2005 AEM Preliminary Exam-Continuum Mechanics

1. A material surface $\eta(\mathbf{x}, t)$ is one such that $\dot{\eta} = 0$. Here, $\mathbf{x} = (x_1, x_2, x_3)$ indicates the Eulerian (spatial) position of a material point, time $t > 0$ and a dot denotes a Lagrangian (material) time derivative.

Consider an ellipsoid,

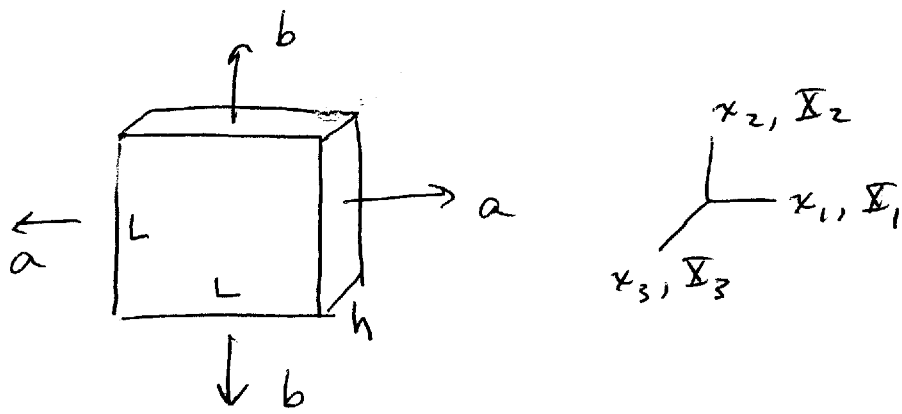
$$\frac{x_1^2}{a_1^2}\left(\frac{t}{\tau}\right)^{\alpha_1} + \frac{x_2^2}{a_2^2}\left(\frac{t}{\tau}\right)^{\alpha_2} + \frac{x_3^2}{a_3^2}\left(\frac{t}{\tau}\right)^{\alpha_3} = 1 \tag{1}$$

where $a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3$ and τ are positive constants.
a) Find a velocity field for which this ellipoid describes a material surface.
b) Find a condition for α_1, α_2 and α_3 such that the resulting motion is isochoric (volume preserving).

2. Consider a plate with dimensions in the reference configuration $L \times L \times h$, where h is the dimension in the thickness direction (see Figure below). Suppose the plate is incompressible with Cauchy stress

$$\mathbf{T} = -p\mathbf{I} + 2\alpha_1 \mathbf{B} - 2\alpha_2 \mathbf{B}^{-1} \tag{2}$$

where α_1 and α_2 are constants and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green strain tensor (such a material is called a Mooney-Rivlin material). Further suppose the plate is under uniform biaxial stress resulting from normal tractions a and b as shown.



Take the motion to be given as $x_i = \lambda_i X_A \delta_{iA}$ (no sum on i), where we use lower case indices for Eulerian quantities and upper case indices for Lagrangian quantities (reconciled by the Kronecker delta δ_{iA}).

a) Calculate $\mathbf{F}$, $\mathbf{B}$ and $\mathbf{T}$.

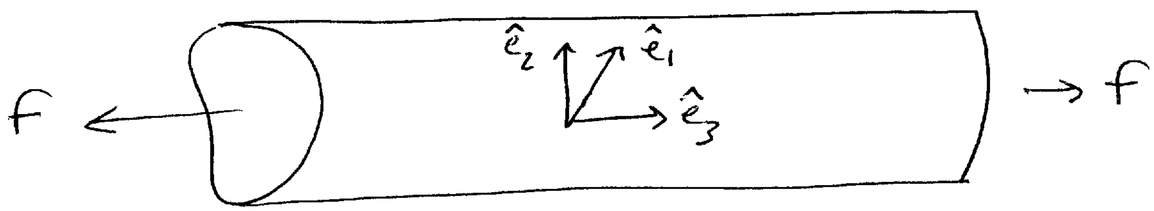
b) Using the traction boundaries conditions on all faces of the plate, derive expressions for the normal tractions a and b in terms of the λ_i and the material constants α_1 and α_2.

3. A constitutive relation for foam rubber is postulated as

$$\mathbf{T} = III_B^{-3/2}\left[(\psi(III_B) - \beta II_B)\,\mathbf{I} + (\alpha III_B + \beta I_B)\,\mathbf{B} - \beta\mathbf{B}^2\right] \qquad (3)$$

where $\mathbf{I}$ is the identity, $\mathbf{T}$ is the Cauchy stress, $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green strain tensor with principal invariants I_B, II_B and III_B, α and β are material constants and $\psi(III_B)$ is an unknown function of the third principal invariant.

A cylindrical piece of the foam rubber is put under a uniform normal traction f as shown below. Note $\hat{\mathbf{e}}_3$ gives the axis of the cylinder and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ define the cross-section of the cylinder. The cross-section is arbitrary and has unit normal $\hat{\mathbf{n}} = n_1\hat{\mathbf{e}}_1 + n_2\hat{\mathbf{e}}_2$.



The principal stretches of the cylinder under the normal traction f are Λ, λ, λ, where Λ is aligned with the axis of the cylinder.

a) Calculare $\mathbf{F}$ and $\mathbf{B}$.
b) Calculate the principal invariants I_B, II_B and III_B of $\mathbf{B}$.
c) Calculate $\mathbf{T}$.
d) Noting that the surface of the cylinder with normal $\hat{\mathbf{n}}$ is traction-free, derive an expression (parameterized by α and β) relating ψ, Λ and λ.
e) Suppose a log-log plot of λ versus Λ is a line with slope $-\nu$, where ν is a positive constant. Use this information together with the result from (d) to find the functional form of $\psi(III_B)$ in terms of α, β and ν.
f) Calculate the axial traction f as a function of Λ, parameterized by α, β and ν.
g) Suppose $\nu = 1/2$. Discuss the implications of this choice on the results above. Hint: consider the differences between problems 2 and 3.