Stability and Disturbance Propagation in Autonomous Vehicle Formations : A Graph Laplacian Approach

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Abstract

Recent works have combined tools from graph theory and control theory to derive stability criteria for vehicle formations. The formulation typically models information exchange between vehicles in a formation by using graph-theoretic tools. This work builds upon these ideas by studying the link between topology of these graphs and the dynamical behavior of a formation subject to disturbances. The problem is re-formulated by using Kronecker products. It is then shown how the Schur decomposition of the graph Laplacian matrix may be used to identify singular disturbance directions. The results we present may be used to simplify the design of robust decentralized controllers for large scale vehicle formations.

1 Introduction

The study of stability and disturbance propagation in vehicle formations is increasingly receiving deserved attention from the controls community. The range of applications which have motivated such research is wide - from car platoons and formation of robots to unmanned air vehicles (UAV) [1, 2].

Vehicle formations belong to a subset of large-scale interconnected systems. The dynamical systems which model the vehicles are assumed to be identical, decoupled and independently controlled. Moreover, each controller is assumed to have only a partial knowledge of the formation status. Because of such peculiar structure, researchers have been trying to decompose the study of stability and robustness of vehicle formations into simple smaller subproblems which depend only on the single dynamics of each vehicle and the interconnection topology. Since an extensive review of all published results in this area goes beyond the scope of this paper, the interested reader is referred to [1] for a good list of references.

Recently, tools from graph theory and control theory were merged in order to derive stability criteria for vehicle formations [3, 4]. The authors model the exchange of information in a vehicle formation by using graph topology. Each node of the graph represents a vehicle and the topology
of the graph captures the information flow between the vehicles. The framework has proven to be useful when all the vehicles are modeled by linear time-invariant dynamics, and each vehicle is controlled by a linear time-invariant controller which minimizes an average error. This error is a function only of each vehicle’s state and the states of the neighboring vehicles. If all the nodes and all the controllers are identical, then the stability of the formation is a function of the single vehicle dynamics and of the Laplacian eigenvalues of the graph. Other recent works in which the same problem formulation has been successfully applied include [5, 6, 7].

In this work, we develop these ideas further by studying the link between graph topology and the dynamical behavior of a formation subject to disturbances. We use the same framework as in [3] and assume additive disturbance on each single node. We show how the Schur decomposition of the graph Laplacian matrix can be used to identify singular disturbance directions which are functions only of the graph topology [11, 12]. In special cases, these singular disturbance directions coincide with the eigenvectors of the graph Laplacian matrix. For this reason, this work can be regarded as the natural extension of the approach of [3] to robustness analysis of a formation. We will make use of certain properties of Kronecker products to re-formulate some of the problems and results presented in [3] in a compact and much simpler form.

There are two main advantages of the approach in this work. First, the study of the robustness to additive disturbance of a formation can be decomposed into a number of studies on smaller subsystems - in the same way as it can be done for stability [3]. Second, the knowledge of the singular directions of disturbance propagation is helpful when the disturbances on the formation are a subset of the possible disturbances, as explained in Section 5.

The rest of this work is organized as follows: Section 2 discusses the vehicle model used and the formulation of the formation topology in terms of Kronecker products. The stability decomposition scheme is discussed in Section 4 as well as the main results of this work. This is then extended to decomposed robustness analysis in the presence of disturbances in Section 5. Illustrative examples to demonstrate these schemes are presented in Section 6 followed by some concluding remarks in Section 7.

2 Problem Formulation

We consider a set of \( N_v \) vehicles with identical linear dynamics. The \( i \)-th dynamical vehicle model is:

\[
\dot{x}_i = Ax_i + Bu_i + Dd_i
\]

where \( x_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^m \), \( d_i \in \mathbb{R}^l \) are states, inputs and disturbances of the \( i \)-th system, respectively.

It is assumed that the \( N_v \) vehicles are appropriately controlled to fly in formation. The control laws are decentralized and the \( i \)-th vehicle controller uses only information from a subset of vehicles in the formation. As in [3], we describe the exchange of information within the team by a graph \( G = (V, E) \). The \( i \)-th node of the graph represents the \( i \)-th vehicle and belongs to the node set \( V = \{1, \ldots, N_v\} \). An arc leaving the \( i \)-th node and entering the \( j \)-th node will be denoted by \((i, j) \in E\) and will correspond to an exchange of information from the \( i \)-th vehicle to the \( j \)-th vehicle. The interested reader is referred to [8] for other standard graph-theoretic terms not defined here.
Denote by \( J_i \subset V \) the set of neighboring nodes to the \( i \)-th node, i.e., the set of all vehicles with which the \( i \)-th vehicle can exchange information. This can be written as \( J_i = \{ j \mid (j, i) \in E \} \). The Laplacian matrix of the graph is defined as \( L = D - Am \) where \( D \in \mathbb{N}^{|N_v| \times |N_v|} \) is the diagonal matrix of vertex in-degrees, i.e., \( D = \text{diag}(|J_1|, \ldots, |J_N|) \) and \( Am \in \mathbb{N}^{|N_v| \times |N_v|} \) is the \((0, 1)\) adjacency matrix, i.e., \( Am_{i,j} = 1 \) if \((i, j) \in E\), otherwise \( Am_{i,j} = 0 \).

Similar to [3], we assume that the vehicle formation is defined in terms of relative distances between the vehicles. In particular, we assume that the coordinate system of the vehicle dynamic (1) has been chosen so that the vectors

\[
z_{ij} = C_2(x_i - x_j), \quad z_{ij} \in \mathbb{R}^m
\]

for all \((i, j) \in E\) denotes the error between the current measurements and their desired values - i.e., the quantities that the formation controllers should bring to zero.

We will assume that each vehicle is controlled by an identical and decentralized linear time-invariant control law

\[
\begin{align*}
\dot{v}_i &= Rv_i + Sz_i \\
u_i &= Tv_i + Vz_i
\end{align*}
\]

where \( z_i \) is assumed to be the sum of the neighboring error signals

\[
z_i = \sum_{j \in J_i} z_{ij}
\]

with \( z_i \in \mathbb{R}^m \).

**Remark 1** In practice, the error in (2) is written as

\[
z_{ij} = C_2(x_i - x_j) - r_{ij}
\]

where \( r_{ij} \) is the reference for the relative error between the \( i \)-th and the \( j \)-th output. This would introduce a constant vector of references in the controller (3)-(4). We will neglect such constant term since it has no relevance in the results that will be presented in this paper.

**Remark 2** The assumption on the control structure (3)-(4) is crucial and in some cases might not be the optimal one. However, several classes of formation flight controllers used in the literature belong to this scheme. For instance an LQR controller may be used in leader-follower formations.

Note also that Fax and Murray in [3] use an average sum in (4). We prefer to remove the scaling factor \( |J_i| \) in order to remain consistent with the more commonly understood definition of the Laplacian matrix. However, such a choice will not affect our results since any scaling factor can be accounted for in the formulation of (4).

**Remark 3** In model (1) the disturbance is assumed to be additive on the state. Often, this is a good model for wind gust disturbances which affect the position of a vehicle. The results presented in this work are therefore applicable to the more general model

\[
\begin{align*}
\dot{x}_i &= Ax_i + Bu_i + Dw_i \\
\dot{w}_i &= A_wx_i + B_wu_i + D_wd_i.
\end{align*}
\]

However, we will continue to use the simpler model (1) for clarity of the exposition.
Denote by \( x \) the state of the vehicle team, \( x = [x_1, \ldots, x_{N_v}] \), by \( u \) the input, \( u = [u_1, \ldots, u_{N_v}] \) and by \( d \) the disturbances, \( d = [d_1, \ldots, d_{N_v}] \). The dynamics of the team can be described in compact form as

\[
\dot{x} = (I_{N_v} \otimes A)x + (I_{N_v} \otimes B)u + (I_{N_v} \otimes D)d
\]

where \( I_{N_v} \in \mathbb{R}^{N_v \times N_v} \) is the identity matrix and \( \otimes \) denotes the Kronecker product.

Let \( \tilde{x}_i \) denote both the states of the \( i \)-th vehicle and controller, i.e., \( \tilde{x}_i = [x_i, v_i] \), then, the \( i \)-th node can be compactly described as

\[
\dot{\tilde{x}}_i = \tilde{A}\tilde{x}_i + \tilde{B}z_i + \tilde{D}d_i
\]

with

\[
z_i = \sum_{j \in J_i} \tilde{C}_2(\tilde{x}_i - \tilde{x}_j)
\]

where the matrices \( \tilde{A}, \tilde{B}, \tilde{D} \) and \( \tilde{C}_2 \) are easily obtained from (1),(3) and (4). Throughout the rest of the paper we will refer to the \( i \)-th vehicle controlled by the \( i \)-th controller as the \( i \)-th node of the graph \( G \).

By using the definition of the Laplacian matrix and the Kronecker product, the model of the team (6) under the action of the decentralized control laws (3)-(4) can be compactly described as

\[
\dot{x} = (I_{N_v} \otimes \tilde{A} + L \otimes \tilde{B}\tilde{C}_2)x + (I_{N_v} \otimes \tilde{D})d
\]

where \( \tilde{x} \) and \( \tilde{d} \) denote the state and the disturbance vectors of the overall formation, respectively, i.e., \( \tilde{x} = [\tilde{x}_1, \ldots, \tilde{x}_{N_v}] \), \( \tilde{d} = [d_1, \ldots, d_{N_v}] \) and \( L \) is the graph Laplacian of dimension \( N_v \times N_v \) as already described.

The stability of the formation and its robustness to the disturbance \( \tilde{d} \) can be studied by applying standard tools to the system (9). The objective of this work is to identify how robustness and stability of the overall system (9) are dependent upon the local dynamics (7) and the properties of the Laplacian matrix \( L \). Before giving the main results we will briefly review the main properties of the eigenvalues of graph Laplacian matrices.

### 3 Graph Laplacians: Eigenvalues and Eigenvectors

This section is a concise review of the relation between eigenvalues and eigenvector of a Laplacian matrix and the topology of a graph. We refer the reader to [9] for a comprehensive treatment of the topic. The eigenvalue results are adapted from [3].

Eigenvalues of Laplacian matrices have been widely studied by graph theorists. Their properties are strongly related to the structural properties of their associated graphs. Denote by \( \Lambda(M) \) the set of eigenvalues of a square matrix \( M \). Let \( \Lambda(L(G)) = \{\lambda_1, \ldots, \lambda_{N_v}\} \) be the set of eigenvalues of the Laplacian matrix \( L \) associated with a graph \( G \) arranged in nondecreasing semi-order. Then,

1. \( \lambda_n = 0 \) and \( \lambda_{n-1} \neq 0 \) iff \( G \) is connected.
2. All the \( \lambda_i, \ i = 1, \ldots, N_v \) are real if \( G \) is undirected.
3. The modulus of $\lambda_i$, $i = 1, \ldots, N_v$ is less then $N_v$.

Some simple relations between the graph topology and eigenvalues location have been discussed in [3]. For instance, a leader-follower configuration corresponds to a graph whose Laplacian matrix has all real eigenvalues equal to one.

Eigenvectors of graph Laplacians have recently started to interest researchers. In [10], Merris describe their main properties. Therein, eigenvectors are used to study the way the spectrum of the Laplacian matrix changes when arcs are added or deleted from the graph. In the next sections, we will show the respective role eigenvalues and eigenvectors play in stability and robustness analysis of a formation.

4 Main Stability Decomposition Scheme

Let $n_c$ be the number of complex and conjugate eigenvalues of $L$ and let $\delta$ be the number of eigenvalues of $L$ when complex and conjugate eigenvalues are counted only once, i.e., $\delta = N_v - n_c/2$. Consider the Schur decomposition of the Laplacian matrix $L$

$$L = U'\Delta U,$$

(10)

where $U$ is a unitary real matrix and $\Delta$ is a block upper triangular real matrix of the form

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1\delta} \\
0 & \Delta_{22} & \cdots & \Delta_{2\delta} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \Delta_{\delta\delta} \end{bmatrix}. \tag{11}$$

The block diagonal elements $\Delta_{ii}$, $i = 1, \ldots, \delta$ in (11) are either the real eigenvalues of $L$ or two-dimensional square matrices with $\Lambda(\Delta_{ii}) \in \Lambda(L)$. If all eigenvalues are real then, $\delta = N_v$ and $\Delta$ is an upper-triangular matrix.

We denote by $n_i$ the dimension of $\Delta_{ii}$, i.e., $\Delta_{ii} \in \mathbb{R}^{n_i \times n_i}$, $n_i \in \{1, 2\}$. Also, we denote by $\Delta_{\lambda_i}$ the block-diagonal element $\Delta_{ii}$ associated with the real eigenvalue $\lambda_i$, i.e., $\Delta_{\lambda_i} = \lambda_i$ and $\Delta_{\lambda_i,\bar{\lambda}_i}$ will denote the block-diagonal element associated with the complex eigenvalue couple $(\lambda_i, \bar{\lambda}_i)$, i.e., $\Delta_{\lambda_i,\bar{\lambda}_i} = \Delta_{kk}$ with $\Lambda(\Delta_{kk}) = \{\lambda_i, \bar{\lambda}_i\}$.

**Theorem 1** Consider the coordinate transformation $\xi = (U \otimes I_n)\tilde{x}$, where $U$ is the schur unitary matrix in (10). Then, the dynamics of the formation (9) in the new coordinate system are

$$\dot{\xi} = (I_{N_v} \otimes \tilde{A} + \Delta \otimes \tilde{B}\tilde{C}_2)\xi + (U' \otimes \tilde{D})\tilde{d} \tag{12}$$

_Proof:_ For the controlled team described by equation (9), the change of variables $\xi = (U \otimes I_n)\tilde{x}$. Then,

$$(U \otimes I_n)^{-1}\dot{\xi} = (I_{N_v} \otimes \tilde{A} + U'\Delta U \otimes \tilde{B}\tilde{C}_2)(U \otimes I_n)^{-1}\xi + (I \otimes \tilde{D})\tilde{d} \tag{13}$$

by using the associative properties of the Kronecker product we obtain

$$\dot{\xi} = (U^{-1}I_{N_v}U \otimes \tilde{A} + U^{-1}U'\Delta U U^{-1} \otimes I_n\tilde{B}\tilde{C}_2 I_n)\xi + (U^{-1} \otimes I_n\tilde{D})\tilde{d} \tag{14}$$

5
which proves the theorem.

In what follows, we will start from equation (12) and draw a series of simple conclusions which will allow us to reduce the study of disturbance propagation and stability of the formation into a number of studies on smaller systems. All the results will be presented as simple corollaries of Theorem 1.

We begin by stating the stability result given in [3]. It is clear that its proof is a direct consequence of Theorem 1 (note that, strictly speaking, Theorem 1 in [3] is mathematically correct only if the eigenvalues of the Laplacian matrix \( L \) are real).

Corollary 1 System (6) controlled by (3)-(4) is stable if and only if the \( \delta \) subsystems

\[
\dot{\xi}_i = (I_{n_i} \otimes \hat{A} + \Delta_{ii} \otimes \hat{B} \hat{C}_2) \xi_i
\]  

are stable.

Corollary 1 relates the stability of the overall team to the stability of \( \delta \) systems where \( \delta \) is the number of eigenvalues of the Laplacian, when complex and conjugate eigenvalues are counted only once.

It is straightforward to show that for real eigenvalues, Theorem 1 is equivalent to Theorem 1 in [3]. In fact, if all the eigenvalues of the Laplacian are real then, \( \delta = N_v \) and each subsystem (15) is

\[
\dot{\xi}_i = (\hat{A} + \lambda_i \hat{B} \hat{C}_2) \xi_i
\]  

Therefore, system (16) will be stable if an only if the \( i \)-th controller (3) stabilizes the \( i \)-th systems (1) with \( \xi_i = \lambda_i C_2 x_i \) for all \( i = 1, \ldots, N_v \) (which is the results of Theorem 1 in [3]).

Remark 4 We will use connected graph topology. This implies that there exists \( i \in V \) such that \( \Delta_{ii} = 0 \). Therefore by Theorem 1, if the \( i \)-th vehicle is unstable, then all the team will be unstable no matter which controller is used (note that \( \hat{A} = \begin{bmatrix} A & B \end{bmatrix} \)). In fact, it is not possible to stabilize a set of identical unstable systems by using identical state-feedback controllers which use only relative measurement errors.

Theorem 1 clearly links the stability of the team with global and local properties of the formations: the dynamics of each vehicle and the eigenvalue of the Laplacian matrix associated with the graph which model the exchange of information between vehicles. The main contribution of this report will be to extend this result to disturbance analysis. We will study how disturbance propagation in a team is related to global and local properties. In particular, we will show how the matrix \( U \) in (10) associated with the Laplacian plays a critical role in defining singular directions of the disturbance propagation.

5 Main Robustness Decomposition Scheme

Depending on the symmetry properties of the graph \( G \) some elements of the matrix \( \Delta \) in (11) can be equal to zero [11, 12]. We will distinguish between two extreme cases: a full upper triangular matrix \( \Delta \) and a block-diagonal matrix \( \Delta \). All the other cases can be studied as a combination of these two.
5.1 $\Delta$ block-diagonal

Assume that $\Delta$ in (11) is block diagonal, i.e., $\Delta_{ij} = 0$ if $i \neq j$, and consider system (12). Denote by $e_i$ the $i$-th row of the matrix $U'$. Consider a disturbance vector $\tilde{d}$ acting on the formation and decompose it as

$$\tilde{d} = \tilde{d}_1 + \cdots + \tilde{d}_N,$$

where

$$\tilde{d}_i = e_i \otimes \alpha_i, \quad i = 1, \ldots, N_v \quad (17)$$

with $\alpha_i \in \mathbb{R}^l$, $i = 1, \ldots, N_v$.

Note that such a decomposition is always possible. That is, it is always possible to find $N_v$ vectors $\alpha_i$ such that equations (17)-(18) hold. Also, note that if $\tilde{d}$ is time variant, $\tilde{d} = d(t)$, we can use the decomposition $\tilde{d}(t) = \tilde{d}_1(t) + \cdots + \tilde{d}_N(t)$ with $\tilde{d}_i(t) = e_i \otimes \alpha_i(t), \ i = 1, \ldots, N_v$.

We will denote by $\alpha_{\lambda_i}$ the vector $\alpha_j$ in (18) with $e_j$ associated with the eigenvalue $\lambda_i$ of the Laplacian $L$. Decomposition (17)-(18) allows one to formulate the following corollary of Theorem 1.

**Corollary 2** The effect of a disturbance $\tilde{d}$ on the team (9) can be studied by analyzing the independent subsystems

$$\dot{\xi}_i = (\bar{A} + \Delta_{\lambda_i} \bar{B} \bar{C}_2)\xi_i + \bar{D}\alpha_{\lambda_i}$$ \quad $(19)$

if $\lambda_i$ is real, or

$$\dot{\xi}_i = (I_2 \otimes \bar{A} + \Delta_{\lambda_i, \lambda_i} \otimes \bar{B} \bar{C}_2)\xi_i + I_2 \otimes \bar{D}[\alpha_{\lambda_i}, \alpha_{\bar{\lambda}_i}]'$$ \quad $(20)$

if $\lambda_i$ is complex for $i = 1, \ldots, \delta$, where the vectors $\alpha_i$ are obtained from (17)-(18) and $\bar{x} = (U' \otimes I_n)\xi$.

Corollary 2 implies that the study of the team dynamics under the effect of the disturbance $\tilde{d}$ on the formation can be equivalently carried out by analyzing the effect of $N_v$ single disturbances $\alpha_i$ on $\delta$ independent and smaller sub-systems in a different coordinate system and bringing it back to the original system through the linear transformation $\bar{x} = (U' \otimes I_n)\xi$.

Therefore, a disturbance on the team can be decomposed into two components: $\alpha_i$ and $e_i$. The component $\alpha_i$ is what enters a simple dynamical system whose dynamics are function of the matrix $\bar{A}$ and of the vector $e_i$ associated with the Laplacian matrix. As an example, assume $\lambda_1$ real. Then, the dynamical behavior of the team (9) under the disturbance $\bar{d} = e_1 \otimes \alpha_1(t) \in \mathbb{R}^{N_v \times l}$ is obtained by studying the behavior of

$$\frac{d\xi_1(t)}{dt} = (\bar{A} + \lambda_1 \bar{B} \bar{C}_2)\xi_1(t) + \bar{D}\alpha_1(t)$$ \quad $(21)$

with $\alpha_1 \in \mathbb{R}^l$ and using the transformation $x = e_i z_i$. The vectors $e_i$ are called “singular” directions for the simple reason that for SISO node systems ($l = 1$, $m = 1$) a disturbance input of $\tilde{d} = e_i \otimes \sin(w t)$ to the formation (9) will produce a steady state output of the overall formation equal to $\bar{x} = e_i \otimes M \sin(w t + \phi)$ with appropriate $M$ and $\phi$.

If all the eigenvalues are real and distinct, then the vectors $e_i$ are the eigenvector of the Laplacian matrix. For this reason the work can be seen as the natural extension of the approach of [3] to robustness analysis of a formation.
In general, there will be two main directions in the transformed space: the direction of the disturbance $\alpha_i$ entering the $i$-th node and the direction $e_i$ which identifies the “mode of the team” which is excited by the disturbance.

Theorem 2 can also be helpful in the case when the disturbance acting on the formation has a certain pattern and is only a subset of the possible disturbances. For instance, if $d$ represents the wind perturbing the position of an UAV and if $D$ is the domain of the wind disturbance which enters a single node, then $\tilde{D} \triangleq D \times D \times \cdots \times D$ is the domain of feasible disturbance on the formation. However, often the real feasible disturbance space on the formation is only a proper subset of $\tilde{D}$. It is in fact realistic to assume that if wind is perturbing a node in a certain direction, the neighbouring node cannot be perturbed at the same instant in the opposite direction. In this case, $\tilde{D}$ will be described as $\tilde{D} = d \cdot [1, 1, \ldots, 1]$, $d \in D$ and it will be enough to study the effect of the disturbance $d$ on simple systems, whose dynamics are a function of the original model and of the components of the vector $[1, 1, \ldots, 1]$ in the basis $e_1, \ldots, e_{N_v}$.

5.2 $\Delta$ full upper-diagonal

We can use the same approach as in Section 5.1 above for full upper-diagonal matrices $\Delta$. The disturbance on the team will be decomposed along the directions of the vectors $e_i$ of the graph Laplacian as in (17)-(18). However, the $i$-th component $\tilde{d}_i$ will enter the $i$-th system (1) and will also affect the state of the $j$-th subsystems through the dynamic coupling for $j < i$. 

**Corollary 3** The effect of a disturbance $\tilde{d}$ on the team (9) can be studied by analyzing the $\delta$ subsystems

$$\dot{\xi}_i = (\hat{A} + \Delta \lambda_i \hat{B} \hat{C}_2)\xi_i + \sum_{j=1}^{i-1} \Delta_{ij}\xi_j + \hat{D}\lambda_\alpha_i$$ (22)

if $\lambda_i$ is real, or

$$\dot{\xi}_i = (I_2 \otimes \hat{A} + \Delta \lambda_i, \bar{\lambda}_i \otimes \hat{B} \hat{C}_2)\xi_i + \sum_{j=1}^{i-1} \Delta_{ij}\xi_j + I_2 \otimes \hat{D}[\alpha_\lambda_i, \bar{\alpha}_\lambda_i]'$$ (23)

if $\lambda_i$ is complex for $i = 1, \ldots, \delta$, where the vectors $\alpha_i$ are given in (17)-(18) and $\tilde{x} = (U \otimes I_n)\xi$.

One should notice that Theorem 3 makes no assumptions on the graph topology and on the disturbance $\tilde{d}$. Therefore, Theorem 3 reformulates the study of the team dynamics under the effect of the disturbance $\tilde{d}$ into the study of $\delta$ smaller sub-systems affected by the disturbances $\alpha_i$ and connected in a tree structure, i.e., the $i$-th node is only a function of the node preceding it. All the remarks and observations in Section 5.1 on the meaning of singular disturbance direction can be readily extended to the decomposition scheme presented in this section.

6 Examples

We consider the following simple model for a node

$$y_i = \frac{0.12}{s^2 + 0.48s + 0.12}u_i + \frac{0.15s}{s^2 + 0.33s + 0.24}d_i,$$ (24)
and a simple decentralized proportional controller

\[ u_i = K z_i \]  \hspace{1cm} (25)  

where \( z_i \) is defined as in (4). System (24) could describe one position coordinate of a controlled UAV which reacts to wind disturbance by oscillating and bringing its position to the original reference. The local controllers (25) are designed in order to keep the UAV in formation with other similar vehicles.

We assume \( N_v = 4 \) and in the following we compare different graph connections.

- **Graph one**

\[
L = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
\end{bmatrix} \hspace{1cm} (26)
\]

- **Graph two**

\[
L = \begin{bmatrix}
1 & 0 & -1 & 0 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 2 & 0 \\
-1 & 0 & 0 & 1 \\
\end{bmatrix} \hspace{1cm} (27)
\]

- **Graph three**

\[
L = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 2 & -1 & -1 \\
-1 & -1 & 3 & -1 \\
-1 & 0 & 0 & 1 \\
\end{bmatrix} \hspace{1cm} (28)
\]

The interconnections of the graph are shown in figure (1). According to the notation used in the previous sections, in the following the vector \( e_i \) will denote the \( i \)-th row of the matrix \( U' \).

- **Graph one**

The Schur decomposition (10) of the Laplacian matrix associated with the graph one is the following:
The matrix $\Delta$ is diagonal, in agreement with the symmetry of the graph connection which has a star structure [11]. Matrices in (29) indicate that there are three independent formation modes which the disturbance can excite. The column of $U'$ represents the singular directions of the disturbance. We can predict that the mode corresponding to the 0 eigenvalue is the one that corresponds the common movement of the all structure (since the components of $e_1$ have same amplitude and same signs). Also, two directions ($e_2$ and $e_3$) will correspond to the same frequency contents. Figure 2 depicts the oscillation of the output of the nodes when a step disturbance is applied to all nodes synchronously in the singular directions.

**Graph two**

The Schur decomposition of the Laplacian matrix associated with the graph two is the following:

$$
\Delta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 4
\end{bmatrix}
$$

$$
U = \begin{bmatrix}
-0.50 & 0.00 & -0.00 & -0.87 \\
-0.50 & -0.76 & -0.29 & 0.29 \\
-0.50 & 0.13 & 0.81 & 0.29 \\
-0.50 & 0.63 & -0.52 & 0.29
\end{bmatrix}
$$

\[(29)\]
(a) Step disturbance $5e_1$ at time 30, all the four states overlap, Integral of relative error: 0, Integral of absolute error: 3.72

(b) Step disturbance $5e_2$ at time 30, Integral of relative error: 3.85, Integral of absolute error: 3.85

(c) Step disturbance $5e_3$ at time 30, Integral of relative error: 3.85, Integral of absolute error: 3.85

(d) Step disturbance $5e_4$ at time 30, Integral of relative error: 38.28, Integral of absolute error: 2.39

Figure 2: Output dynamics of the team (24), (25), (26) subject to different step disturbances

\[
\Delta = \begin{bmatrix}
2.00 & -1.15 & -0.35 & 0.20 \\
0.00 & 0.00 & 0.61 & -0.35 \\
0.00 & 0.00 & 3.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 1.00 \\
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
0.50 & -0.87 & 0.00 & 0.00 \\
-0.50 & -0.29 & -0.71 & -0.41 \\
-0.50 & -0.29 & 0.71 & -0.41 \\
-0.50 & -0.29 & 0.00 & 0.82 \\
\end{bmatrix}
\]
The matrix $\Delta$ is upper-triangular with all real eigenvalues. In this cases there are 4 distinct modes but they are not all decoupled. The common mode corresponding to the zero eigenvalue $\lambda_2$ is coupled with the mode corresponding to $\lambda_1$, all in one direction by exiting only one mode. Also, since a disturbance in the direction of $e_1$ will not excite any other modes, it is reasonable to expect $\lambda_1$ to have a smaller absolute error and $\lambda_4$ the largest error.

![Graph three](image)

Figure 3: Output dynamics of the team (24), (25), (26) subject to different step disturbances

- **Graph three**

  The Schur decomposition of the Laplacian matrix associated with the graph 3 is the
following:

\[
\Delta = \begin{bmatrix}
3.84 & 0.25 & 0.28 & -0.50 \\
0.00 & 0.00 & -0.86 & 0.37 \\
0.00 & 0.00 & 1.58 & -0.91 \\
0.00 & 0.00 & 0.40 & 1.58
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
-0.29 & -0.52 & -0.68 & -0.43 \\
-0.50 & -0.53 & 0.68 & -0.09 \\
0.81 & -0.45 & 0.20 & -0.31 \\
0.10 & -0.49 & -0.20 & 0.84
\end{bmatrix}
\]

(31)

The matrix \( \Delta \) is block upper-triangular, with a couple of complex and conjugate eigenvalues. In this cases there are 4 distinct modes, two of which are complex and conjugate.

7 Conclusions

We have shown how the study of robustness to additive disturbance of a formation can be decomposed into a number of studies for smaller subsystems. The knowledge of singular directions of the disturbance propagation are helpful when the disturbances on the formation are a subset of possible disturbances.

Future endeavors will include a study of the relation between the singular directions and the magnitude of the error propagations in a formation. Also, we plan to investigate the way the spectrum of the Laplacian changes when nodes are added to, removed from or change positions in the graph (i.e. formation re-configuration).

References


Figure 4: Output dynamics of the team (24), (25), (26) subject to different step disturbances

(a) Step disturbance $5e_1$ at time 10, all the four states overlap, Integral of relative error: 36.05, Integral of absolute error: 2.44

(b) Step disturbance $5e_2$ at time 10, Integral of relative error: 0.15, Integral of absolute error: 5.39

(c) Step disturbance $5e_3$ at time 10, Integral of relative error: 12.76, Integral of absolute error: 3.67

(d) Step disturbance $5e_4$ at time 10, Integral of relative error: 12.4600, Integral of absolute error: 3.72


