Pendulum Waves: A lesson in aliasing

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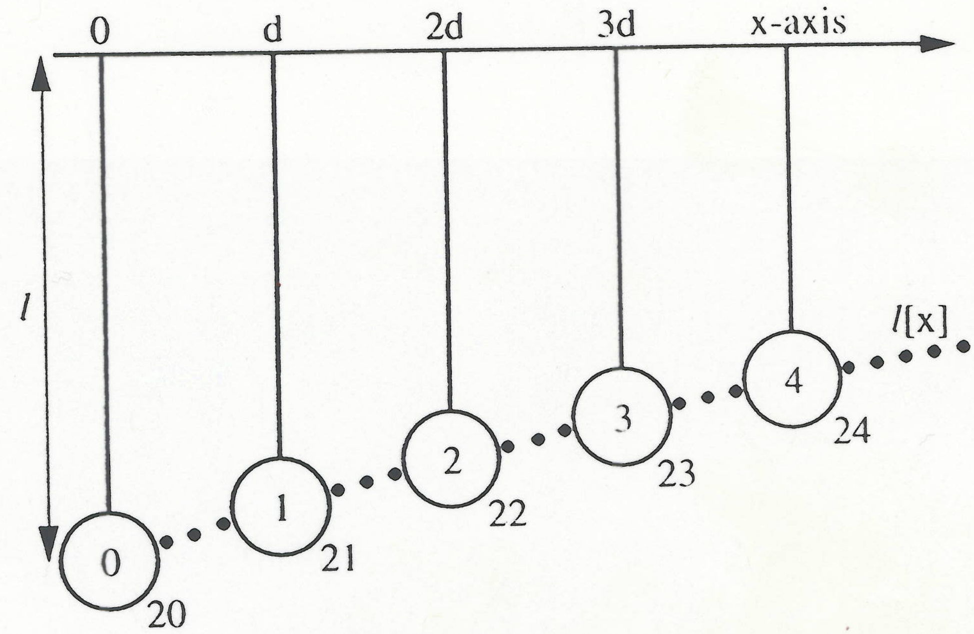
The demonstration.

- uncoupled pendula
- different lengths → diff. periods
- start in phase, return to in-phase, beautiful intermediate patterns dubbed "pendulum waves"
- reminiscent of traveling waves (that reverse direction), standing waves, and chaos at diff. times

contrived patterns betw. indep. oscill.

- Richard Berg, AAPT meeting at Univ. of Maryland
- origins murky -- Russia? USA?

How to build one:



Tune pendula so that in time Γ (about 20 s for our apparatus) the longest pendulum goes through N cycles ($N = 20$), the next goes through $N+1$, the next $N+2$, etc.

When started in phase at $t=0$ the pendula will come back in phase at $t = \Gamma, 2\Gamma$, etc. The intermediate patterns are very beautiful too.

Examine patterns using Mathematica.

- balls oscillate independently
- if started out of phase but all have same angular freq, can illustrate traveling waves

travelingwaves.nb

- if started in phase but all have different (special) angular freqs, can illustrate "pendulum wave" sequence of patterns

pendulumwaves.nb

Notice how patterns evolve to the out-of-phase point then run through exact same series in reverse order.

Traveling Waves (with a twist)

$$y[x,t] = A\cos[kx + \omega t + \phi]$$

sets value at $x=0, t=0$

sets cycling in time $\omega = 2\pi\text{ rad}/T$

sets cycling in space $k = 2\pi\text{ rad}/\lambda$

Here ω is not fixed but varies with x since the pendula get shorter & shorter.

$$\therefore\ \omega \longrightarrow \omega[x]$$

One observes that λ is not fixed but gets smaller and smaller as time goes by

$$\therefore\ k \longrightarrow k[t]$$

Proposal:

$$y[x,t] = A\cos\left[k[t]x + \omega[x]t + \phi\right]$$

set to zero

Actually the variation of k with time is a consequence of the fact that ω varies with x so it is sufficient to write

$$y[x,t] = A\cos\left[k_0 x + \omega[x]t\right]$$

needed to dictate shape when $t=0$

OR

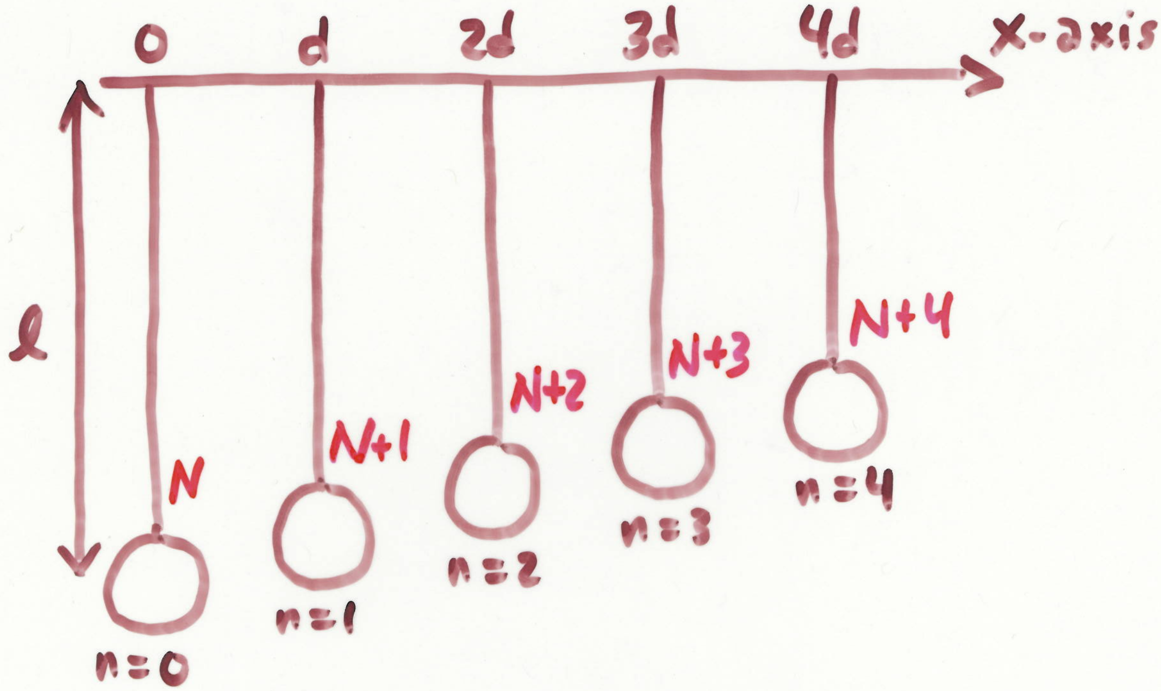
$$y[x,t] = A\cos\left[k[t]x + \omega_0 t\right]$$

needed to dictate motion of function at $x=0$

Pursue first form...

$$y[x,t] = A\cos[k_0 x + \omega[x]t]$$

Since $y[x, t=0] = A$ for all values of x, apparently $k_0 = 0$.



$x_n = nd$ ... location of n^{th} pendulum

$T_n = \dfrac{\Gamma}{N+n}$ ... period of n^{th} pendulum

$\omega_n = 2\pi\,\text{rad}/T_n = 2\pi\,\text{rad}\,\dfrac{N+n}{\Gamma}$ ... ang. freq. of n^{th} pendulum

$\omega[x] = 2\pi\,\text{rad}\,\dfrac{(N+x/d)}{\Gamma} = 2\pi\,\text{rad}\,\dfrac{(x+Nd)}{\Gamma d}$ ... continuous version

Thus we have the following continuous function underlying the pendula patterns.

$$y[x,t] = A\cos[\overset{\text{zero}}{\cancel{k_0}}x + \omega[x]t]$$

$$y[x,t] = A\cos\left[\left(2\pi\,\text{rad}\,\frac{(x+Nd)}{\Gamma d}\right)t\right]$$

If the x-dependence is separated out this becomes

$$y[x,t] = A\cos\left[\left(2\pi\,\text{rad}\,\frac{t}{\Gamma d}\right)x + \left(2\pi\,\text{rad}\,\frac{N}{\Gamma}\right)t\right]$$

which is the other proposed form

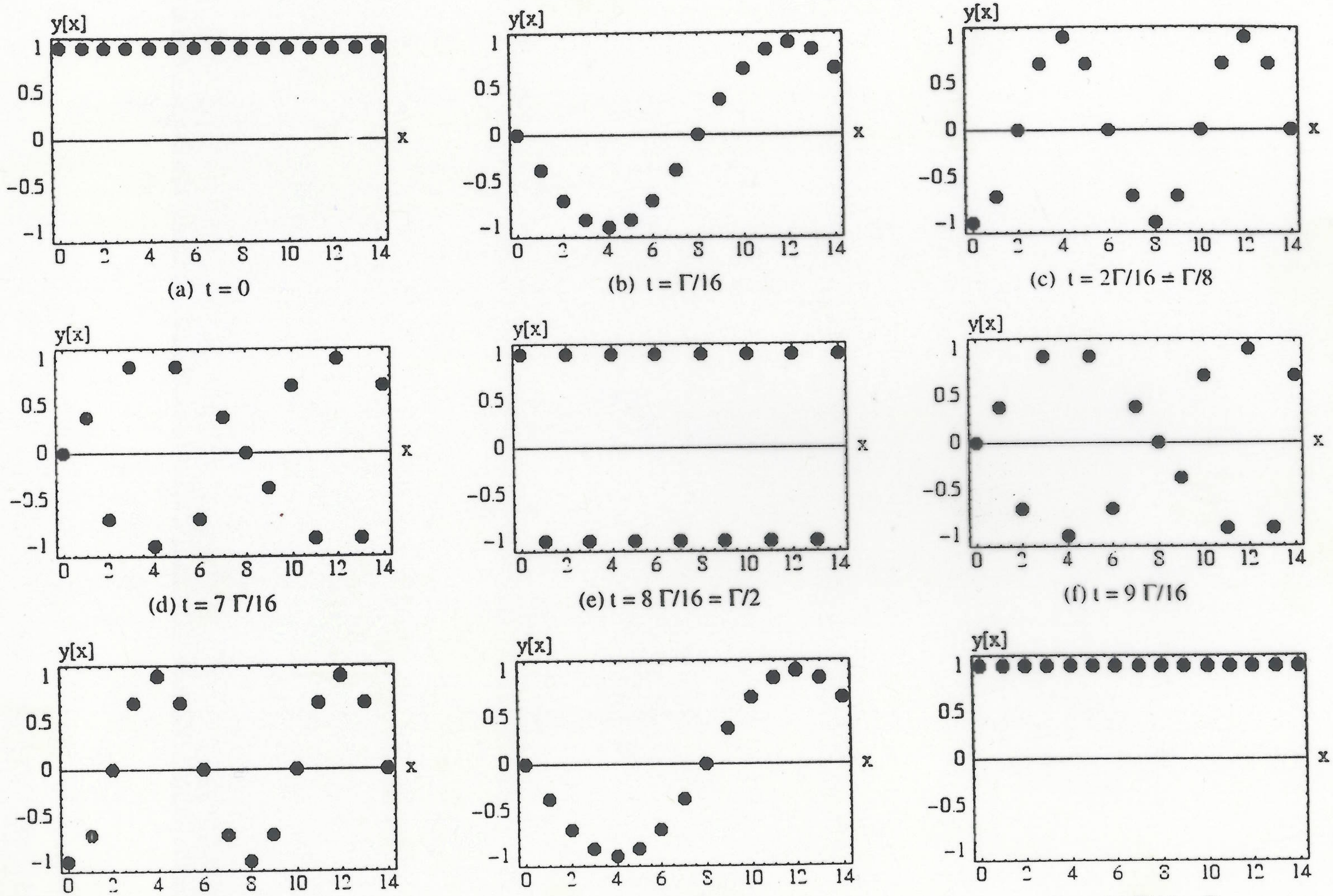
$$y[x,t] = A\cos[k[t]x + \omega_0 t]$$

$$k[t] = 2\pi\,\text{rad}\,\frac{t}{\Gamma d} \Rightarrow \lambda[t] = \frac{2\pi\,\text{rad}}{k[t]} = \frac{\Gamma d}{t}$$

$$\omega_0 = 2\pi\,\text{rad}\,\frac{N}{\Gamma} = \frac{2\pi\,\text{rad}}{T_0} \quad \text{(as expected)}$$

Notice $\lambda[t]$ starts at ∞ then shrinks as $1/t$.

Some snapshots.

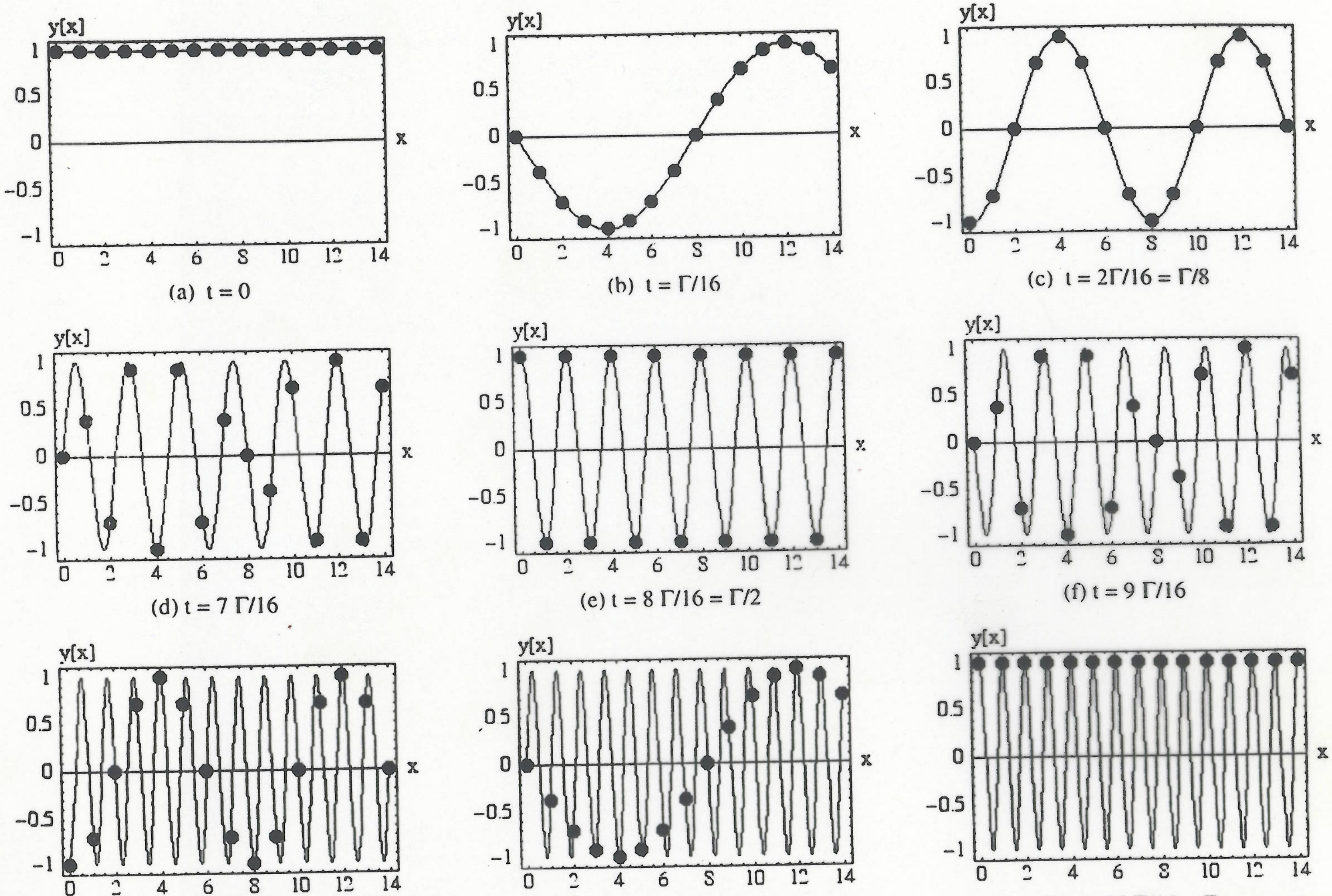


(a) $t = 0$ (b) $t = \Gamma/16$ (c) $t = 2\Gamma/16 = \Gamma/8$

(d) $t = 7\,\Gamma/16$ (e) $t = 8\,\Gamma/16 = \Gamma/2$ (f) $t = 9\,\Gamma/16$

(g) $t = 14\,\Gamma/16 = 7\,\Gamma/8$ (h) $t = 15\,\Gamma/16$ (i) $t = 16\,\Gamma/16 = \Gamma$

y[x,t] on the snapshots. This is aliasing.



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Spatial aliasing -- information is available at all times but only at specific locations. Details between locations can be hidden.

Question 1.

Does $y[x,t]$ at the specific x_n's where the pendula are located oscillate sinusoidally in time with the appropriate angular frequency ω_n?

$$\text{i.e.} \quad y[x_n,t] \stackrel{?}{=} A\cos[\omega_n t]$$

Yes! Here is one possible proof.

$$y[x_n,t] = A\cos\left[2\pi\,\text{rad}\,\frac{t}{\Gamma d}\,x_n + 2\pi\,\text{rad}\,\frac{N}{\Gamma}\,t\right]$$

defn. of $y[x,t]$

$$y[x_n,t] = A\cos\left[2\pi\,\text{rad}\,\frac{t}{\Gamma d}\,nd + 2\pi\,\text{rad}\,\frac{N}{\Gamma}\,t\right]$$

used $x_n = nd$

$$y[x_n,t] = A\cos\left[2\pi\,\text{rad}\left(\frac{N+n}{\Gamma}\right)t\right]$$

combined terms

$$y[x_n,t] = A\cos[\omega_n t] \qquad \text{QED}$$

used $\omega_n = 2\pi\,\text{rad}\left(\frac{N+n}{\Gamma}\right)$

Question 2

Does $y[x, t+m\Gamma]$ at the specific x_n's where the pendula are located equal $y[x,t]$? Here m is an integer. That is to say, is the <u>pendulum</u> pattern identical every time Γ has elapsed, even though $y[x,t]$ itself gets more and more complicated?

i.e. $y[x_n, t+m\Gamma] \stackrel{?}{=} y[x_n, t]$

Yes! Here is one possible proof.

$$y[x_n, t+m\Gamma] = A\cos\left[2\pi\,\text{rad}\,\frac{(t+m\Gamma)}{\Gamma d}x_n + 2\pi\,\text{rad}\,\frac{N}{\Gamma}(t+m\Gamma)\right]$$ defn. of $y[x,t]$

$$y[x_n, t+m\Gamma] = A\cos\left[2\pi\,\text{rad}\,\frac{t}{\Gamma d}x_n + 2\pi\,\text{rad}\,\frac{N}{\Gamma}t + 2\pi\,\text{rad}\,m\,(x_n/d + N)\right]$$ sep. m-depend.

$$y[x_n, t+m\Gamma] = A\cos\left[2\pi\,\text{rad}\,\frac{t}{\Gamma d}nd + 2\pi\,\text{rad}\,\frac{N}{\Gamma}t + 2\pi\,\text{rad}\,m\,(n+N)\right]$$ used $x_n = nd$

$$y[x_n, t+m\Gamma] = A\cos\left[2\pi\,\text{rad}\,\frac{t}{\Gamma d}x_n + 2\pi\,\text{rad}\,\frac{N}{\Gamma}t\right]$$

since $m(n+N)$ is an integer, adding $2\pi\,\text{rad}\,m\,(n+N)$ to the arguement of the cosine has no effect

$y[x_n, t+m\Gamma] = y[x_n, t]$ QED defn of $y[x,t]$

Question 3

Does $y[x, t = \Gamma/2 + \epsilon]$ at the specific x_n's where the pendula are located equal $y[x, t = \Gamma/2 - \epsilon]$? That is to say, are the pendulum patterns symmetric in time about the out-of-phase pattern, even though $y[x, t]$ itself gets more and more complicated?

i.e. $y[x_n, \frac{\Gamma}{2} + \epsilon] \stackrel{?}{=} y[x_n, \frac{\Gamma}{2} - \epsilon]$

Yes! The proof of this is left to the interested listener.

Let's see that function superimposed on the pendula from $t=0$ up to $t=\Gamma/2$ (out-of-phase pattern).

pwavesfcnfirst.nb

That worked but can it possibly be right beyond $t=\Gamma/2$ when the patterns run backward and get less & less complicated while y[x,t] continues to collapse?

Hint: $\lambda[t=\Gamma]=\frac{\Gamma d}{\Gamma}=d$ ... suggests one full cycle between every two pendula ... not enough pend. to show details of y[x,t]

pwavesfcnsecond.nb

Wow! That works too!

The math is a bit simpler if the origin of the position axis is shifted.

remember: $\omega[x] = 2\pi \text{ rad } \frac{(x+Nd)}{\Gamma d}$

define $\xi = x + Nd$ (shift origin in the minus x-direction by a distance Nd)

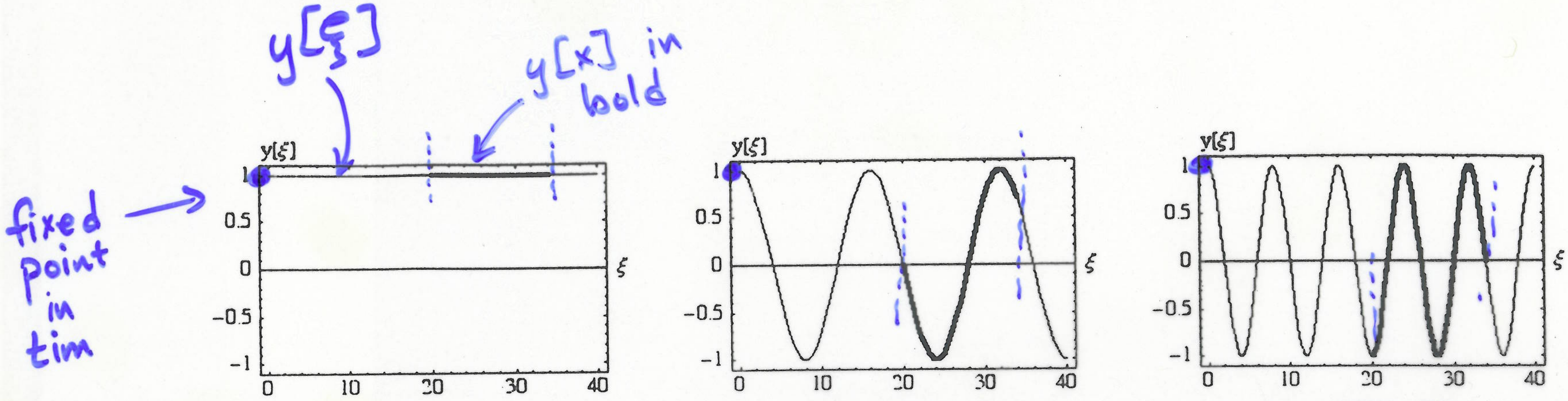
$\therefore \omega[\xi] = 2\pi \text{ rad } \xi/\Gamma d$

$y[\xi, t] = A\cos\left[2\pi\left(\xi/\Gamma d\right)t\right]$

Physically, this corresponds to building longer and longer pendula: $N=19$, $N=18$, ... $N=0$

tough since $T_0 = \infty$ so this one must be infinitely long

$y[\xi, t]$ looks like a collapsing accordian with $y[\xi=0, t] = A$, fixed for all times.



(a) $t = 0$ (b) $t = \Gamma/16$ (c) $t = 2\Gamma/16 = \Gamma/8$

$y[x]$ is a subset from $\xi = Nd$ to $\xi = (N + n_{max} - 1)d$ of the more general $y[\xi]$ function. That is, $y[x]$ is a window on $y[\xi]$

For more details, watch for

Pendulum Waves: A lesson in aliasing
J. Flaten & K. Parendo, to be
published in AJP this summer

Video clips and animations may be found at

http://www.mrs.umn.edu/~flatenja/pendulumwaves.shtml