i,\ \rho_{i,2},\ \ldots,\ \rho_{i,N_i} = \overline{\rho}_i\}$ with $N_i \gg 1$ and $\rho_{i,j} < \rho_{i,j+1}$ for all $1 \le i \le m$ and $1 \le j \le N_i$. Moreover, let $\Phi := \Phi_1 \times \Phi_2 \times \ldots \times \Phi_m$ where $\Phi_i := \{\underline{\mu}_i, \overline{\mu}_i\}$ Then the infinite number of constraints in Lemma 1 and Lemma 2 can be converted to a finite set by replacing ρ and $\dot\rho$ in $P(\rho)$, $N(\rho,\dot\rho)$ and $M(\rho,\dot\rho)$ by the elements of $\Gamma \times \Phi$. The total number of the constraints is $2^m \prod_{i=1}^m N_i$. It is clear that the finite LMI constraints better approximate the original problem if the grid is dense, i.e. N_i is "large". On the other hand, refining the grid involves significant complexity increase.

It is clear that the number of LMI conditions generated in Lemma 1 and Lemma 2 can be extremely large if a lot of scheduling variables and/or dense parameter grids are considered. In addition, the number of decision variables in the SDP grows rapidly with the number of states of G_ρ and the number of considered basis functions. Hence, the SDP problems are very computationally demanding for large scale LPV systems. Using the ideas from the previous section, the conditions $P(\rho), -N(\rho,\dot\rho), -M(\rho,\dot\rho) \in \mathcal{PD}$ can be relaxed to $P(\rho), -N(\rho,\dot\rho), -M(\rho,\dot\rho) \in \mathcal{SDD}$. This effectively converts the constraints in Lemma 1 and Lemma 2 into second order cones, which are numerically cheaper, albeit at the cost of conservatism.

The paper is looking for answers to the following questions. Assuming that (5) is quadratically stable, when do (6) and (7) have solutions with the stronger conditions $P, -N, -M \in \mathcal{SDD}$? Is there a state transformation such that these problems are feasible for the transformed system? Let γ_{SDD} and γ_{PD} be denote the minimal γ values satisfying (7) in case of SDD and semidefinite relaxations, respectively. Since $\mathcal{SDD} \subset \mathcal{PD}$ then it is clear that $\gamma_{SDD} \ge \gamma_{PD}$. How large is the gap between the two upper bounds? If $\gamma_{SDD} \gg \gamma_{PD}$, is it possible to construct an (iterative) algorithm that guarantees $\gamma_{SDD}[k] \to \gamma_{PD}$, where $\gamma_{SDD}[k]$ denotes the solution of Problem (7) at the consecutive iteration steps?

IV. MAIN RESULTS

In this section we consider first the stability and $\mathcal{L}_2$ gain analysis problems for linear, time-invariant systems. Some interesting theoretical results are presented. Then we examine how these results can be extended to the parameter-varying case.

A. LTI systems

Assume that an LTI system is given in the following state-space form:

$$\dot x = Ax + Bw \qquad z = Cx + Dw. \tag{8}$$

Note that an LTI system can be seen as a special case of an LPV system by fixing ρ to a constant value. As such, Lemma 1 and Lemma 2 are applicable in a simplified form to the LTI case. Specifically, P, N and M are only constant matrices in this case.

First, we prove two simple lemmas, that will be needed in the forthcoming derivations.

Lemma 3. If $M = [M_{ij}]$ is block triangular, then $M \in \mathcal{SDD}$ if $M_{ii} \in \mathcal{SDD}$, i.e. if the diagonal blocks are scaled diagonally dominant.

Proof. Assume M is upper block triangular and for simplicity, let it be a 3×3 block matrix:

$$M = \begin{bmatrix} M_1 & N_{12} & N_{13} \\ & M_2 & N_{23} \\ & & M_3 \end{bmatrix}$$

Let S_i be the positive definite, diagonal scaling matrices satisfying $M_i S_i \in \mathcal{DD}$. Define $\overline{S} = \operatorname{diag}(\lambda_1 S_1, \lambda_2 S_2, \lambda_3 S_3)$. Then

$$MS = \begin{bmatrix} \lambda_1 M_1 S_1 & \lambda_2 N_{12} S_2 & \lambda_3 N_{13} S_3 \\ & \lambda_2 M_2 S_2 & \lambda_3 N_{23} S_3 \\ & & \lambda_3 M_3 S_3 \end{bmatrix}.$$

It is clear, if λ_3 is chosen s.t. $\lambda_3 N_{23} S_3$ is "small", i.e. it does not destroy the diagonal dominance of $\lambda_2 M_2 S_2$; and λ_2, λ_3 are chosen such that both of $\lambda_2 N_{12} S_2$ and $\lambda_3 N_{13} S_3$ are "small" compared to $\lambda_1 M_1 S_1$ then MS will be diagonally dominant. Note that while the proof was only given for the case 3×3, it trivially extends to the $n \times n$ case. □

Lemma 4. If A is a stable, 2×2 matrix with a complex eigenvalue pair $r \pm sj$, then there exists a diagonal Lyapunov matrix $P \succ 0$ such that $A^T P + PA \prec 0$ and diagonal.

Proof. It follows from the characteristic equation that $a_{11} < 0$, $a_{22} < 0$ and $a_{12}a_{21} < 0$. If $P = \mathrm{diag}(p_1, p_2) \succ 0$ then

$$A^T P + PA = \begin{bmatrix} 2p_1 a_{11} & p_2 a_{21} + p_1 a_{12} \\ \star & 2p_2 a_{22} \end{bmatrix}$$

Since a_{11}, a_{22} are negative and a_{12}, a_{21} have opposite sign, if p_1 and p_2 are chosen s.t. $p_2 a_{12} + p_1 a_{21} = 0$ then $A^T P + PA$ will be a diagonal negative definite matrix. □

The following two lemmas provide sufficient conditions for the feasibility of (6).

Lemma 5. If $A \in -\mathcal{SDD}$ is an irreducible or block triangular matrix then there exist diagonal state transformation F and a diagonal Lyapunov function $\tilde{P}$ such that $\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} \in -\mathcal{DD}$, where $\tilde{A} = F^{-1}AF$.

Proof. Since $A \in -\mathcal{SDD}$ and A is irreducible or block triangular then by Theorem 2 of [13], $A^T \in -\mathcal{SDD}$, as well. Consequently, there exist diagonal, positive definite matrices F, E s.t. $AF \in -\mathcal{DD}$ and $A^T E \in -\mathcal{DD}$. Applying the state transformation defined by F, let $\tilde{A} := F^{-1}AF$. Since multiplication from left by a diagonal matrix does not affect the diagonal dominance, so $\tilde{A} \in -\mathcal{DD}$. If $\tilde{P} := FE$, then

$$\begin{aligned} \tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} &= FA^T F^{-1}(FE) + (FE)F^{-1}AF \\ &= FA^T E + EAF \end{aligned}$$

where both $FA^T E$ and EAF are in $-\mathcal{DD}$. Consequently, the sum is also diagonally dominant. □

The trivial consequence of this lemma is that if $A \in -\mathcal{DD}$ then there exists a diagonal, positive definite P, such that $A^T P + PA \in -\mathcal{DD}$.

Lemma 6. If A is stable then there exists a state transformation T such that there exists a diagonal matrix $\tilde{P}$ for the transformed system matrix $\tilde{A} = T^{-1}AT$ satisfying $\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} \in -\mathcal{DD}$.

Proof. We can assume that A is in upper block triangular form with maximum 2 by 2 blocks along the diagonal. If this is not the case we can apply the modal transformation (see e.g. chapter 2.5 in [14]), which transforms A to modal canonical form: the transformed matrix is upper block triangular and every diagonal block corresponds to a pole or complex pole pair. Since A is block triangular, due to Lemma 3, it is enough to focus on the 2 by 2 diagonal blocks. These blocks are $\mathcal{DD}$ or not. In the latter case, e.g. if for some k, $A_{kk} \notin -\mathcal{DD}$ then by Lemma 4 there always exist a diagonal, positive definite matrix P_k s.t. $A_{kk}^T P_k + P_k A_{kk}$ is diagonal and negative definite. For simplicity, consider the case, when A is a 3 by 3 block-matrix with diagonal blocks $A_{ii} \in \mathbb{R}^{2\times 2}$ and off-diagonal blocks B_{ij} $i, j = 1 \ldots 3$, $i < j$. Assume $A_{11}, A_{33} \in -\mathcal{DD}$ and $A_{22} \notin -\mathcal{DD}$. Then it can be easily checked that the matrix

$$\hat{A} = \begin{bmatrix} A_{11} & B_{12} & B_{13} \\ & \frac{1}{2}(A_{22}^T P_2 + P_2 A_{22}) & P_2 B_{23} \\ & & A_{33} \end{bmatrix}$$

is in $-\mathcal{SDD}$ and the scaling matrix rendering $\hat{A}$ diagonally dominant has the form $\hat{F} = \mathrm{diag}(\lambda_1 I, \lambda_2 I, \lambda_3 I)$. Since for $\hat{A}$ the row and column dominance are equivalent, thus $\hat{A}^T$ is also in $-\mathcal{SDD}$ and its scaling has the structure $\hat{E} = \mathrm{diag}(\gamma_1 I, \gamma_2 I, \gamma_3 I)$. Now we can apply Lemma 5. Due to the structure of $\hat{F}$ and $\hat{E}$, if $\hat{P}$ denotes the diagonal Lyapunov matrix proposed by Lemma 5, i.e. $\hat{P} = \hat{F}\hat{E}$, then

$$\begin{aligned} \hat{A}^T \hat{P} + \hat{P}\hat{A} &= \hat{F}\hat{A}^T \hat{F}^{-1}(\hat{F}\hat{E}) + (\hat{F}\hat{E})\hat{F}^{-1}\hat{A}\hat{F} \\ &= \hat{F}\hat{A}^T \hat{E} + \hat{E}A\hat{F} = \tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} \end{aligned}$$

where $\tilde{P} = \hat{P} \cdot \mathrm{diag}(I, P_{22}, I)$ and $\tilde{A} = \hat{F}^{-1}A\hat{F}$. □

The next results prove that the feasibility of (6) implies the feasibility of (7).

Lemma 7. If for some stable LTI system there exists $P \in \mathcal{SDD}$ s.t. $A^T P + PA \in -\mathcal{SDD}$ then there exists $\gamma > 0$ such that

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \in -\mathcal{SDD}$$

Proof. Let R be a positive diagonal matrix such that $(A^T P + PA)R \in -\mathcal{DD}$. Assume that S, Z are positive diagonal matrices of suitable dimension and consider the following matrix product:

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \begin{bmatrix} R & & \\ & S & \\ & & Z \end{bmatrix} = \\ = \begin{bmatrix} (A^T P + PA)R & PBS & C^T Z \\ B^T PR & -\gamma S & D^T Z \\ CR & DS & -\gamma Z \end{bmatrix}$$

It is clear if we choose S and Z so that the off-diagonal entries are 'small' then we can pick a suitable large γ s.t. the diagonal entries γS and γZ are 'larger' than the off diagonal blocks. □

Lemma 7 guarantees only the feasibility of (7), but does not say anything about the gap between γ_{SDD} and the optimal γ_{PD}. Numerical examples prove that γ_{SDD} is, in general, significantly larger than γ_{PD}. We propose, therefore, an iterative algorithm, which improves the γ_{SDD} bound by applying a "diagonalizing" transformation on the P and M matrices in each step. The transformation is determined from the previous solution of (7). The algorithm can be formulated precisely as follows:

1) Let $R[1] = I$ and $S[1] = I$, $\gamma_{SDD}[0] = \infty$, $\kappa = 1$
2) Minimize γ with respect to constraints $R[\kappa]^T PR[\kappa] \in \mathcal{SDD}$ and $S[\kappa]^T MS[\kappa] \in -\mathcal{SDD}$, where P, M are defined in (7). Let the result be denoted by $\gamma_{SDD}[\kappa]$.

3) if $\gamma_{SDD}[\kappa-1] - \gamma_{SDD}[\kappa] \leq \epsilon$ then STOP, otherwise continue to step 4.
4) Determine $P[\kappa]$ and $M[\kappa]$ matrices and let $R[\kappa+1]$ and $S[\kappa+1]$ be the unitary matrices diagonalizing $P[\kappa]$ and $M[\kappa]$. These matrices can be determined by spectral decomposition. Set $\kappa := \kappa + 1$ and go to step 2.

B. LPV systems

Most theorems presented in the previous section cannot be extended easily to LPV systems. The LTI results, however, help understanding the properties of the SDD relaxation and give hints how to treat the parameter varying case.

Considering the quadratic stability problem, the results of Lemma 5 remains applicable for the parameter varying case if the system matrix $A(\rho)$ satisfies the following properties: $A(\rho) \in -\mathcal{SDD}$ for all $\rho \in \mathcal{P}$ and the associated diagonal scaling matrices E, F are *parameter independent*. If these conditions hold then the derivations used to prove Lemma 5 can be repeated. For a general LPV system the conditions above may prove to be too restrictive. Even if this is the case, i.e. there does not exist constant E, F rendering $A(\rho)$ diagonally dominant, it is still reasonable to find a suitable (parameter-dependent) state transformation, which transforms $A(\rho)$ to a diagonal or upper block triangular forms. If the matrix is diagonally structured it is much easier to find solution to the stability problem (6). Finding strict conditions for the feasibility of the stability problem in LPV case requires further investigations.

As for the $\mathcal{L}_2$ norm analysis, Lemma 7 remains applicable as well in parameter-varying case. It is also true that $\gamma_{SDD} > \gamma_{PD}$ in general, but then the iterative algorithm introduced in the previous section can be applied to refine the bound. The only difference between the LTI and LPV implementations is that a separate $S^{(r)}[\kappa], R^{(r)}[\kappa]$ transformation pair has to be computed and stored in the LPV case for each $P(\rho_k), M(\rho_k, \mu_\ell)$ matrix pair. The numerical example presented in the next section will demonstrate an important property of this algorithm: only a few iteration steps (2-3) are enough in general to obtain an upper bound close to γ_{PD}.

V. NUMERICAL EXAMPLES

A. LTI example

First, the scalability of the SDD relaxation was examined by checking the stability of a set of randomly generated, large-dimensional LTI systems. The systems were constructed in the form

$$G_K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] := F_1(s) \cdot \ldots \cdot F_K(s)$$

where $F_i(s)$ are randomly generated SISO transfer functions. Each $F_i(s)$ had 10 states. Due to the construction, A_K is upper block triangular. In order to destroy this special structure K random entries of the lower triangular part of A_K were replaced by randomly generated numbers. The stability analysis was performed at $K = 5, 7, 9, 11, 13$ and was solved both by semidefinite and scaled diagonal dominant relaxations. The numerical computations were performed in MATLAB by using the SDP/SOCP solver Mosek [15]. The results are collected in Table I.

TABLE I
RESULTS OF THE STABILITY ANALYSIS PERFORMED ON RANDOMLY GENERATED LTI SYSTEMS

K	Solve time (PD)	Solve time (SDD)	Speedup	Result
5	1.77	1.35	1.31	stable
7	6.22	3.17	1.96	stable
9	18.35	8.70	2.11	stable
11	48.22	22.09	2.18	stable
13	106.7	30.09	3.54	stable

It can be seen that for small dimensional problems there is no significant difference between the two formulations. As the dimension increases the SDD approach scales much better: the relative speedup (the ratio of the solve times) increases. This result certifies that the SDD relaxation is a reasonable alternative to the semidefinite program in the case of stability analysis for large dimensional linear systems.

B. LPV example

A simple LPV example is used demonstrate the applicability of the proposed method to the induced L_2 norm analysis. The example is a simple 6$^{\text{th}}$ order LPV system with single in- and output. The system depends on 4 scheduling parameters ρ_1, ρ_2, ρ_3 and ρ_4. The plant is an interconnection of simpler plants as shown in Fig. 1.

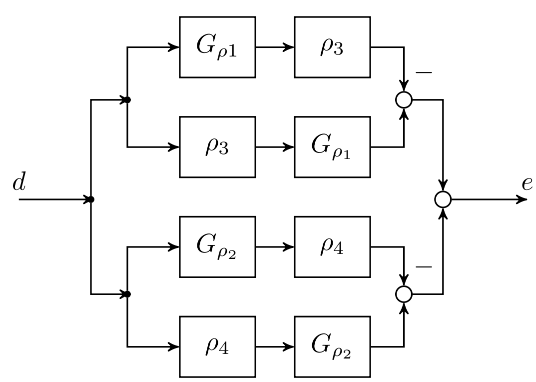

Fig. 1. Interconnection of LPV Systems

The systems G_{ρ_1} and G_{ρ_2} are LPV systems that depend on ρ_1 and ρ_2 respectively. G_{ρ_1} has the following form:

$$\begin{aligned} \dot{x}_{G_{\rho_1}} &= \begin{bmatrix} -\frac{(2-0.35\tau(\rho_1))}{\tau(\rho_1)} & \frac{1}{\tau(\rho_1)} \\ -0.25^2\tau(\rho_1) & 0 \end{bmatrix} x_{G_{\rho_1}} + \begin{bmatrix} \frac{(0.35\tau(\rho_1)-1)}{\tau(\rho_1)K(\rho_1)} \\ 0.25^2\,\frac{\tau(\rho_1)}{K(\rho 1)} \end{bmatrix} u_{G_{\rho_1}} \\ y_{G_{\rho_1}} &= \begin{bmatrix} -K(\rho_1) & 0 \end{bmatrix} x_{G_{\rho_1}} + u_{G_{\rho_1}} \end{aligned} \quad (9)$$

with

$$\tau(\rho_1) = \sqrt{133.6 - 16.8\rho_1}, \qquad K(\rho_1) = \sqrt{4.8\rho_1 - 8.6}. \quad (10)$$

G_{ρ_2} is the following first order LPV system:

$$\begin{aligned} \dot{x}_{G_{\rho_2}} &= -\sin(\rho_2)x_{G_{\rho_2}} + u_{G_{\rho_2}} \\ y_{G_{\rho_1}} &= \cos(\rho_2)x_{G_{\rho_2}} - 2u_{G_{\rho_2}} \end{aligned} \tag{11}$$

The parameters ρ_1 and ρ_2 are assumed to be time varying and are assumed to satisfy the following magnitude and rate bounds:

$$\begin{aligned} \rho_1(t) \in [2,\ 7], \quad \dot{\rho}_1(t) \in [-1,\ 1] \\ \rho_2(t) \in [-1,\ 1], \quad \dot{\rho}_2(t) \in [-1,\ 1] \end{aligned} \tag{12}$$

The results of the study are summarized in Tab. II. All computations are done using the solver Mosek [15]. Relaxing Lemma 2 by diagonally dominant conditions and solving the corresponding LP does not result in a feasible solution for any of the test cases. Using the SDD relaxation with a single iteration reduces solve time between 39% for the constant Lyapunov function case and 91% for quadratic Lyapunov. This comes at the expense of an increase in γ between 8% and 55%.

TABLE II

PERFORMANCE BOUNDS OF THE LPV SYSTEM

Relaxation	Solve Time	Perf. Index γ
Constant Lyapunov Function		
PD	1.42 s	23.7
SDD (1 Iter.)	0.86 s	25.7
SDD (2 Iter.)	1.57 (0.62/0.95) s	23.7
Affine Lyapunov Function		
PD	80.9 s	16.4
SDD (1 Iter.)	21.8 s	20.2
SDD (2 Iter.)	155.9 (21.8/134.1) s	16.8
Quadratic Lyapunov Function		
PD	624.5 s	11.9
SDD (1 Iter.)	56.4 s	18.0
SDD (2 Iter.)	355.3 (57.4/297.9) s	13.5

It can be seen that the iterative approach converges quickly to the solution of the SDP in this example. After only a second run of the SOCP, γ is already in the worst scenario within 15% of the solution of the SDP. However, it shall be noted that consecutive iterations result in a much larger solve time of the SOCP. In fact, for the constant and affine Lyapunov function, this results in an even higher solve time than the SDP. The numbers in brackets for the two iteration case are the solve times of the first and second iteration respectively. The reason behind the increase in computational effort is that the diagonalizing step in the iteration destroys the structure of the optimization problem. This can be seen by looking at the solve time for each iteration. The second iteration needs a lot longer to solve than the first. Therefore, in future work it is contemplated to search for state transformations of the LPV system that reduce the conservatism of the SDD relaxation instead of transforming the whole constraint. Still it shall be noted that for large problems, e.g. the quadratic Lyapunov function case, the iterative approach results in a 43% decrease in solve time.

VI. CONCLUSION

This paper introduced a relaxation method for the analysis of large scale LPV systems based on scaled diagonally dominance. This significantly improves the scalability of semidefinite optimization problems which appear commonly in the LPV framework at the cost of some conservatism. Two specific problems were considered in the paper, namely the stability analysis and $\mathcal{L}_2$ norm computation but the results carry over for many problems that can be posed as a semidefinite program. First theoretical results are given to show when the proposed relaxation is tight. The potential of the new approach is demonstrated with simple numerical examples, one being the stability analysis of a large LTI system and the other the $\mathcal{L}_2$ norm computation of an LPV system. Future work will explore potential state transformations for LPV systems that can reduce the conservatism of the results, as well as consider more realistic applications like aeroservoelastic systems or wind turbines.

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