Problem 1: Gain and Phase Margins

Gain and phase margins correspond to the amount of independent gain and phase variation the closed-loop system can tolerate before destabilizing the system. Consider the interconnection structure shown in the figure below. We are interested in relating the perturbation $\delta$ to the gain and phase margin calculations.

![Feedback Loop Diagram](image)

Problem 1a:
What is the transfer function from $q_{cmd}$ to $q$? How is this transfer function related to the loop transfer function $G(s)K(s)$ of the system? How is this transfer function related to the calculation of the gain and phase margins?
Problem 1b:
Show that if $\delta$ is a real variable, the value of $\delta$ that causes the closed-loop system to be destabilized is $GM - 1$ where $GM$ is the gain margin of the loop transfer function.
Problem 1c:
How should $\delta$ be defined such that it would correspond to the phase margin of the system? Show how the perturbed closed-loop system is related to the phase margin calculation defined for $\delta$. 
Problem 2: Orion Crew Exploration Vehicle
NASA is currently developing booster rockets and a crew exploration vehicle to replace the Space Shuttle. One main component of the Orion Crew Exploration Vehicle (CEV) is a conical crew module. This module has several thrusters to control the vehicle attitude on re-entry to the earth’s orbit. A linear model for the short-period mode of the CEV pitch dynamics during re-entry is given by:

\[ \dot{x} = \begin{bmatrix} Z_\alpha & 1 \\ M_\alpha & M_q \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]

where \( x \) := \begin{bmatrix} \delta \alpha \\ q \end{bmatrix}^T \) and \( u \) is the pitch torque generated by the thrusters.

Problem 2a: CEV Transfer Function
What is the transfer function \( G(s) \) from \( u \) to \( q \)?
Problem 2b: Open Loop Poles/Zeros
Compute the locations of the poles and zeros of $G(s)$ if $Z_\alpha = 2$, $M_q = 2$, and $M_\alpha = -36$. 
Problem 2c: Controller Architecture
Consider the 2 control architectures shown below.

1. Calculate the transfer functions from $q_{cmd} \rightarrow q$ and $q_{cmd} \rightarrow e$ for architectures A and B.

2. Which is the preferred architecture, A or B, for tracking control? Why? Discuss your decision in the context of the transfer functions calculated in the previous part.

3. For what control problems is architecture A preferred? architecture B?

![Diagram of architectures A and B]
Problem 2d: Root Locus
Consider the rate feedback loop in the figure shown below. Assume a proportional control law, \( K(s) \equiv K \), and sketch the root locus of closed loop poles for positive controller gains. Label the open loop poles and zeros.

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q_{cmd}  \rightarrow e \rightarrow K(s) \rightarrow u \rightarrow G(s) \rightarrow q
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Figure 2: Rate Feedback Loop
Problem 2e: Proportional Control
What values of the proportional gain $K$ achieve closed loop stability?
Design a proportional controller, $K_1$ such that the closed loop poles have natural frequency $\omega_n = 4 \text{ rad/sec}$. Design another proportional controller, $K_2$ such that the closed loop poles have natural frequency $\omega_n = 5 \text{ rad/sec}$. Label the locations of the closed loop poles for both $K_1$ and $K_2$ on your root locus plot.
Problem 2f: Bode Plots

On the figures below, sketch the Bode plots of $G(s)K_1$ and $G(s)K_2$. 

![Bode Plot](image-url)
Problem 2g: Gain Margin
What are the gain margins associated with $K_1$ and $K_2$?
Problem 2h: Time Delay
The controller that you designed will be implemented on a microcontroller in discrete-time. The implementation sample rate is 25 Hz. This will introduce a 0.04 sec time delay in the control signal. Recall a pure time-delay can be represented via a Laplace transform as $e^{-sT}$ where $T$ is the time delay.

1. Graph the Bode plot of the 0.04 sec time delay.

2. How does the introduction of the 0.04 sec time delay effect the closed-loop performance and robustness of your control design.

3. Graph the Bode plot of the revised loop transfer function and describe the resulting gain and phase margins.

4. If your controller no longer achieves the desired performance and robustness specifications with the introduction of the 0.04 sec time delay, redesign your controller.
Problem 2i: Step Response
Sketch the response of $q$ due to a unit step of the reference command, $q_{cmd}$, for both gains. You can assume zero time delay when making these plots. What are some issues with these step responses and how can the design be modified to deal with these issues? In particular, discuss the steady-state error, the over/undershoot, and settling time.
Problem 2j: RHP Zero Constraints
The plant \( G(s) \) has a RHP zero, i.e. a value \( z \) with \( \text{Re}(z) > 0 \) such that \( G(z) = 0 \). This places fundamental constraints on the closed loop performance. One fundamental constraint relates the relative undershoot and the settling time:

1. Let \( T(z) := \frac{G(s)K(s)}{1+G(s)K(s)} \) be the complementary sensitivity function for an arbitrary stabilizing controller \( K(s) \). What is the constraint on \( T(z) \)?

2. Let \( Q(s) \) denote the Laplace Transform of the response \( q(t) \) due to a unit step in the reference command \( q_{\text{cmd}} \). What is the constraint on \( Q(z) \)?

3. Let \( t_s \) denote settling time such that \( q(t) \approx q_{ss} \) for all \( t \geq t_s \). One characteristic of a system with a RHP zero is that its step response initially undershoots. Define the relative undershoot, \( ru \), as:

\[
ru := \max_{0 \leq t \leq t_s} \left[ \frac{q(t)}{q_{ss}} \right].
\]

Use the constraint on \( Q(z) \) and the definition of the Laplace Transform to show that the relative undershoot must satisfy

\[
ru \geq \frac{1}{e^{zt_s} - 1}
\]

4. What can you say about the relative undershoot if we try to make the settling time very small (\( t_s << 1 \))?