1. For the linear advection equation \[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \] with \( a > 0 \), consider the ‘Leapfrog’ method:

\[ \frac{u^{n+1}_i - u^{n-1}_i}{2\Delta t} + a \left( \frac{u^{n+1}_{i+1} - u^{n+1}_{i-1}}{2\Delta x} \right) = 0. \]

a) Derive the modified equation \( \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \frac{\partial^5 u}{\partial t^5} \). Convert all spatial derivatives into temporal derivatives and consider temporal derivatives up to the fifth-order.

b) What does this modified equation indicate about the dissipation and dispersion of the Leapfrog method? Without performing stability analysis, does this modified equation reveal any information about the stability of the method? What is the order of accuracy of the method?

c) Are there any special CFL numbers where any/all of the terms on the right-hand-side of the modified equation vanish? What would the implication of this be for a practical numerical simulation?

d) Sketch the numerical solution (obtained when using the Leapfrog method) in the vicinity of a discontinuity.

e) Now use von Neumann analysis to determine the stability limits on the Courant number, \( c = (a\Delta t/\Delta x) \), for the ‘Leapfrog’ method applied to the linear advection equation.

f) Given an initial discretized solution, \( u_i^{n=0} \), explain how you would implement the Leapfrog method to obtain a solution at some later time (i.e. \( u_i^{n+1}, u_i^{n+2}, \ldots, u_i^n \)).

g) One of the problems with the Leapfrog method is “odd-even decoupling” of the solution. Explain how this could occur using the Leapfrog stencil (sketch an enlarged stencil of the method using multiple points in x and t).

h) Is this method useful for the simulation of practical problems? Why or why not?