Part II:

Model Predictive Control

Manfred Morari
Alberto Bemporad
Francesco Borrelli
Unconstrained Optimal Control

Unconstrained infinite time optimal control

\[ V(x_0) = \inf_U \sum_{k=0}^{\infty} \left[ x'_k Q x_k + u'_k R u_k \right] \]

subj. to \( x_{k+1} = A x_k + B u_k, \ k \geq 0, \ x_0 = \bar{x}_0 \)  \( (1) \)

where \( U = \{u_k\}_{k=0}^{\infty} \)

Unconstrained finite time optimal control

\[ V(x_0) = \inf_{U^N} \sum_{k=0}^{N-1} \left[ x'_k Q x_k + u'_k R u_k \right] + x'_N P x_N \]

subj. to \( x_{k+1} = A x_k + B u_k, \ k \geq 0, \ x_0 = \bar{x}_0 \)  \( (2) \)

where \( U^N = \{u_k\}_{k=0}^{N-1} \)

The solution is known!
Constrained Optimal Control

Constrained infinite time optimal control

\[
V(x_0) = \inf_{U} \sum_{k=0}^{\infty} \left[ x'_k Q x_k + u'_k R u_k \right]
\]

subj. to \( x_{k+1} = Ax_k + Bu_k, \ k \geq 0, \ x_0 = \bar{x}_0 \)
\( U \in U \)
\( x \in \mathcal{X} \)

where \( U = \{u_k\}_{k=0}^{\infty}, \ x = \{x_k\}_{k=0}^{\infty}, \)
\( U = \{U|u_k \in \mathbb{U}, \ \forall k \geq 0\}, \mathcal{X} = \{x|x_k \in \mathbb{X}, \ \forall k \geq 0\} \)

Constrained finite time optimal control

\[
V(x_0) = \inf_{U^{N-1}} \sum_{k=0}^{N-1} \left[ x'_k Q x_k + u'_k R u_k \right] + x'_N P x_N
\]

subj. to \( x_{k+1} = Ax_k + Bu, \ k \geq 0, \ x_0 = \bar{x}_0 \)
\( U \in U^N \)
\( x \in \mathcal{X}^N \)

where \( U = \{u_k\}_{k=0}^{N-1}, \ x = \{x_k\}_{k=0}^{N}, \)
\( U^N = \{U|u_k \in \mathbb{U}, \ \forall 0 \leq k \leq N-1\}, \)
\( \mathcal{X}^N = \{x|x_k \in \mathbb{X}, \ \forall 0 \leq k \leq N\} \)

How to solve it ??
Model Predictive Control

- **MODEL**: a model of the plant is needed to predict the future behaviour of the plant
- **PREDICTIVE**: optimization is based on the predicted future evolution of the plant
- **CONTROL**: control complex constrained multivariable plants
Compute the optimal sequence of manipulated inputs which minimizes

- tracking error $= \text{output} - \text{reference}$
- subject to constraints on inputs and outputs

On-line optimization $\Rightarrow$ Receding Horizon
Receding Horizon

- Optimize at time $t$ (new measurements)
- Only apply the first optimal move $u(t)$
- Repeat the whole optimization at time $t+1$

Advantage of on-line optimization:

FEEDBACK!
Receding Horizon - An Example

Chess Game
MPC Based on QP

\[
\min_{U \triangleq \{u_t, \ldots, u_t + Nu - 1\}} \sum_{k=0}^{Ny-1} \left[ x'_{t+k|t} Q x_{t+k|t} + u'_{t+k} R u_{t+k} \right] + x'_{t+Ny|t} P x_{t+Ny|t}
\]

subj. to

\[
\begin{align*}
y_{\text{min}} & \leq y_{t+k|t} \leq y_{\text{max}}, \ k = 1, \ldots, N_c \\
u_{\text{min}} & \leq u_{t+k} \leq u_{\text{max}}, \ k = 0, 1, \ldots, N_u \\
x_{t|t} & = x(t) \\
x_{t+k+1|t} & = A x_{t+k|t} + B u_{t+k}, \ k \geq 0 \\
y_{t+k|t} & = C x_{t+k|t}, \ k \geq 0 \\
u_{t+k} & = K x_{t+k|t}, \ Nu \leq k < Ny
\end{align*}
\]

(5)

- \(x_{t+k|t}\): predicted state vector at time \(t + k\), obtained by applying the input sequence \(u_t, \ldots, u_t + k - 1\) to system model starting from \(x(t)\).
- \(Q = Q' \succeq 0, \ R = R' \succ 0, \ P \succeq 0, \ Q = C' C\) with \((C, A)\) detectable
- \(Ny, Nu, Nc\) are the output, input, and constraint horizons, \((Nu \leq Ny\) and \(Nc \leq Ny - 1\))
- **The optimization problem (5) can be translated into a QP.**
MPC Based on QP

By substituting

\[ x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j Bu_{t+k-1-j} \]  

(6)

in (5), this can be rewritten in the form

\[
V(x(t)) = \min_U \frac{1}{2} U' H U + x'(t) F U \\
\text{subj. to } G U \leq W + E x(t)
\]  

(7)

where

- \( U \triangleq [u_t', \ldots, u_{t+Nu-1}']' \in \mathbb{R}^s, \ s \triangleq mNu, \) is the optimization vector,
- \( H = H' \succ 0, \) and \( H, F, Y, G, W, E \) obtained from \( Q, R, \) and (5)–(6)

Control policy:

- At time \( t \) compute \( U^*(t) = \{u_t^*, \ldots, u_{t+Nu-1}^*\} \)
- Apply \( u(t) = u_t^* \)  

(8)
- repeat the at time \( t + 1 \), based on the new state \( x(t+1) \).
MPC Based on QP - Tracking

At time $t$:

- Measure (or estimate) current state $x(t)$
- Find the input sequence
  \[
  \Delta U^* \triangleq \{\delta u^*(t), \delta u^*(t+1), \ldots, \delta u^*(t+N_u)\} \text{ solving}
  \]
  \[
  \min_{\Delta U} J(\Delta U, t) \triangleq \sum_{k=0}^{N_y} w_y^2 \|y(t+k|t) - r(t+k|t)\|^2 + w_{\delta u}^2 \|\delta u(t)\|^2
  \]
  \[
  \begin{bmatrix} \delta u(t) \triangleq u(t) - u(t-1) \end{bmatrix}
  \]
  subj. to
  \[
  u_{\text{min}} \leq u(t+k) \leq u_{\text{max}}
  \]
  \[
  \delta u_{\text{min}} \leq \delta u(t+k) \leq \delta u_{\text{max}}
  \]
  \[
  y_{\text{min}} \leq y(t+k|t) \leq y_{\text{max}}
  \]
- Apply only $u(t) = \delta u^*(t) + u(t-1)$, and discard $\delta u^*(t+1), \delta u^*(t+1), \ldots$

Repeat the same procedure at time $t+1$
Linear MPC - Example

- Plant:
  \[ G(s) = \frac{1}{s^2 + 0.4s + 1} \]
  \[ T_s = 0.5 \text{ sec.} \]

- Model:
  \[
  \begin{align*}
  x(t + 1) &= \begin{bmatrix} 0.7115 & -0.4345 \\ 0.4345 & 0.8853 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\
  y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)
  \end{align*}
  \]

- Performance index:
  \[ J(U, t) \equiv \sum_{k=0}^{10} [y(t + k|t) - r(t + k|t)]^2 + 0.04\delta u^2(t) \]
Linear MPC - Example

- Constraint $0.8 \leq u(t) \leq 1.2$ (amplitude)

- Constraint $-0.2 \leq \delta u(t) \leq 0.2$ (slew-rate)
Anticipating/Causal MPC

- Reference not known in advance (causal):

\[ r(t + k|t) \equiv r(t), \ \forall k = 0, 1, \ldots, N_y \]

- Future reference samples (partially) known in advance (anticipating):

\[ r(t + k|t) = \begin{cases} r(t + k) & \text{if } k = 0, \ldots, N_r \\ r(t + N_r) & \text{if } k = N_r + 1, \ldots, N_y \end{cases} \]

- Example:
MPC Based on LP

\[
\begin{align*}
\min_{U \triangleq \{u_t, \ldots, u_{t+N_u-1}\}} & \quad \sum_{k=0}^{N_y-1} \|Q x_{t+k|t}\|_{\infty} + \|R u_{t+k}\|_{\infty} + \|P x_{t+N_y|t}\|_{\infty} \\
\text{subj. to} & \quad y_{\text{min}} \leq y_{t+k|t} \leq y_{\text{max}}, \quad k = 1, \ldots, N_c \\
& \quad u_{\text{min}} \leq u_{t+k} \leq u_{\text{max}}, \quad k = 0, 1, \ldots, N_u \\
& \quad x_{t|t} = x(t) \\
& \quad x_{t+k+1|t} = A x_{t+k|t} + B u_{t+k}, \quad k \geq 0 \\
& \quad y_{t+k|t} = C x_{t+k|t}, \quad k \geq 0 \\
& \quad u_{t+k} = K x_{t+k|t}, \quad N_u \leq k < N_y
\end{align*}
\] (9)

- \(x_{t+k|t}\) : predicted state vector at time \(t + k\), obtained by applying the input sequence \(u_t, \ldots, u_{t+k-1}\) to system model starting from \(x(t)\).
- \(Q, R, P\) nonsingular
- The optimization problem (9) can be translated into a LP.
MPC Based on LP

Introduce slack variables \( \{ \varepsilon_{x1}, \ldots, \varepsilon_{xNy}, \varepsilon_{u1}, \ldots, \varepsilon_{uNu} \} \)

\[
\begin{align*}
-1_n \varepsilon_{xk} & \leq Q_{xt+k|t} k = 1, 2, \ldots, N_y - 1 \\
-1_n \varepsilon_{xk} & \leq -Q_{xt+k|t} k = 1, 2, \ldots, N_y - 1 \\
-1_r \varepsilon_{xNy} & \leq P_{xt+N_y|t} \\
-1_r \varepsilon_{xNy} & \leq -P_{xt+N_y|t} \\
-1_m \varepsilon_{uk+1} & \leq R_{ut+k} k = 0, 1, \ldots, N_u - 1 \\
-1_m \varepsilon_{uk+1} & \leq -R_{ut+k} k = 0, 1, \ldots, N_u - 1 \\
\end{align*}
\]

where \( 1_k \triangleq [1 \ldots 1]^T \), By substituting

\[
x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{t+k-1-j}
\]

in (9), this can be rewritten in the form

\[
V(x(t)) = \min_z c_1^t z
\]

subj. to \( Gz \leq W + E x(t) \)

where \( z \triangleq \{ \varepsilon_{x1}, \ldots, \varepsilon_{xNy}, \varepsilon_{u1}, \ldots, \varepsilon_{uNu}, u_t, \ldots, u_{t+N_u-1} \} \)

- The implementation of MPC requires the on-line solution of a LP at each time step
- Same policy
Nonlinear MPC

$$\min_{u_t^N} J(x_t, u_t^N)$$

$$J(x_t, u_t^N) = \sum_{i=0}^{N-1} L(x_{t+i|t}, u_{t+i}) + T(x_{t+N|t})$$

subject to:

$$x_{k+1} = f(x_k, u_k)$$  \hspace{1cm} \text{system dynamics}$$

$$x_t \text{ given}$$  \hspace{1cm} \text{initial condition}$$

$$u_{t+l} \in U$$  \hspace{1cm} \text{input constraints}$$

$$x_{t+l+1|t} \in \mathcal{X}$$  \hspace{1cm} \text{state constraints}$$

$$l = 0, \ldots, N - 1$$

with:

$$u_t^N = \{u_t, u_{t+1}, \ldots, u_{t+N-1}\}$$

$$N \quad \text{horizon length}$$

usually

$$U := \{u_k \in \mathbb{R}^m | u_{\text{min}} \leq u_k \leq u_{\text{max}}\}$$

$$\mathcal{X} := \{x_k|t \in \mathbb{R}^n | x_{\text{min}} \leq x_k|t \leq x_{\text{max}}\}$$
Model

Represent the plant Simple for computations ⇒ Capture most significant dynamics

- **Linear**
  - step/impulse response
  - state space/transfer function
- **Nonlinear**
  - first principle models
  - multiple linear models
  - artificial neural networks
  - NARMAX models
- **Robust**
  - Linear models + uncertainty description
- **Hybrid**
  - Mixed logical dynamical systems
Features

- Multivariable “non-square” systems (i.e. \#inputs \neq \#outputs)
- “Optimal” delay compensation
- Anticipating action for future reference changes
- “Integral action”, i.e. no offset for step-like inputs
Features

• Any model
  – linear (state space, finite impulse response, . . . )
  – nonlinear (state space, neural network, . . . )
  – single variable/multivariable
  – time delays
  – constraints on inputs, outputs, states

• Any objective function
  – sum of squared errors
  – sum of absolute errors
  – worst error over time horizon
  – economic objective function

BUT ...
Features

... BUT

• depending on model/objective function
  
  – different computational complexity

  – different (theoretical) performance guarantees, e.g. stability
Present Industrial Practice

- linear impulse/step response models
- sum of squared errors objective function
- executed in supervisory mode

Particularly suited for problems with

- many inputs and outputs
- constraints
- varying objectives (e.g. because of fault)

Tariq Samad, Honeywell (1997):
“For us multivariable control is predictive control”
Present Industrial Practice

• MPC is capable of handling single variable loops with
  – deadtime
  – inverse response
  – ...

BUT

• equivalent performance can be obtained with other simpler techniques

HOWEVER MPC allows (in principle)

UNIFORMITY

(i.e. same technique for wide range of problems)

⇒ reduce training
⇒ reduce cost
MPC - Theory

• Historical Goal :
  Explain the success of DMC

• Present Goal :
  Improve, simplify, and extend industrial algorithms

• Areas :

  Linear MPC  ⇐ linear model
  Nonlinear MPC  ⇐ nonlinear model
  Robust MPC  ⇐ uncertain (linear) model
  MPC+Logic  ⇐ model integrating logic, dynamics, and constraints
               NEW! ↑

• Issues :
  – Feasibility
  – Stability
  – Open loop vs closed loop
  – Computation
Feasibility Issues - An Example

Consider the system
\[
\begin{align*}
    x(t + 1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\end{align*}
\]
and the MPC problem
\[
\sum_{k=0}^{1} x_{t+k+1|t} L \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{T} \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_{t+k+1|t} + 0.1 u_{t+k}^2
\]
subject to the input constraints
\[-1 \leq u_{t+k} \leq 1, \ k = 0, 1\]
and the state constraints
\[-5 \leq x_{t+k|t} \leq 5, \ k = 1, 2\]

- Step 1 (feasible) \( x(0) = [-4; 4] \)
  \[
  U^* = [-1; -1], \ x_{pred}^*(1) = [0; 3]; \ x_{pred}^*(2) = [3; 2]
  \]
- Step 2 (feasible) \( x(0) = [0; 3] \)
  \[
  U^* = [-1; -1], \ x_{pred}^*(1) = [3; 2] \ x_{pred}^*(2) = [5; 1]
  \]
- Step 3 \( x(0) = [3, 2] \): problem infeasible!
MPC Feasibility

When $N_c < \infty$ there is no guarantee that the optimization problem (5) will remain feasible at all future time steps $t$.

- Setting $N_c = \infty$, optimization problem with an infinite number of constraints $\implies$ impossible to handle.
- If the set of feasible state-input vectors is bounded and contains the origin in its interior, $N_y$ can be chosen ensuring feasibility of the MPC problem at all time instants. (theory of maximal output admissible set)

References:
Predicted and Actual Trajectories in Finite Horizon MPC

Even assuming perfect model, no disturbances:

\[
\begin{array}{c}
\text{predicted open-loop trajectories} \\
\neq \\
\text{closed-loop trajectories}
\end{array}
\]

⇒ **Performance**

Goal: \( \min \sum_{i=0}^{\infty} L(x_{k+i}, u_{k+i}) \)

What is achieved by repeatedly minimizing \( \sum_{i=0}^{N-1} L(x_{k+i}, u_{k+i}) \)?

⇒ **Stability**

Why should the closed-loop be stable?
MPC Stability

Stability is a complex function of the tuning parameters $N_u$, $N_y$, $N_c$, $P$, $Q$, and $R$.

- Impose conditions on $N_y$, $N_c$ and $P$ so that stability is guaranteed for all $Q \succeq 0$, $R > 0$. $Q$ and $R$ can be freely chosen as tuning parameters to affect performance.

- Problem (5) is augmented with a so called “stability constraint” imposing over the prediction horizon explicitly forces the state vector either to shrink in some norm or to reach an invariant set at the end of the prediction horizon.
MPC Stability

Additional constraints to ensure stability:

- **Infinite Output/Prediction Horizon**
  
  \[ N_y \to \infty \]

  (Rawlings and Muske, 1993)

- **End constraint:**
  
  \[ x(t + N_y|t) = 0 \]

  (Kwon and Pearson, 1977)

  (Keerthi and Gilbert'88)

- **Relaxed terminal constraint**
  
  \[ x(t + N_y|t) \in \Omega \]

  (Scokaert and Rawlings, 1996)

- **Contraction Constraint constraint**
  
  \[ \|x(t + 1|t)\| \leq \alpha \|x(t)\|, \alpha < 1 \]

  (Polak and Yang, 1993)

  (Bemporad, 1998)

All the proofs use the value function \( V(t) = \min_U J(U,t) \) as a Lyapunov function.
Stability Theorem

**Theorem 1** Consider the system

\[
\begin{align*}
    x(t+1) &= \phi(x(t), \delta u(t)), \quad 0 = \phi(0,0) \\
    y(t) &= \eta(x(t)), \quad 0 = \eta(0,0)
\end{align*}
\]

and let for simplicity \( r(t) \equiv 0 \). Then the MPC law based on

\[
\min_U J(U,t) = \sum_{k=0}^{N_y} w_y^2 \| y(t+k|t) \|^2 + w_{\delta u}^2 \| \delta u(t) \|^2
\]

subj. to constraints

for either \( N_y \to \infty \) or with the extra constraint \( x(t+N|t) = 0 \) is such that

\[
\begin{align*}
    \lim_{t \to \infty} y(t) &= 0 \\
    \lim_{t \to \infty} \delta u(t) &= 0
\end{align*}
\]

for all \( w_y, w_u > 0 \).

(Keerthi and Gilbert, 1988), (Bemporad et al., 1994)
Stability - Proof

IDEA: Use $V(t) = J(t, U_t^*)$ as a Lyapunov function. Recall:

$$J(U, t) = \sum_{k=0}^{N_u} w_y^2 \|y(t + k|t)\|^2 + w_{\delta u}^2 \|\delta u(t)\|^2$$

At time $t + 1$, extend the previous sequence and consider

$$U_{\text{shift}} \triangleq \{\delta u_t^*(t), \delta u_t^*(t + 1), \ldots, \delta u_t^*(t + N_u), 0\}$$

By construction, $U_{\text{shift}}$ is feasible at time $t + 1$, and

$$V(t + 1) = J(t + 1, U_{t+1}^*) \leq J(t + 1, U_{\text{shift}}) =$$

$$= J(t, U_t^*) - w_y \|y(t)\|^2 - w_u \|\delta u(t)\|^2 =$$

$$= V(t) - w_y \|y(t)\|^2 - w_u \|\delta u(t)\|^2 =$$

$\Rightarrow$ $V(t)$ is nonnegative and decreasing

$\Rightarrow \exists \lim_{t \to \infty} V(t)$

$\Rightarrow \lim_{t \to \infty} V(t) - V(t + 1) = 0$

$\Rightarrow w_y \|y(t)\|^2 + w_u \|\delta u(t)\|^2 \leq V(t) - V(t + 1) \to 0$

$\Rightarrow \|y(t)\|, \|\delta u(t)\| \to 0$  \[\square\]
Example: AFTI F16

Linearized model:

\[
\begin{align*}
\dot{x} &= \begin{bmatrix}
-0.0151 & -60.5651 & 0 & -32.174 \\
-0.0011 & -1.3411 & 0.9929 & 0 \\
0.0018 & 43.2541 & -0.86939 & 0 \\
0 & 0 & 1 & 0 \\
-2.516 & -13.136 \\
-1.689 & -0.2514 \\
-17.251 & -1.5766 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} x + \\
& \begin{bmatrix}
-2.516 & -13.136 \\
-1.689 & -0.2514 \\
-17.251 & -1.5766 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} u \\
y &= \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} x,
\end{align*}
\]

- Inputs: elevator and flaperon angle
- Outputs: attack and pitch angle
- Sampling time: \( T_s = 0.05 \) s (+ zero-order hold)
- Constraints: max 25° on both angles
- Open-loop response: unstable
  (open-loop poles: \(-7.6636, -0.0075 \pm 0.0556j, 5.4530\))
\[ N_y = 10, \ N_u = 3, \ w_y = \{10, 10\}, \ w_{\delta u} = \{.01, .01\}, \]
\[ u_{\text{min}} = -25^\circ, \ u_{\text{max}} = 25^\circ \]

\[ N_y = 10, \ N_u = 3, \ w_y = \{100, 10\}, \ w_{\delta u} = \{.01, .01\}, \]
\[ u_{\text{min}} = -25^\circ, \ u_{\text{max}} = 25^\circ \]
• $N_y = 10$, $N_u = 3$, $w_y = \{10, 10\}$, $w_{\delta u} = \{.01, .01\}$, $u_{\text{min}} = -25^\circ$, $u_{\text{max}} = 25^\circ$, $y_{1,\text{min}} = -0.5^\circ$, $y_{1,\text{max}} = 0.5^\circ$

• $N_y = 10$, $N_u = 3$, $w_y = \{10, 10\}$, $w_{\delta u} = \{.01, .01\}$, unconstrained MPC ($\Rightarrow$ linear controller) $+$ actuator saturation $\pm 25^\circ$ $\Rightarrow$ UNSTABLE!
Tuning Parameters — Guidelines

Control more aggressive when

\( N_u \uparrow \) : input horizon increased
\( N_y \downarrow \) : output horizon decreased
\( w_y \uparrow \) : output weight increased
\( w_{\delta u} \downarrow \) : input weight decreased
MPC Extensions

• $\delta u$-formulation
  – introduces $m$ new states (the last input $u(t-1)$)

• Reference Tracking
  – provides offset-free fracking for constant reference
  – the translated problem is

$$\min_U \frac{1}{2} U' H U + [x'(t) \ u'(t-1) \ r'(t)] F U$$

subject to $G U \leq W + E \begin{bmatrix} x(t) \\ u(t-1) \\ r(t) \end{bmatrix}$

• Measured Disturbances
MPC Extensions

- **Soft Constraints**
  - A slack variable $\epsilon$ is introduced into the constraints $Ax \leq b + \epsilon M$, where $M$ is a matrix of suitable dimensions, and the term $\epsilon' E \epsilon$ is added to the objective to penalize constraint violations.

- **Variable Constraints**
  - The bounds $y_{\text{min}}$, $y_{\text{max}}$, $\delta u_{\text{min}}$, $\delta u_{\text{max}}$, $u_{\text{min}}$, $u_{\text{max}}$ may change depending on the operating conditions.
  - The bounds can be treated as parameters in the mp-QP and added to the vector $x$.

- **Output Feedback**
  - State-feedback MPC controller and a linear state observer $\hat{x}(t+1) = A\hat{x}(t) + L[y(t) - C\hat{x}(t)] + Bu(t)$ can be combined together.
MPC - Computations

- The on-line optimization problem is a Quadratic Program (QP)

- No need to reach global optimum (see proof of the theorem)

- Algorithms:
  - Simplex method (small/medium size) ⇐
  - Interior point methods (large size)

Commercial and public domain software available

Alternative: Linear Programming (with $\| \cdot \|_1$, $\| \cdot \|_\infty$ norms), but worse performance
- Non-optimized Dantzig’s routine
- Interior-point methods might be better for large scale
- Tests in Matlab 5.3 on Pentium II 300 MHz (MEX file from C code)
**FHCOC and IHCOC**

**FHCOC**: Finite-Horizon Constrained Optimal Control  
**IHCOC**: Infinite-Horizon Constrained Optimal Control

Problem $\mathcal{P}$

$$V(x_0) = \inf_U \sum_{k=0}^{\infty} [x_k Q x_k + u_k^T R u_k]$$

subj. to

$$x_{k+1} = A x_k + B u_k, \quad k \geq 0, \quad x_0 = \bar{x}_0$$

$$U \in U, \quad x \in \mathcal{X}$$

where $U = \{u_k\}_{k=0}^{\infty}$, $x = \{x_k\}_{k=0}^{\infty}$,

$U = \{U | u_k \in U, \forall k \geq 0\}$, $\mathcal{X} = \{x | x_k \in \mathcal{X}, \forall k \geq 0\}$

(12)

Problem $\mathcal{P}^N$

$$V(x_0) = \inf_{U^N} \sum_{k=0}^{+\infty} [x_k Q x_k + u_k^T R u_k]$$

subj. to

$$x_{k+1} = A x_k + B u_k, \quad k \geq 0, \quad x_0 = \bar{x}_0$$

$$U \in U^N, \quad x \in \mathcal{X}^N$$

where $U = \{u_k\}_{k=0}^{N-1}$, $x = \{x_k\}_{k=0}^{N}$,

$U^N = \{U | u_k \in U, \forall 0 \leq k \leq N-1\}$,

$\mathcal{X}^N = \{x | x_k \in \mathcal{X}, \forall 0 \leq k \leq N\}$

(13)

Problem $\mathcal{P}^N$ can be equivalently rewritten as

$$
V(x_0) = \inf_{U^N} \sum_{k=0}^{N-1} \left[ x_k'Q x_k + u_k'R u_k \right] + x_N' P x_N
$$

subj. to

$$
\begin{align*}
x_{k+1} &= Ax_k + B u_k, \quad k \geq 0, \quad x_0 = \bar{x}_0 \\
U &\in U^N \\
x &\in X^N
\end{align*}
$$

where $U = \{u_k\}_{k=0}^{N-1}$, $x = \{x_k\}_{k=0}^N$, $U^N = \{U | u_k \in \mathbb{U}, \forall \ 0 \leq k \leq N - 1\}$, $X^N = \{x | x_k \in X, \forall \ 0 \leq k \leq N\}$

where $P$ is the positive-definite solution to the algebraic Riccati equation

Assumptions:

1. $Q > 0, R > 0$
2. $X$ and $\mathbb{U}$ are closed, bounded and convex
3. $0 \in \text{int} X$, $0 \in \text{int} X$
4. $x_0 \in X_0 = \{x^0 \in \mathbb{R}^n | \exists u \in \mathbb{U} \text{ with } x \in \mathcal{X} \text{ and } V(x_0) < \infty\}$
FHCOC and IHCOC

Result:

- For a given $x_0$ exists $N$ such that $\mathcal{P}^N = \mathcal{P}$
- For all $x_0 \in \bar{X}_0$ exists such $N$

- The proof is constructive.
- Uses the sets
  \[
  O_\infty = \{ x \in \mathbb{R}^n | (A - BK_{LQ})^k x \in \bar{X}; k \geq 0 \} \\
  \bar{X} = \{ x \in \mathbb{R}^n | x \in \bar{X} K_{LQ} x \in \mathcal{U} \}
  \]
- An upper bound on $N$ can be found for every given polyhedral set of initial conditions
Example: Servo Motor

Linear Model:

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{k_\theta}{J_L} & -\beta_L/J_L & \frac{k_\theta}{\rho J_L} & 0 \\
0 & 0 & \frac{k_\theta}{\rho J_M} & 0 \\
\frac{k_\theta}{\rho J_M} & 0 & -\frac{k_\theta}{\rho^2 J_M} & -\beta_M + \frac{k_T^2}{R J_M}
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
0 \\
\frac{k_T}{R J_M}
\end{bmatrix} V
\]

\[\theta_L = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x\]

\[T = \begin{bmatrix} k_\theta & 0 & -\frac{k_\theta}{\rho} & 0 \end{bmatrix} x\]

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value (MKS)</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_S$</td>
<td>1.0</td>
<td>shaft length</td>
</tr>
<tr>
<td>$d_S$</td>
<td>0.02</td>
<td>shaft diameter</td>
</tr>
<tr>
<td>$J_S$</td>
<td>negligible</td>
<td>shaft inertia</td>
</tr>
<tr>
<td>$J_M$</td>
<td>0.5</td>
<td>motor inertia</td>
</tr>
<tr>
<td>$\beta_M$</td>
<td>0.1</td>
<td>motor viscous friction coefficient</td>
</tr>
<tr>
<td>$R$</td>
<td>20</td>
<td>resistance of armature</td>
</tr>
<tr>
<td>$K_T$</td>
<td>10</td>
<td>motor constant</td>
</tr>
<tr>
<td>$\rho$</td>
<td>20</td>
<td>gear ratio</td>
</tr>
<tr>
<td>$k_\theta$</td>
<td>1280.2</td>
<td>torsional rigidity</td>
</tr>
<tr>
<td>$J_L$</td>
<td>20$J_M$</td>
<td>nominal load inertia</td>
</tr>
<tr>
<td>$\beta_L$</td>
<td>25</td>
<td>load viscous friction coefficient</td>
</tr>
</tbody>
</table>
Servo Motor - I

- Finite shear strength of steel shaft \( (\tau_{adm} = 50 \text{ N/mm}^2) \) \( \Rightarrow |T| \leq 78.5398 \text{ Nm} \)

- DC voltage limits \(|V| \leq 220 \text{ V}\)

- Sampling time: \( T_s = .1 \text{ s} \) (+ zero-order hold)
\[ N_y = 10 \]
\[ N_u = 3 \]
\[ w_y = \{10, 0\} \]
\[ w_{\delta u} = 0.05 \]
\[ u_{\text{min}} = -220 \text{ V} \]
\[ u_{\text{max}} = 220 \text{ V} \]
\[ y_{\text{min}} = \{-\infty, -78.5398\} \text{ Nm} \]
\[ y_{\text{max}} = \{\infty, 78.5398\} \text{ Nm} \]
$N_y = 10$

$N_u = 3$

$w_y = \{10, 0\}$

$w_{\delta u} = .05$

$u_{\text{min}} = -220$ V

$u_{\text{max}} = 220$ V

$y_{\text{min}} = \{-\infty, -78.5398\}$ Nm

$y_{\text{max}} = \{\infty, 78.5398\}$ Nm
Model Predictive Control
Toolbox for Matlab/Simulink

Manfred Morari
N. Lawrence Ricker
Alberto Bemporad
MPC Toolbox

- On the market
  - MPC Toolbox v1.0

- Under beta-testing
  - MPC Graphical User Interface (GUI)
  - MPC Simulink Library

- Under development
  - MPC Toolbox v2.0
MPC Graphical User Interface

- Load and configure models (LTI, Simulink, Identification, MPC formats)
- Open-loop model testing
- MPC Controller design from model
- Closed-loop simulation
- Export controller to MPC object and Simulink

Easy Design/Simulation of MPC (no M-script)
MPC Graphical User Interface

Software and documentation:

http://depts.washington.edu/control/LARRY/GUI/
Software and documentation:

bemporad@aut.ee.ethz.ch