Stress in Viscous Potential Flow

\[ T_{ij} + p \delta_{ij} = \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] = 2\mu \frac{\partial^2 \phi}{\partial x_i \partial x_j} \]

- Viscosity may generate a significant contribution to the stress in regions of irrotational flow


Viscous Potential Flow

- The theory of potential flow of an inviscid fluid can be replaced with the theory of potential flow of a viscous fluid. The drag on a body in potential flow is independent of the viscosity but there is an additional viscous moment in two-dimensional flow with circulation. It is evident that the various vorticity and circulation theorems which are at the foundation of the theory of inviscid potential flow hold equally when the viscosity is not zero. In addition, the theory of viscous potential flow admits approximations to real flows in certain situations. Perhaps if the theory of potential flow had developed after 1850, no one would put $\mu = 0$. 
Viscous Potential Flow with Interfaces

\[ u = \nabla \phi, \quad \nabla^2 \phi = 0, \quad \text{Bernoulli equation} \]

Interface conditions

\[ \omega = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y}, \quad \text{Kinematics of moving interfaces} \]

\[ -[[p]] + 2\mu \left[ \frac{\partial u_n}{\partial n} \right] = \gamma \nabla \pi \cdot n, \quad \text{Jump of normal stress}\]

balanced by surface tension

Continuity of tangential component of velocity continuity of the shear stresses are not enforced

- Shear stresses are neglected
- Extensional stresses are not neglected
Viscous Potential flow analysis of Rayleigh-Taylor Instability

Rayleigh –Taylor Waves in Water. The tick marks on the photographs locate wave troughs.


Ms = 2

Ms = 3
$n$ (sec$^{-1}$) vs. $k$ (cm$^{-1}$) for Viscous Potential Flow; Shock Mach no. = 3
Values of the wave number, wave length, and growth rate of the most dangerous wave for the experimental conditions given in tables 1 and 2.

\( M_s = 3 \)

<table>
<thead>
<tr>
<th>Liquid</th>
<th>Viscosity (kg/msec)</th>
<th>Viscosity (sec^-1)</th>
<th>Fully Viscous ( k (cm^1) )</th>
<th>( \lambda (mm) )</th>
<th>Viscous Potential ( n (sec^-1) )</th>
<th>Viscous Potential ( k (cm^1) )</th>
<th>( \lambda (mm) )</th>
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<td>SO 10000</td>
<td>100</td>
<td>17790</td>
<td>9.5</td>
<td>6.61</td>
<td>19342</td>
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<td>0.12</td>
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</table>

\( M_s = 2 \)

<table>
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<tr>
<th>Liquid</th>
<th>Viscosity (kg/msec)</th>
<th>Viscosity (sec^-1)</th>
<th>Fully Viscous ( k (cm^1) )</th>
<th>( \lambda (mm) )</th>
<th>Viscous Potential ( n (sec^-1) )</th>
<th>Viscous Potential ( k (cm^1) )</th>
<th>( \lambda (mm) )</th>
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</thead>
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</tbody>
</table>

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**Maximum Growth Rate Curve**

- Maximum growth rate
  - \( \lambda > 2\pi \sqrt{\gamma/\rho ca} \)
  - \( a = 10^5 \text{g} \)

- Unstable
- Drop diameters which are completely stable to RT instability are less than \( \lambda_c \)
- 23 to 65 microns \( M_S = 3 \)
- 46 to 135 microns \( M_S = 2 \)

- Most dangerous wave
- Universal cut-off, independent of viscosity
  - \( k = \sqrt{\rho ca/\gamma} \)
  - \( a = 10^5 \text{g} \)
Viscous Potential Analysis of Kelvin-Helmholtz Instability in a channel

Kelvin Helmholtz instability due to a discontinuity of velocity of air above liquid in a rectangular channel. The no-slip condition is not enforced in viscous potential flow so that the two-dimensional solution satisfies the side-wall boundary conditions. There is no exact viscous solution.

The critical velocity is the minimum value on the neutral curve. The vertical line is $\hat{\mu} = \hat{\rho} = 0.0012$ and the horizontal line at $V = 635.9$ cm/sec is the critical value for inviscid fluids. The vertical dashed line at $\hat{\mu} = 0.018$ is for air and water. Typical values for a high-viscosity liquid are given in table 3 below.
Mandhane flow chart for PDVSA-Intevep data from 0.508 m I.D. flow loop with air and 0.480 Pa.s lube oil. The identified flow patterns are smooth stratified (SS, open circles), wavy stratified (SW, open squares), slug (SL, open triangles) and annular (AN, open diamonds). Stratified- and non-stratified transition theories after different authors are compared; Taitel and Dukler (1976) [TD]: stars, Barnea and Taitel (1993) [BT]: +, inviscid Kelvin-Helmholtz with [IKH]: ×, Funada and Joseph (2001) [FJ]: broken line, Funada and Joseph multiplied by $\alpha$ (2001) [FJ]×$\alpha$: heavy line. Notice that the curves [FJ] and [FJ]×$\alpha$ sharply drop around $U_{SG} \approx 5$ m/s.
The force $\gamma = r$ forces fluid from the throat, decreasing $r$ leading to collapse.

No.  | material (fluid $l$ – fluid $a$)  
---|---
1   | mercury – air  
2   | mercury – water  
3   | water – air  
4   | benzene – air  
5   | water – benzene  
6   | SO100 – air  
7   | glycerin – mercury  
8   | glycerin – air  
9   | oil – air  
10  | goldensyrup – CC4 and paraffin  
11  | SO10000 – air  
12  | goldensyrup – BB oil  
13  | goldensyrup – Black lubrication oil  
14  | tar pitch mixture – goldensyrup
The growth rate $\sigma$ vs. $k$ for case 1, mercury in air

The growth rate $\sigma$ vs. $k$ for case 4, benzene in air

The growth rate $\sigma$ vs. $k$ for case 2, mercury in water

The growth rate $\sigma$ vs. $k$ for case 5, water in benzene

The growth rate $\sigma$ vs. $k$ for case 3, water in air

The growth rate $\sigma$ vs. $k$ for case 6, SO100 in air
The growth rate $\sigma$ vs. $k$ for case 7, glycerin in mercury

The growth rate $\sigma$ vs. $k$ for case 10, goldensyrup in paraffin

The growth rate $\sigma$ vs. $k$ for case 8, glycerin in air

The growth rate $\sigma$ vs. $k$ for case 11, SO10000 in air

The growth rate $\sigma$ vs. $k$ for case 9, oil in air

The growth rate $\sigma$ vs. $k$ for case 12, goldensyrup in BB oil
The growth rate $\sigma$ vs. $k$ for case 13, goldensyrup in black lubrication oil

The growth rate $\sigma$ vs. $k$ for case 2, (inverse) water in mercury

The growth rate $\sigma$ vs. $k$ for case 14, tar pitch mixture in goldensyrup

The growth rate $\sigma$ vs. $k$ for case 3, (inverse) air in water

The growth rate $\sigma$ vs. $k$ for case 1, (inverse) air in mercury

The growth rate $\sigma$ vs. $k$ for case 4, (inverse) air in bezine
The growth rate $\sigma$ vs. $k$ for case 5, *inverse* benzene in water

The growth rate $\sigma$ vs. $k$ for case 8, *inverse* air in glycerin

The growth rate $\sigma$ vs. $k$ for case 6, *inverse* air in SO100

The growth rate $\sigma$ vs. $k$ for case 9, *inverse* air in oil

The growth rate $\sigma$ vs. $k$ for case 7, *inverse* mercury in glycerin

The growth rate $\sigma$ vs. $k$ for case 10, *inverse* paraffin in goldensyrup
The growth rate $\sigma$ vs. $k$ for case 11, \text{(inverse)} air in SO10000

The growth rate $\sigma$ vs. $k$ for case 13, \text{(inverse)} black lubrication oil in goldensyrup

The growth rate $\sigma$ vs. $k$ for case 12, \text{(inverse)} BB oil in goldensyrup

The growth rate $\sigma$ vs. $k$ for case 14, \text{(inverse)} goldensyrup in tar pitch mixture