2007 AEM Preliminary Exam—Continuum Mechanics

1. It can be shown that for isotropic materials the most general stress-strain relation is

$$\boldsymbol{\sigma} = c_0\mathbf{I} + c_1\mathbf{B} + c_2\mathbf{B}^2, \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\mathbf{B} = \mathbf{F}\bullet\mathbf{F}^{\mathrm{T}}$ is the left Cauchy-Green deformation tensor, $\mathbf{F}$ is the deformation gradient, and c_i are general functions of the principal invariants of $\mathbf{B}$,

$$c_i = c_i(I_1, I_2, I_3),$$

with $I_1 = \mathrm{tr}\,(\mathbf{B})$, $I_2 = \frac{1}{2}((\mathrm{tr}\,(\mathbf{B}))^2 - \mathrm{tr}\,(\mathbf{B}^2))$, and $I_3 = \det(\mathbf{B})$.

   (a) Use the Cayley-Hamilton theorem (a second-order tensor satisfies its own characteristic equation) to prove that an equivalent expression to Eq. (1) is

$$\boldsymbol{\sigma} = \delta\mathbf{I} + \alpha\mathbf{B} + \gamma\mathbf{B}^{-1}, \tag{2}$$

   where

$$\delta = c_0 - I_2c_2, \qquad \alpha = c_1 + I_1c_2, \qquad \gamma = I_3c_2.$$

   (b) Given Eq. (2), what is the most general *linear* relation between the components of $\boldsymbol{\sigma}$ and the components of $\mathbf{B}$?

   (c) Show that for infinitesimal straining the relation found in item 1b reduces to the well-known generalized Hooke's law for isotropic materials,

$$\boldsymbol{\sigma} = (\sigma_0 + \lambda\,\mathrm{tr}\,(\boldsymbol{\epsilon}))\mathbf{I} + 2\mu\boldsymbol{\epsilon}, \tag{3}$$

   where $\boldsymbol{\epsilon}$ is the small strain tensor and λ and μ are Lamé's constants.
   Hint: Recall the Gâteaux derivative of a tensor $\mathbf{G}(\mathbf{x})$ is

$$< D\mathbf{G}, \mathbf{U} >= \frac{d}{d\eta}\left(\mathbf{G}(\mathbf{x} + \eta\mathbf{U})\right)\Big|_{\eta=0},$$

   where $\mathbf{U}$ is a small displacement perturbation about the current configuration.

   (d) The coefficients δ, α, γ in Eq. (2) are usually restricted to values satisfying

$$\delta \leq 0, \qquad \alpha > 0, \qquad \gamma \leq 0.$$

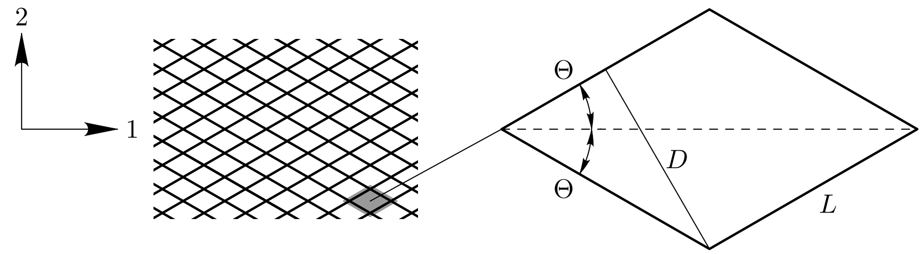
   Explain the motivation behind the use of these restrictions.

   (e) A thin sheet of an *incompressible* isotropic material is subjected to a bi-axial deformation characterized by the in-plane stretches Λ_1 and Λ_2 along the horizontal and vertical directions respectively. The resulting out-of-plane stretch is Λ_3. Assuming a condition of plane stress along the 3-direction perpendicular to the sheet, show that the most general state of stress possible in the plane of the sheet is

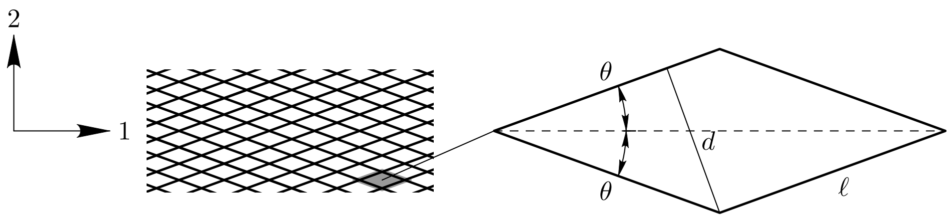
$$\begin{aligned}\sigma_{11} &= (\Lambda_1^2 - \Lambda_3^2)(\alpha - \gamma\Lambda_2^2),\\ \sigma_{22} &= (\Lambda_2^2 - \Lambda_3^2)(\alpha - \gamma\Lambda_1^2),\\ \sigma_{12} &= \sigma_{21} = 0.\end{aligned} \tag{4}$$

   Hint: Recall that for an incompressible material an arbitrary hydrostatic pressure $-p\mathbf{I}$ can be superposed on the stress relation.

2. Consider a plane network formed by two families of straight, inextensible wires woven together to form a parallelogram structure. The wires are symmetrically arranged about the 1-direction with angles $\pm\Theta$. The distance between the wires is D. The length of a wire segment between intersections is L.



When stretched the network adopts its deformed configuration, described by the current angles $\pm\theta$, distance d, and length ℓ.



(a) Show that stretches Λ_1 and Λ_2 applied to the wire network must satisfy the relation

$$\Lambda_1^2 \cos^2(\Theta) + \Lambda_2^2 \sin^2(\Theta) = 1. \tag{5}$$

(b) What is the magnitude of the forces required to hold the network in a state of stretch consistent with Eq. (5)?

(c) Just as an incompressible material is able to support an arbitrary hydrostatic pressure, an inextensible wire can support an arbitrary tension along its length. Assume that the wires oriented along $+\theta$ and $-\theta$ support tensions τ_p and τ_m respectively. Show that the effective tension per unit length for the network is given by the tensor components

$$\begin{aligned} n_{11} &= (\tau_p + \tau_m)\frac{\Lambda_1 \cos^2(\Theta)}{\Lambda_2 D}, \\ n_{22} &= (\tau_p + \tau_m)\frac{\Lambda_2 \sin^2(\Theta)}{\Lambda_1 D}, \\ n_{12} &= (\tau_p - \tau_m)\frac{\sin(\Theta)\cos(\Theta)}{D}. \end{aligned} \tag{6}$$

3. For an engineering application it is necessary to stiffen the incompressible sheet in question 1e by embedding within it the inextensible network of wires discussed in question 2. Assume the components of the composite interact like a set of nonlinear springs in parallel and that the thickness of the incompressible sheet is given by $2T$ (assumed to be a small quantity).

   (a) Find the average tension $\mathbf{s}$ (force per unit current length) in the composite sheet when it is subjected to stretches Λ_1, Λ_2, and Λ_3, in terms of the Cauchy stress $\boldsymbol{\sigma}$ of the incompressible sheet, the tension $\mathbf{n}$ in the inextensible network of wires, and the thickness T.

   (b) Find the tension in the wires τ_p and τ_m for uniaxial tension ($s_{11} \neq 0$, $s_{12} = s_{22} = 0$).

   (c) For the loading described in item 3b, find the angles $\pm\Theta$ that are required for the composite sheet to have a reference stiffness in the 1-direction,

$$K = \left. \frac{\partial s_{11}}{\partial \Lambda_1} \right|_{\mathbf{F}=\mathbf{I}}, \tag{7}$$

   that is double that of the incompressible sheet on its own.