

Cavitation and the state of stress in a flowing liquid

Daniel D. Joseph
University of Minnesota
Dept. of Aerospace Engineering & Mechanics
110 Union St.
Minneapolis, MN 55455
May, 1997

The problem of the inception of cavitation is formulated in terms of a comparison of the breaking strength or cavitation threshold at each point of liquid sample with the principal stresses there. A cavity will open in the direction of the maximum tensile stress which is 45° from the plane of shearing in pure shear of a Newtonian fluid. A criterion of maximum tension is proposed which unifies the theory of cavitation, the theory of maximum tensile strength of liquid filaments and the theory of fracture of amorphous solids. Experiments which support these ideas are discussed and some new experiments are proposed.

It is argued that outgassing is a phase change for condensed gas in solution under conditions in which the gas not in solution could not condense, and that the liquid which is supersaturated with condensed gas is apt to cavitate.

New explanations of cavitation inhibition due to polymer additives are considered. Analyses of the second order fluid shows that explanations based on irrotational strain fields are not consistent with inhibition but cavitation on solid bodies can be inhibited by compressive normal stresses due to shear which give rise to increased pressures preventing the nucleation of gas on bodies. Boundary layer separation is controlled by the variation of the total normal stress which has a substantial viscoelastic component. The observed size effects on cavitation in the presence of polymer additives in which inhibition is seen on small but not large bodies is implied by the elasticity number which is independent of velocity and increases with the reciprocal of the square of particle radius.

1 Introduction

In previous papers (Joseph [1995], Joseph, Huang and Candler [1996]) I drew attention to the fact that the pressure in a flowing incompressible liquid is not a fundamental dynamic variable; at each point of the liquid the state of stress is determined by three principal stresses. In Newtonian

fluids the pressure is the negative of the mean of these stresses (6); in non-Newtonian fluids the pressure is an unknown field variable whose relation to the principal stresses depends on the choice of a constitutive equation.

We may generally express the stress \mathbf{T} by a constitutive equation of the form

$$\mathbf{T} = -p\mathbf{1} + \boldsymbol{\tau}[\mathbf{u}] \quad (1)$$

where the part $\boldsymbol{\tau}$ of \mathbf{T} which is characterized by a constitutive equation can be regarded as functional of the velocity \mathbf{u} . For incompressible liquids, the conservation of mass is expressed by

$$\text{div } \mathbf{u} = 0 \quad (2)$$

and the conservation of momentum by

$$\rho \left[\frac{\partial \mathbf{u}}{\partial \tau} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \text{div } \boldsymbol{\tau}[\mathbf{u}]. \quad (3)$$

Equations (2) and (3) are four equations for three components of velocity and the pressure p is an additional unknown which we need to close the system.

For Newtonian liquids

$$\boldsymbol{\tau}[\mathbf{u}] = 2\eta \mathbf{D}[\mathbf{u}] \quad (4)$$

where $\mathbf{D}[\mathbf{u}]$, the rate of strain, is the symmetric part of the gradient of velocity, η is the viscosity, and

$$\text{Trace } \mathbf{D}[\mathbf{u}] = \text{div } \mathbf{u} = 0 \quad (5)$$

As a consequence of (5),

$$p = -\frac{1}{3} \text{Trace } \mathbf{T} \quad (6)$$

More generally, $\text{Trace } \boldsymbol{\tau} \neq 0$ and

$$p = -\frac{1}{3} \text{trace } (\mathbf{T} - \boldsymbol{\tau}) \quad (7)$$

depends on the constitutive equation, the choice of the functional relating $\boldsymbol{\tau}$ to \mathbf{u} .

Though it is true that a liquid at rest, in which all the stresses are all equal to $-p$, can make sense of (6), a moving liquid cannot average the principal stresses as is required by (6), and (7) is even more a consequence how we choose to define $\boldsymbol{\tau}$ than a fundamental quantity which can be felt at a point by the liquid.

2 Downgrade of the cavitation index

The idea is that the state of stress at each point of a moving liquid is determined by the three principal stresses

$$T_{11} \geq T_{33} \geq T_{22} \quad (8)$$

and not by the pressure given by (7). The utility of a cavitation index based on pressure is not evident. In one formulation, the index is given by

$$k = \frac{\tilde{p} - p_c}{\frac{1}{2}\rho U^2} \quad (9)$$

where \tilde{p} is the static pressure in the main stream, U is the bulk velocity of the fluid and \tilde{p} and U are measured in the instant that cavitation commences.

Milne-Thompson [1965, Chap XII] considers the vapor cavity behind a moving cylinder and he forms an index

$$k = \frac{\tilde{p} - p_c}{\frac{1}{2}\rho U^2} = \frac{V^2 - U^3}{U^2} \quad (10)$$

which he attributes to Prandtl. By Bernoulli's theorem

$$\tilde{p} + \frac{1}{2}\rho U^2 = p_c + \frac{1}{2}\rho V^2 \quad (11)$$

where \tilde{p} is the pressure at ∞ , p_c is the pressure in the cavity and V is the fluid speed on the cavity wall. This index shows that a cavity will form on the top of the cylinder where the flow is fastest.

The origin of the cavitation index potential flow makes perfect sense for high speed flow over bodies, where the flow outside a thin boundary is potential flow. However, this kind of index has nothing whatever to say about the rheology of the fluid or the cavitation of flow other than those over bodies, like the tensile strength of thin liquid filaments in extensional flow. In reading the literature on cavitation, one senses that experts in this field do not feel confident about the value of the index.

3 Principal stresses and cavitation

Even though criteria for cavitation ought to be based on the principal stresses and not the pressure, it is useful to introduce a pressure as the

mean normal stress as in a Newtonian liquid and to define it that way for Non-Newtonian liquids. If we write

$$\mathbf{T} = -\pi\mathbf{1} + \mathbf{S} = -p\mathbf{1} + \boldsymbol{\tau} \quad (12)$$

where \mathbf{S} is the stress deviator

$$\pi = -\frac{1}{3} \text{Trace } \mathbf{T}, \quad \text{Trace } \mathbf{S} = S_{11} + S_{22} + S_{33} = 0 \quad (13)$$

and

$$p = -\frac{1}{3} \text{Trace } (\mathbf{T} - \boldsymbol{\tau}) \quad (14)$$

Since $S_{11} \geq S_{33} \geq S_{22}$ we have

$$S_{11} > 0 \text{ and } S_{22} < 0 \quad (15)$$

where

$$S_{11} - S_{22} > 0$$

is largest in the coordinate system in which \mathbf{T} is diagonal.

Consider now the opening of a small cavity. It is hard to imagine very large differences in the pressure of the vapor in the cavity so that the cavity should open in the direction where the tension is greatest. The idea that vapor cavities open to tension is endemic in the cavitation community, but it seems not to have been noticed before that this idea requires one to consider the state of stress at a point and, at the very least, to determine the special principal axes coordinates in which the tension is maximum. To remind us of this important point we shall call $\zeta(\theta)$ the special coordinate system in which the orthogonal transformation \mathbf{Q} diagonalizes \mathbf{T} (and \mathbf{S}):

$$\mathbf{Q}^T \mathbf{T} \mathbf{Q} = \text{diag}(T_{11}, T_{22}, T_{33}) \quad (16)$$

Here θ in $\zeta(\theta)$ stands for the direction cosines in the diagonalizing transformation, and θ is the diagonalizing angle for the two-dimensional rotation. The rotation of \mathbf{T} is an important part of the theory of cavitation which has not been considered before.

In two dimensions the components of the stress deviator in $\zeta(0)$ are given by

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & -S_{11} \end{bmatrix}; \quad (17)$$

The angle θ that diagonalizes \mathbf{S} , $S'_{12} = 0$ is given by

$$\begin{cases} \sin 2\theta = S_{12}/\sqrt{S_{12}^2 + S_{11}^2}, \\ \cos 2\theta = S_{11}/\sqrt{S_{12}^2 + S_{11}^2} \end{cases} \quad (18)$$

and

$$[\mathbf{S}] = \sqrt{S_{12}^2 + S_{11}^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (19)$$

The largest stress component in the principal value coordinate system is

$$T_{11} + \frac{1}{2}(T_{11} + T_{22}) = S_{11}; \quad (20)$$

the smallest component is

$$T_{22} + \frac{1}{2}(T_{11} + T_{22}) = -S_{11} \quad (21)$$

and

$$T_{11} - T_{22} = 2S_{11} \quad (22)$$

We call T_{11} the maximum tension and T_{22} is the minimum tension. If the maximum tension is negative, it is compressive; the minimum tension is even more compressive.

If the cavitation (outgassing) threshold p_c is above $\Pi - S_{11}$ but below Π the cavity will appear when and where the tension due to motion is large enough; if this threshold is greater than $\Pi - S_{22}$ ($S_{22} < 0$) then the cavity will open only at those points where no component of the total stress is larger than the cavitation threshold; this is the minimum tension criterion and in neither case is the criterion framed in terms of the pressure Π alone.

For Newtonian fluid $S_{11} = 2\eta\partial u_1/\partial x_1$, $S_{22} = -S_{11}$ and $S_{12} = \eta(\partial u_1/\partial x_2 + \partial u_2/\partial x_1)$ where η is the viscosity. In principal coordinates, $S_{12} = 0$.

If a cavitation bubble opens up, it will open in the direction of maximum tension. Since this tension is found in the particular coordinate system in which the stress is diagonal, the opening direction is in the direction of maximum extension, even if the motion is a pure shear.

The features in the two dimensional problem which were just discussed have an immediate and obvious extension to three dimensions.

4 Cavitation criteria

It is well known that cavitation occurs at weak spots (nucleation sites) in the fluid. It is not necessary to form vapor bubbles; outgassing may occur and in a severe cavitation even a vacuum cavity may open up (see section 8). It is nearly impossible to know where the nucleation sites are or what the outgassing or breaking stress p_c may be, especially in pure and carefully prepared liquids. In section 9, I argue that outgassing may be regarded as a phase change for condensed gas in solution under conditions for which the gas not in solution could not condense. The liquid can saturate with condensed gas; cavitation is apt to occur in the supersaturated case at pressures of the order of the vapor pressure. Here, we shall assume that the breaking stress is a given parameter which can be defined at each point of a liquid; we then compare the state of stress in a moving liquid at the point with p_c to form a cavitation criteria.

The cavitation threshold used in the prior literature is framed in terms of a mean stress

$$\pi = -\frac{1}{3}(T_{11} + T_{22} + T_{33}) \quad (23)$$

cavitation will occur when $\pi - p_c < 0$ and will not occur when $\pi - p_c > 0$.

Two cavitation thresholds based on the maximum tension T_{11} and minimum tension T_{22} in three dimensions ought to be considered. The criterion for maximum tension can be framed in terms of

$$B_{11} = -T_{11} - p_c = -S_{11} + \pi - p_c \quad (24)$$

and the criterion for minimum tension expressed by

$$B_{22} = -S_{22} + \pi - p_c \quad (25)$$

where $S_{11} > 0$ and $S_{22} < 0$; all stresses are evaluated in the principal value coordinate system.

The **maximum tension** criteria is given by

$$B_{11} < 0 \quad (26)$$

If (26) holds, then $-S_{11} + \pi - p_c < 0$, so if $\pi > p_c$ tending to close the cavity the deviatoric tension $S_{11} > 0$ would open it again.

Since $S_{22} < 0$ it is possible to have

$$B_{22} > 0, B_{11} < 0.$$

A **minimum tension** criterion

$$B_{22} < 0$$

can be considered. In this case $-S_{22} + \pi - p_c < 0$; if $\pi \leq p_c$ tending to open the cavity, the deviatoric compress $S_{22} < 0$ would close it again. A cavity which is being opened in the direction of the stress S_{11} is being closed by S_{22} in the orthogonal direction.

Definite statements about cavitation which seem compelling and which are probably consistent with experimental facts can be framed in terms of B_{11} and B_{22} . If

$$B_{11} > 0 \tag{27}$$

then $B_{22} > 0$ (even more) and p_c is larger than any principal stress and the **cavity will open**. In the other hand, if

$$B_{22} > 0 \tag{28}$$

the p_c is smaller than all principal stresses and the **cavity will close**.

The mean stress which we identify as pressure may be a good estimate for breaking thresholds, but it does not enter into the criteria since it has no physical meaning in a moving fluid.

The minimum tension criterion has not been proposed before. It may be possible to distinguish between these two criteria in the shear experiments described in section 6 when the shear rates are very large and the difference of principal tension and compression 45° from the direction of shear is also large.

The archival literature on cavitation allows only for breaking in tension, though the state of stress at a point which ought to be considered, has not been considered. Typically the discussion of cavitation is framed in the context of the breaking strength of liquids; the main conclusion is that liquids may withstand very large tensions if impurities and nucleation sites are suppressed. A convenient and readable discussion of this point has been given by Batchelor [1967]. There is a vast literature on the tensile strength of liquids some of which may be found in the book by Knapp, Daily & Hammitt [1970] who say that "... Measurements have been made by several different methods and are too numerous to report completely" and in other books on cavitation.

Knapp et al. [1970] have considered whether the cavitation threshold ought to be framed in terms of the vapor pressure or the tensile strength of liquids, concluding for the latter. They say that

... the elementary concept of inception is the formation of cavities at the instant the local pressure drops to the vapor pressure of the liquid. However, the problem is not so simple. Although experiments show inception to occur near the vapor pressure, there are deviations of various degrees with both water and other liquids that are not reconcilable with the vapor-pressure concept. We define the vapor pressure as the equilibrium pressure, at a specified temperature, of the liquid's vapor which is in contact with an *existing* free surface. If a cavity is to be created in a homogeneous liquid, the liquid must be ruptured, and the stress required to do this is not measured by the vapor pressure but is the *tensile strength* of the liquid at that temperature. The question naturally arises then as to the magnitudes of tensile strengths and the relation these have to experimental findings about inception.

A similar point of view was expressed by Plesset [1969]

... A central problem in cavitation and boiling is how macroscopic vapor cavities can form when moderate tensions are applied to the liquid. The theory of the tensile strength of pure liquids predicts that a vapor cavity will form only when the liquid is under extremely large tensions; as an equivalent effect the theory also predicts that vapor bubbles appear in boiling only when the liquid has very large superheats. Since these large tensile strengths and superheats are not observed, the idea of nuclei has been introduced. These nuclei are in some sense holes in the liquid which are already beyond molecular dimensions and which may therefore grow into macroscopic bubbles under moderate liquid tensions.

Brenner [1995] notes that "... This ability of liquids to withstand tension is very similar to the more familiar property exhibited by solids and is a manifestation of the elasticity of a liquid." Of course the elasticity of liquids, solid-like behavior, could occur only in time so short that the configurations of molecules is not changed by flow, as could be expected in a cavitation event. Fischer [1948] notes that "... Glass and other undercooled liquids may fail by the nucleation and propagation of cracks, rather than of bubbles as do more mobile liquids." He notes that "... A satisfactory analysis of cavitation problems requires a reliable estimate of the maximum negative

pressure that a given liquid can withstand, as well as an examination of the conditions that may reduce this value in practice.”

The theory of cavitation, the tensile strength of liquids and the fracture of amorphous solids may be framed in a unified manner in which the breaking strength of the material is defined in terms of tensile stresses along the principal axes of stress. The formation of cracks or bubbles is probably controlled by comparing rapidity of flow with the propagation speed of fracture. Glass at different temperatures is a perfect material for these considerations. At high temperatures the molten glass flows and under the right conditions, flow bubbles ought to open at a weak spot in the direction of the principal tension. Low temperature glass is an amorphous solid and we can imagine a crack to be initiated under tension at the same weak spot. Very tacky glass at intermediate temperatures may exhibit as yet unknown properties between cavity formation and crack propagation.

5 Cavitation in shear

Consider plane shear flow between parallel plates as in figure 1.

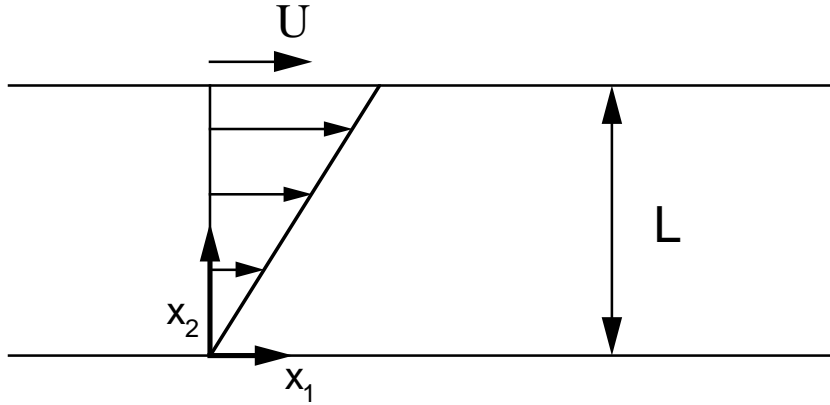


Figure 1: Plane Couette flow between walls

The stress in this flow is given by

$$\begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} = -\pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \eta \begin{bmatrix} 0 & \frac{U}{L} & 0 \\ \frac{U}{L} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (29)$$

where $\pi = \frac{1}{3}(T_{11} + T_{22} + T_{33})$ is determined by the “pressurization” of the