1. Problem 1

Given a simple pendulum (mass $m$ attached from above to a mass-less rod of length $l$) with state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$. The pendulum is swinging according to a nominal trajectory $\theta(t) = 2\pi t$.

Determine the linearized equations of motion that describe the pendulum’s dynamics around this nominal condition.

Figure 1: Illustration of simple pendulum.
2. Problem 2

Given the system from Problem 1 but with static equilibrium $\ddot{\theta} = 0$ and a horizontal force $u$ acting on the mass $F_x = l \cos \theta(t) u(t)$. The output of the system is the angular position of the arm $y = \theta$.

(a) Show that the linearized equations are:

$$\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{q}{m} & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{l}{m} \end{bmatrix} \delta u$$ \hspace{1cm} (1)

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$ \hspace{1cm} (2)

(b) Assuming you have a state estimator, you are implementing a state-feedback controller:

$$\delta u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \delta x$$ \hspace{1cm} (3)

Write out the state-space model for the closed-loop system and draw the closed-loop block diagram.

(c) Convert the state-space equations into the Laplace domain.

(d) Determine the characteristic equation.

(e) Draw the Bode plot.
(f) Repeat the same (b)-d) but with an output feedback controller.
(g) Based on this example, describe the differences between the type of control performance you can achieve with the state feedback and the output feedback. Make sure to back up your statement by analysis.
3. Problem 3

In the linear system's model the control parameter $k_2$ is uncertain.

Express the family of plants resulting from the uncertainties using a feedback uncertainty (around the nominal plant) shown in the Figure bellow. $\Delta$ is a scalar perturbation $-1 \leq \Delta \leq 1$ and $W_2$ is a weight (which can be a function of $s$).

![Block diagram of a feedback uncertainty](image)

Figure 2: Block diagram of a feedback uncertainty.