Mathematical problems for miscible, incompressible fluids with Korteweg stresses

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Abstract. It is shown that the equations governing the motion and diffusion of miscible liquids can be reduced to a form like the Navier-Stokes equations when the equation of state is for the density of a simple mixture. In particular, in this case, $\mathbf{W}=C\mathbf{u}+D\rho\phi$ where C and D are constant, is solenoidal. This allows one to introduce a generalized stream and diffusion function which may be useful in the study of two-dimensional problems. Problems of unidirectional shearing flows in the presence of gradients of composition are briefly considered. Korteweg terms do not enter these problems. We consider the problem of the stability of a vertically stratified incompressible motionless Korteweg fluid of variable concentration analogous to the classical Bénard problem. In general the stability problem is not self-adjoint and it may be possible to have complex eigenvalues at criticality. One and only one Korteweg constant enters into this calculation.

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1. Introduction

There are a number of papers which are related to Korteweg's theory of compressible fluids. A fairly complete list of references to this work can be found in the [1986] paper of J.E. Dunn. These theories rely strongly on thermodynamic arguments which do not seem appropriate to the incompressible case. As far as we know, the paper "Fluid dynamics of two miscible liquids with slow diffusion and gradient stresses" by Joseph [1990] which precedes this paper and this paper are the first to explore Korteweg's ideas in the setting of generalized incompressible liquids whose density and volume change with composition and temperature, but not with pressure.

2. Equations governing a simple mixture

The aim of this section is to write Korteweg's equations (24J*), (25J) and (20J) in terms of a solenoidal vector field **W**. This can be done provided that $\rho(\phi)$ satisfies the equation (8J) governing a simple mixture. Specifically, we are able to introduce a new velocity field **W**, linearly related to **u** and $\rho\phi$, which is divergence-free and to which it is possible to associate a pressure field P in such a way that the functions **W**, P, ϕ obey a set of equations (2.8), (2.9) and (2.10) which resemble the Navier-Stokes equations in several respects. We observe that, in such a case, the pressure P can be interpreted as the usual dynamical variable corresponding to the constraint div **W**=0. We may also find a solenoidal field **W** in the case that (14J) instead of (20J) is assumed to govern provided that the equation of state (8J) is replaced by $\rho(\phi)=\exp(\phi/C)$ with a constant C.

First consider the case in which (20J) is assumed to hold. Subtracting (20J) from (24J) divided by ρ_ϕ we have

* The notation (17J), etc. refers to equation (17) in the preceding paper by Joseph.

$$\left(\frac{\rho(\phi)}{\rho_\phi} - \phi\right) \text{div}\, \mathbf{u} + \text{div}\,(D\rho\phi) = 0\ . \tag{2.1}$$

From (8J) it follows that

$$\frac{\rho(\phi)}{\rho_\phi} - \phi = [\rho_A\phi + (1-\phi)\rho_B]\ \ / (\rho_A - \rho_B) - \phi$$

$$= \frac{\rho_B}{\rho_A - \rho_B} + C^{‡} \tag{2.2}$$

and (2.1) reduces to

$$\text{div}\,(\mathbf{u} + \boldsymbol{\beta}) \overset{\text{def}}{=} \text{div}\,\mathbf{W} = 0\ . \tag{2.3}$$

where

$$\boldsymbol{\beta} \overset{\text{def}}{=} \frac{D}{C}\ \rho\phi\ . \tag{2.4}$$

and

$$\mathbf{u} = \mathbf{W} - \boldsymbol{\beta}\ . \tag{2.5}$$

Now we may use (2.5) to eliminate **u** with **W** from (20J) and (24J). We find that

$$\frac{D\phi}{Dt} = (C+\phi)\ \text{div}\, \boldsymbol{\beta} + \frac{C}{D}\ \ |\boldsymbol{\beta}|^2 \tag{2.6}$$

where

$$\frac{D}{Dt} \overset{\text{def}}{=} \frac{\partial}{\partial t} + (\mathbf{W}\bullet\rho) \tag{2.7}$$

and

‡ The equation of state (8J) for a simple mixture assume that ϕ [0, 1] is a volume fraction. Equation (2.2) holds also when ϕ is interpreted as a mass fraction, provided that $\rho(\phi)$ is linear.

$$\rho \frac{D\mathbf{W}}{Dt} = -\rho P + 2 \operatorname{div} (\mu \mathbf{D}[\mathbf{W}]) + \rho \mathbf{g}$$
$$- 2 \operatorname{div} (\mu \mathbf{D}[\boldsymbol{\beta}]) + \rho \frac{D\boldsymbol{\beta}}{Dt} + \rho (\boldsymbol{\beta} \bullet \rho) (\mathbf{W} - \boldsymbol{\beta})$$
$$+ \operatorname{div} \mathbf{T}^{(2)} \qquad (2.8)$$

where $P = p - \lambda \operatorname{div} \mathbf{u}$, $\mathbf{D}[\mathbf{f}]$ is the symmetric part of $\rho\mathbf{f}$ for any vector $\mathbf{f}$ and

$$\mathbf{T}^{(2)} = \hat{\delta} \rho\phi \quad \rho\phi + \hat{\gamma} \rho \quad \rho\phi \qquad (2.9)$$

where, after using (8J),

$$\hat{\delta} = (\rho_A - \rho_B)^2 \delta_1 + \delta_2 + 2\nu(\rho_A - \rho_B) ,$$
$$\hat{\gamma} = \gamma_1(\rho_A - \rho_B)^2 + \gamma_2 .$$

If $\hat{\gamma}$ is independent of ϕ, then the second term on the right hand side of (2.9) can be expressed as a gradient and folded into the pressure. Then the only active Korteweg coefficient is $\hat{\delta}$.

Equations (2.3), (2.6), (2.8) and (2.9) are the dynamical equations governing our simple mixture (8J).

The reader may observe that (2.3) and (2.8) resemble the Navier-Stokes equations for an *incompressible* fluid with suitable coupling to the concentration field ϕ. Moreover, the pressure field P given by (2.7) is associated with the solenoidal velocity field $\mathbf{W}$. This may be said to give rise to a "concentration pressure" which is to be added to the dynamical variable p.

From now on we shall suppose that $\hat{\delta}$ and $\hat{\gamma}$ are constants. Then

$$-\rho P + \operatorname{div} \mathbf{T}^{(2)} = -\rho\pi + \hat{\delta} \rho\phi\rho^2\phi \qquad (2.10)$$

where $\pi = P - \hat{\gamma} \rho^2\phi - \frac{\hat{\delta}}{2} |\rho\phi|^2$

If, instead of (14J), we adopt (20J) as the law of diffusion, then from (20J) and (23J) divided by ρ_ϕ we deduce

$$\frac{\rho(\phi)}{\rho_\phi} \text{ div } \mathbf{u} + \text{grad } (D\rho\phi) = 0 \tag{2.11}$$

and so the field $\mathbf{W}= \frac{\rho(\phi)}{\rho_\phi} \mathbf{u}+D\rho\phi$ is solenoidal, provided $\rho(\phi)/\rho_\phi$=C,that is, $\rho=\rho_0 \exp(\phi/C)$. In such a case, one can show that **W**, ϕ verify equations similar to (2.6) and (2.8).

In two-dimensional problems and in axisymmetric problems we may introduce a stream and diffusion function ψ arranged so as to satisfy div **W**=0 identically.

If (x, z) is the plane of our problem, then

$$\mathbf{W} = \mathbf{j} \quad \rho\psi ,$$

$$\frac{\partial\psi}{\partial z} = u + \frac{D}{C}\frac{\partial\phi}{\partial x} , \tag{2.12}$$

$$\frac{\partial\psi}{\partial x} = -w - \frac{D}{C}\frac{\partial\phi}{\partial z} . \tag{2.13}$$

where **j** is in direction y, out of plane, and y=(u,v,w). Similar expressions for ψ in axisymmetric flow can be written down in different coordinate systems. Using these relations we may satisfy (2.6) identically and eliminate **W** in (2.6) and (2.8) with ψ. Then (2.6) and (2.8) are three equations for three scalar fields ψ, ϕ and P.

3. Some mathematical problems

Many of the problems of classical incompressible fluids can be reworked in the present setting. Suppose our problems are two-dimensional, in the x, z plane, with

$$\mathbf{u} = \mathbf{i}u(x, t) , \quad \phi = \phi(x, t) , \quad \mathbf{g} = -\mathbf{k}g$$

where **k** is a unit vector in the direction z. Then div **u**, **u**•ρ**u** and **u**•ρϕ are identically zero. Equation (20J) then shows that ϕ=Ex+F with a time-independent E and F. We may therefore write ϕ(x)=Ex, E=1/L where ϕ(0)=0 and ϕ(L)=1 is prescribed. We may now calculate the entries in the matrix $\mathbf{T}^{(2)}$ giving the compositional stresses; $T^{(2)}_{xx} = \hat{\delta} E^2$ and the other three entries are zero. The density of our simple mixture is

$$\rho(\phi) = \rho_A\phi + (1-\phi)\rho_B = E\rho_A x + (1-Ex)\rho_B . \qquad (3.1)$$

Recalling now the **u**=**k**u(x, t) we deduce

$$\rho(\phi)\frac{\partial u}{\partial t} = -P' + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) - \rho(\phi)g \qquad (3.2)$$

with suitable conditions on u at x=0 and x=L. Various elementary problems, Couette flow, Poiseuille flow, etc. are contained and generalized in the unidirectional problems which satisfy (3.1) and (3.2).

More interesting, but more difficult problems can be found for motions such that div **u**≠0 but div **W**=0. An interesting example of this is the problem of dispersion of soluble matter flowing slowly through a tube which was studied in two asymptotic limits by G.I. Taylor [1953]. Taylor considers the problem of pipe flow of a pure liquid in which initially a slug of another miscible liquid is located. He works with div **u**=0 and the usual equation of diffusion assuming that the velocity profile in the pipe is given by $u_0(1-r^2/a^2)$ where a is the radius of the pipe and u_0 is independent of x. In fact u_0 cannot be independent of x because the viscosity μ depends on the concentration c, in his notation, which varies with x, so that the formula relating u_0 to the pressure gradient in Hagen-Poiseuille flow will involve an x dependence through the viscosity. This is not necessarily a small effect; for example, we could think of the problem for two silicone oils with a viscosity ratio of 10^4 or more. If then the axial velocity depends on x, the div **u** will not vanish on the basis of a kinematic assumption, and since $d\rho/dc \neq 0$ and $\partial c/\partial t \neq 0$

then div $\mathbf{u}\neq 0$, in general. Of course, Taylor did not include Korteweg-type forces and we certainly do not yet know if they should be included. The approximations introduced by Taylor, as well as the ones he did not explicitly acknowledge, are evidently appropriate for the description of the dispersion of the 1% aqueous potassium permanganate which was used in his experiments. The neglected effects should be more important in experiments in which concentration, density and viscosity gradients are stronger. In these cases we might expect to see secondary motions resembling those generated by immiscible displacements (see Kafka and Dussan V [1979]).

Another class of important problems which might be reworked are those associated with miscible displacements in porous media and Hele-Shaw cells. These problems have heretofore been treated under the assumption that div $\mathbf{u}=0$ and with a modified form of the classical diffusion equation (see Homsy [1989]).

4. Dimensionless equations for the stability of motionless solutions

Before introducing a dimensionless form of our equations suitable for studying the stability of motionless solutions it is useful to remove the hydrostatic pressure π_S from (2.8) and (2.10) by writing $\pi=\pi^*+\pi_S$ and

$$-\rho\pi + \rho\mathbf{g} = -\rho\pi^* - \rho\pi_S + [\rho_B + (\rho_A-\rho_B)\phi]\mathbf{g} = -\rho\pi^* + (\rho_A-\rho_B)\phi\mathbf{g} \ . \tag{4.1}$$

Motionless solutions have no intrinsic scale of velocity and it is appropriate to use L/D as the scale for velocity. We are now assuming that D is a constant independent of ϕ. Then (2.8), (2.9) and (2.10) may be written in terms of dimensionless variables (with a hat)

$$\mathbf{W} = D\hat{\mathbf{W}}/L\ , \quad \mathbf{u} = D\hat{\mathbf{u}}/L\ , \quad \mathbf{x} = L\hat{\mathbf{x}}\ , \quad t = L^2\hat{t}/D\ ,$$
$$\phi = \Delta\phi\hat{\phi}\ , \quad \boldsymbol{\beta} = D\hat{\boldsymbol{\beta}}/L\ , \quad \hat{\boldsymbol{\beta}} = \frac{1}{E}\hat{\nabla}\hat{\phi}\ , \quad E = \Delta\phi/C\ , \tag{4.2}$$
$$\rho = \rho_B\hat{\rho}\ , \quad \hat{\rho} = 1+\hat{\phi}/E\ , \quad \mu = \mu_B\hat{\mu}(\hat{\phi})\ , \quad \pi^* = \frac{\mu_B D}{L^2}\ \hat{\pi}$$

where L is a reference length and ρ_B and μ_B are the density and viscosity of the fluid B and $\nu_B=\mu_B/\rho_B$ and $C=\rho_B/(\rho_A-\rho_B)>0$. The dimensionless equations (with hats omitted) are

$$\mathbf{W} = \mathbf{u} + \frac{1}{E}\ \rho\phi \overset{\text{def}}{=} \mathbf{u} + \boldsymbol{\beta}\ , \tag{4.3}$$

$$\rho = 1 + \phi/E\ , \tag{4.4}$$

$$\text{div}\ \mathbf{W} = \text{div}\ \mathbf{u} + \frac{1}{E}\ \rho^2\phi = 0\ , \tag{4.5}$$

$$\frac{\partial\phi}{\partial t} + [(\mathbf{W}-\boldsymbol{\beta})\cdot\nabla]\ \phi = \left(1 + \frac{\phi}{E}\right)\ \rho^2\phi\ , \tag{4.6}$$

$$\begin{aligned}\frac{\rho}{S}&\left\{\frac{D(\mathbf{W}-\boldsymbol{\beta})}{Dt} - (\boldsymbol{\beta}\cdot\nabla)\ (\mathbf{W}-\boldsymbol{\beta})\right\}\\ &= -\rho\pi + 2\ \text{div}\ [\mu\mathbf{D}(\mathbf{W}-\boldsymbol{\beta})]\\ &+ K_1(\rho\phi)\ (\rho^2\phi) + R^2\phi\ \mathbf{g}/|\mathbf{g}|\end{aligned} \tag{4.7}$$

where D/Dt is defined by (2.7).

The dimensionless variables of our problem are

$$E = \Delta\phi(\rho_A-\rho_B)\ /\ \rho_B\ , \qquad \text{(Inhomogeneity number)}\ ,$$
$$S = \nu_B\ /\ D\ , \qquad \text{(Schmidt number)}\ ,$$
$$K_1 = \hat{\delta}(\Delta\phi)^2\ /\ \mu_B D\ , \qquad \text{(Korteweg number)}\ ,$$
$$R = \left[\frac{(\rho_A-\rho_B)gL^3\Delta\phi}{D\nu_B}\right]^{1/2}\ , \qquad \text{(``Rayleigh'' number)}\ . \tag{4.8}$$

Equation (4.7) shows that the only coefficient of the Korteweg tensor $\mathbf{T}^{(2)}$ entering the equations of motion is $\hat{\delta}$, through the Korteweg number K_1. The other coefficients are in gradient form embedded in the "dynamic pressure" π^*.

The Navier-Stokes equations are recovered in the cases in which the fluid is well-mixed so that prescribed concentration differences are zero and ϕ=constant everywhere. There is another case of fast mixing which is of interest characterized by a large diffusion coefficient D. In this case the scale we have chosen for t, **W**, **u** should be based on ν_B rather than D. We would change variables again, writing

$$\mathbf{W} = S\,\tilde{\mathbf{W}} \ , \quad \mathbf{u} = S\,\tilde{\mathbf{u}} \ , \quad t = \tilde{t}\,/S \ . \tag{4.9}$$

Then (4.6) becomes

$$\frac{\partial\phi}{\partial\tilde{t}} + (\tilde{\mathbf{W}} \bullet \rho) = \frac{1}{S}\left\{\left(1 + \frac{\phi}{E}\right)\nabla^2\phi + \frac{1}{E}\,|\nabla\phi|^2\right\} \tag{4.10}$$

Since $S \varnothing 0$ as $D \varnothing \infty$, we get $\rho^2\phi=0$ and $|\rho\phi|=0$ everywhere except in boundary layers in which more careful analysis is required.

We call the equations which arise from (4.3)–(4.6) when $E \varnothing 0$ and $R^2=\rho_B g L^3 E/D\nu_B$ and $K_1=\hat{\delta}(\Delta\phi)^{\,2}/\mu_B D$ are finite, Korteweg-Boussinesq equations. This means that density differences are negligible except when multiplied by the large value of gravity. In this limit $\rho \varnothing 1$, $\boldsymbol{\beta} \varnothing 0$, $\mathbf{W} \varnothing \mathbf{u}$, $D/Dt \varnothing d/dt$, and

$$\text{div}\,\mathbf{u} = 0 \ , \tag{4.11}$$

$$\frac{d\phi}{dt} = \rho^2\phi \ , \tag{4.12}$$

$$\frac{1}{S}\,\frac{d\mathbf{u}}{dt} = -\rho\pi + 2\,\text{div}\,\mu\mathbf{D}[\mathbf{u}] + K_1(\rho\phi)\,(\rho^2\phi) + R^2\phi\,\mathbf{g}\,/\,|\mathbf{g}| \tag{4.13}$$

The effects of gradients of the composition in the Korteweg term could be important say when the two different and miscible liquids have nearly or exactly the same density. When they have the same density they will usually have different viscosities so that the variation of $\mu(\phi)$ cannot be neglected.

5. Stability of a vertically stratified incompressible motionless Korteweg fluid of variable concentration

We could have called the problem being studied here a Korteweg-Bénard problem. This short name for our problem might remind the reader of the Oberbeck-Boussinesq (OB) equations which do not apply here. There are two important differences between equations (4.3) through (4.7) and the analogous OB equations. First and foremost, our fluids may undergo volume changes by diffusion of species with different densities. In this case div $\mathbf{u}\neq 0$ and $\rho_\phi = \dfrac{d\rho}{d\phi}$ is not small, but may be of the same order as ρ, as is true of glycerin and water mixtures. In addition, Korteweg terms are missing from OB equations.

Assume we have a layer of Korteweg fluid between horizontal planes $\mathbf{g}/|\mathbf{g}| = -\mathbf{k}$ separated by a dimensionless distance of one. The bottom plate at z=0 has $\phi=0$, corresponding to $\rho=\rho_B$ there. The top plate has more of fluid A, $\Delta\phi>0$ there. This is a top heavy situation. There is a steady diffusion solution with no motion of (4.3) through (4.7)

$$\mathbf{u} = 0\ , \quad \phi = z\ , \quad \rho = 1+z/E\ , \quad d\pi/dz = R^2 z \tag{5.1}$$

We linearize these equations around (5.1) using $\mathbf{u}'$, ϕ', π'' as perturbations which satisfy, dropping primes,

$$\text{div}\,\mathbf{u} + \frac{1}{E}\,\rho^2\phi = 0\ , \tag{5.2}$$

$$\frac{\partial \phi}{\partial t} + (\mathbf{k}\cdot\mathbf{u})\phi = \left(1 + \frac{z}{E}\right) \rho^2\phi , \tag{5.3}$$

$$\frac{1+z/E}{S} \frac{\partial \mathbf{u}}{\partial t} = -\rho\pi + 2 \text{ div } (\mu(z)\mathbf{D}[\mathbf{u}]) + (K_1\rho^2\phi - R^2\phi)\mathbf{k} \tag{5.4}$$

where $\mu(z)=\hat{\mu}(\hat{\phi})$ with $\hat{\phi}=z$ according to recipe (4.2). These are five equations for $\mathbf{u}$, ϕ, π and they can be studied for different boundary conditions, say for Dirichlet conditions $(\phi,\mathbf{u})=(0,\mathbf{0})$, z=0 and z=1.

We draw the reader's attention to the fact that the system (5.2)–(5.4) is not generally self-adjoint even when the Korteweg effects are absent, $K_1=0$. It may be that for some values of the parameters, this system with $K_1=0$ gives rise to complex eigenvalues, overstability leading to Hopf bifurcation. Another possible case of overstability at the other extreme of parameter values, when the non-Boussinesq terms are inactive $E\varnothing\infty$ but the Korteweg terms are active $K_1\neq 0$, is considered below.

The reader will notice that the system (5.2)–(5.4) is more complicated than in the Boussinesq case (div $\mathbf{u}$=0) partly because the coefficients of some terms are z-dependent. This z-dependence also arises in the classical case of a fluid layer heated from below when the Boussinesq approximation is relaxed, as the following argument shows. Suppose the fluid is homogeneous of constant composition, ϕ=constant, but $\rho=\rho(\hat{\theta})$ where $\hat{\theta}(z) =\Theta_0(z)+\theta$ is the temperature and $\Theta_0(z)$ is the temperature field of the motionless solution and $\theta(\mathbf{x}, t)$ is a small perturbation. The linearized continuity equation,

$$\rho_\theta[\Theta_0(z)] \left\{\frac{\partial \theta}{\partial t} + w\Theta_0'(z)\right\} + \rho[\Theta_0(z)] \text{ div } \mathbf{u} = 0 \tag{5.11}$$

where $\rho_\theta=d\rho/d\hat{\theta}$ evaluated at $\Theta_0(z)$, has z dependent coefficients even when ρ is a linear function of $\hat{\theta}$. We are aware of two different mathematical papers which lay down conditions under which the Oberbeck-Boussinesq equations are a valid approximation (Mihaljan [1962], Spiegel and Veronis [1960]). In general it is required that ρ_θ be small, and other things, which may hold up well in some cases and less well in others. When all is said and done, we have to recognize that the approximations of Oberbeck and Boussinesq were introduced long before the use of computers as a pragmatic procedure to make tractable problems which now border on trivial.

6. Stability of a vertically stratified Korteweg-Boussinesq equation of variable concentration

Nothing is yet known about the Korteweg coefficients. Their sign has not been determined, and it is not even known if any of them are different from zero. Some information may be obtained from doing simple studies to see if implausible physical results can be eliminated by putting suitable restrictions on the range of the Korteweg constraints. Now we are going to prove an unusual result, namely that the motionless solution (5.1) is unconditionally stable provided that $K_1 < -R^2/\pi^2$. We may avoid this unusual conclusion by requiring that

$$K_1 + R^2 / \pi^2 > 0 \,. \tag{6.1}$$

We are going to perturb the system (4.11), (4.12) and (4.13) around (5.1), but we do not linearize. Thus

$$\operatorname{div} \mathbf{u} = 0 \,, \tag{6.2}$$

$$\frac{\partial \phi}{\partial t} + (\mathbf{u} \bullet \rho)\phi = -\mathbf{u} \bullet \mathbf{k} + \rho^2 \phi \,, \tag{6.3}$$

$$\frac{1}{S}\left(\frac{\mathbf{u}}{\partial t} + (\mathbf{u} \bullet \nabla)\mathbf{u}\right) = -\rho\pi + \operatorname{div}\,(\mu(\phi)\mathbf{D}[\mathbf{u}]) + K_1(\rho\phi)(\rho^2\phi) - (R^2\phi - K_1\rho^2\phi)\mathbf{k} \,. \tag{6.4}$$

The spectral problem corresponding to (6.2, 3, 4) is obtained by linearizing for small disturbances proportional to exp(σt). The functions of proportionality are again called (**u**, ϕ, π) but are functions of **x** alone, and not of t. Thus, we have (6.2) and

$$\sigma\phi = -\mathbf{u}\bullet\mathbf{k} + \rho^2\phi\ , \tag{6.5}$$

$$\frac{\sigma}{S}\ \mathbf{u} = -\rho\pi + 2\ \mathrm{div}\ \{\mu(z)\mathbf{D}[\mathbf{u}]\} - (R^2\phi - K_1\rho^2\phi)\mathbf{k}\ . \tag{6.6}$$

Let Ω be any domain with flat top and bottom and vertical side-walls compatible with the motionless solution (5.1). A plane layer with specified cells of periodicity is one such domain. Multiply (6.5) by $\overline{\phi}$, the complex conjugate of ϕ, and integrate over Ω

$$-\sigma\bullet|\phi|^2\circledR = \bullet\overline{\phi}\ w\circledR + \bullet|\rho\phi|^2\circledR \tag{6.7}$$

where $w = \mathbf{u}\bullet\mathbf{k}$ and the boundary conditions on ϕ are such that $\bullet\overline{\phi}\ \rho^2\phi\circledR = \bullet|\rho\phi|^2\circledR$, for example, ϕ=0 on ∂Ω. Now multiply the complex conjugate of (6.6) by **u** and integrate over Ω_0. Thus

$$\frac{\overline{\sigma}}{S}\ \bullet|\mathbf{u}|^2\circledR = \bullet\mu(z)|\mathbf{D}[\mathbf{u}]|^2\circledR + R^2\bullet\overline{\phi}\ w\circledR + K_1\bullet\rho\overline{\phi}\ \bullet\rho w\circledR\ . \tag{6.8}$$

To deal with the last term at the right hand side of (6.8) we form one more identity multiplying (6.5) by $\rho^2\overline{\phi}$ and integrating over Ω. This gives

$$-\sigma\bullet|\rho\phi|^2\circledR = \bullet\rho w\bullet\rho\overline{\phi}\ \circledR + \bullet|\rho^2\phi|^2\circledR \tag{6.9}$$

Now multiply (6.7) by R^2 and (6.9) by K_1. Then we subtract this weighted sum of R^2(6.7) + K_1(6.9) from (6.8) and get

$$-\frac{\overline{\sigma}}{S}\ \bullet|\mathbf{u}|^2\circledR = +\ R^2\sigma\bullet|\phi|^2\circledR + K_1\sigma\bullet|\rho\phi|^2\circledR$$

$$= \bullet\mu(z)|\mathbf{D}[\mathbf{u}]|^2\circledR - K_1\bullet|\rho^2\phi|^2\circledR - R^2\bullet|\rho\phi|^2\circledR \tag{6.10}$$

Hence

$$\mathrm{Im}\sigma \left\{\frac{1}{S} \bullet|\mathbf{u}|^2\circledR + R^2\bullet|\phi|^2\circledR + K_1\bullet|\nabla\phi|^2\circledR\right\} = 0 \tag{6.11}$$

$$-\mathrm{Re}\sigma = \frac{\bullet\mu(z)|\mathbf{D}[\mathbf{u}]|^2\circledR - K_1\bullet|\nabla^2\phi|^2\circledR - R^2\bullet|\nabla\phi|^2\circledR}{\bullet|\mathbf{u}|^2\circledR - R^2\bullet|\phi|^2\circledR - K_1\bullet|\nabla\phi|^2\circledR} \tag{6.12}$$

The Poincaré inequalities (see Galdi [1985])

$$\bullet|\rho^2\phi|^2\circledR \geq \pi^2\bullet|\rho\phi|^2\circledR \geq \pi^4\bullet|\phi|^2\circledR \tag{6.13}$$

imply that

$$-R^2\bullet|\rho\phi|^2\circledR - K_1\bullet|\rho^2\phi|^2\circledR \geq -\left(\frac{R^2}{\pi^2} + K_1\right) \bullet|\rho\phi|^2\circledR \tag{6.14}$$

and

$$-R^2\bullet\phi^2\circledR - K_1\bullet|\rho\phi|^2\circledR \geq -\left(\frac{R^2}{\pi^2} + K_1\right) \bullet|\rho\phi|^2\circledR \tag{6.15}$$

If

$$K_1 \leq -R^2/\pi^2 , \tag{6.16}$$

then $\mathrm{Re}\sigma<0$ and the basic state is stable. In other words, a sufficiently negative Korteweg constant can always stabilize an otherwise unstable motionless solution.

If $K_1\geq 0$, then $\mathrm{Im}\sigma=0$ for all eigenvalues. The form of (6.11) suggests that $\mathrm{Im}\sigma=0$ for all eigenvalues even when $K_1\leq 0$. Put $\sigma=0$ in (6.5) and (6.6), replace $\rho^2\phi$ in (6.6) with $w=\mathbf{u}\bullet\mathbf{k}$ from (6.5) and write $\phi=\psi/R$. Then we have

$$Rw - \rho^2\psi = 0 , \tag{6.17}$$

$$\rho\pi - 2 \text{ div } \{\mu(z)\mathbf{D}[\mathbf{u}]\} + (R\psi - K_1 w)\mathbf{k} = 0 . \tag{6.18}$$

Equations (6.17) and (6.18) are the Euler equations for the variational function

$$R[\mathbf{u}, \psi] = \frac{\bullet\mu(z)|\mathbf{D}^2[\mathbf{u}]|\circledR + \bullet|\nabla\psi|^2\circledR - K_1\bullet w^2\circledR}{-2\bullet w\psi\circledR} \tag{6.19}$$

on solenoidal vector fields **u**, for **u** and ψ which satisfy suitable boundary conditions. Equation (6.17) shows the $R\bullet w\psi\circledR = -\bullet|\rho\psi|^2\circledR$ is negative. We may seek R as a minimum value on a space H of functions. If

$$\tilde{R}(K_1) = \min_{\mathbf{u},\psi\ H} R[\mathbf{u}, \psi] = R[\tilde{\mathbf{u}}, \tilde{\psi}] \tag{6.20}$$

then

$$\frac{d\tilde{R}}{dK_1} = \frac{\bullet w^2\circledR}{2\bullet w\psi\circledR} = -\frac{\bullet w^2\circledR}{2R\bullet|\nabla\psi|^2\circledR} . \tag{6.21}$$

This shows that the stability limit is a decreasing function of $K_1 \geq 0$. The Korteweg terms are destabilizing.

7. Energy stability of the Korteweg-Bénard problem

We shall now perform a nonlinear energy stability theory of the Korteweg-Bénard problem, limiting ourselves to the case when the Korteweg number K_1 satisfies

$$K_1 \leq -\frac{R^2}{\pi^2} . \tag{7.1}$$

Specifically, we shall prove that if (7.1) holds with the strict inequality sign and, moreover

$$\mu(\phi) \geq \mu_0 > 0 \tag{7.2}$$

for some constant μ_0, the basic state (5.1) is unconditionally, asymptotically stable. If, otherwise, at least one of the following relations holds

$$K_1 = -\frac{R^2}{\pi^2} \quad \text{or} \quad \mu_0 = 0 \tag{7.3}$$

we only have that (5.1) is monotonically stable. However, in all cases, we recover nonlinearly the same result proved in the previous section by means of the linearized theory.

To show all the above, we introduce the following generalized energy (Lyapunov) functional:

$$\mathrm{E} = \frac{1}{2}\int_\Omega \left(\frac{1}{\mathrm{S}}\,\mathbf{u}^2+\lambda\phi^2+\sigma|\nabla\phi|^2\right)\,d\Omega \tag{7.4}$$

where λ, σ are parameters to be chosen appropriately. We evaluate $d\mathrm{E}/dt$ along (6.2), (6.4) and use integration by parts together with the Dirichlet conditions (say) $(\mathbf{u},\phi)=(0,0)$ at z=0,1 to obtain

$$\frac{d\mathrm{E}}{dt} = (\mathrm{I}-1)\,\mathrm{D}(\mathbf{u},\phi) + \mathrm{N}(\mathbf{u},\phi) \tag{7.5}$$

where

$$\mathrm{I} = -\frac{(R^2+\lambda)\bullet\phi w ® + (\sigma+K_1)\bullet\nabla\phi\cdot\nabla w ®}{\mathrm{D}(\mathbf{u},\phi)}$$

$$\mathrm{D}(\mathbf{u},\phi) = \bullet\mu(\phi)|\mathbf{D}[\mathbf{u}]|^2 ® + \lambda\bullet|\rho\phi|^2 ® + \sigma\bullet|\rho^2\phi|^2 ®$$

$$\mathrm{N}(\mathbf{u},\phi) = (K_1+\sigma)\,\bullet\mathbf{u}\cdot\rho\phi\rho^2\phi ® \,. \tag{7.6}$$

Choosing

$$\lambda = -R^2\,, \quad \sigma = -K_1\,,$$

from (7.6), it follows $\mathrm{I} = \mathrm{N} = 0$ and the energy identity (7.5) reduces to

$$\frac{dE}{dt} = -D(\mathbf{u},\phi) \ . \qquad (7.7)$$

We next observe that, since **u** is divergence-free and vanishes at the boundary, it obeys the well-known identity:

$$\bullet|\mathbf{D[u]}|^2 \circledR = \bullet|\rho \mathbf{u}|^2 \circledR \ .$$

Thus, from (7.1) with the strict inequality sign, from (7.2), and from the Poincaré inequality (6.13), we recover

$$2E \geq \frac{1}{S} \ \bullet|\mathbf{u}|^2 \circledR + \delta\bullet|\phi|^2 \circledR$$

$$D(\mathbf{u},\phi) \geq \mu_0 \bullet|\rho \mathbf{u}|^2 \circledR + \delta\bullet|\rho\phi|^2 \circledR$$

where $\delta = -K_1\pi^2 - R^2 (>0)$. Therefore, E and D are always positive definite and, furthermore, by (6.13) (applied to ϕ and **u**) it follows

$$D(\mathbf{u},\phi) \geq 2\gamma E$$

with

$$\gamma = \min\left\{\mu_0\pi^2 S_1 - \frac{\delta}{K_1}\right\} \ . \qquad (7.8)$$

The energy identity (7.7) and this latter inequality yield

$$\frac{dE}{dt} \leq -2\gamma E$$

which, upon integration, in turn implies

$$E(t) \leq E(0)\exp\{-2\gamma t\} \ , \quad \text{for all } t>0 \ ,$$

implying unconditional and exponential stability.

If we now assume that at least one of conditions (7.3) is satisfied, by (7.8) it is γ=0 and we can no longer deduce asymptotic stability. Nevertheless, $(7.6)_2$ and (7.7) still furnish monotonic stability

$$\frac{dE}{dt} = -\mathcal{D}(\mathbf{u},\phi) \le 0 .$$

We conclude by observing that, following the recent methods introduced by Galdi and Rionero [1985] and Galdi [1985], an energy theory could possibly be developed also in the case K_1>0, by selecting *other* suitable values of the parameters, λ, σ. As a consequence, in this case, the nonlinear term appearing in the energy identity (7.5) would no longer be identically zero and therefore it is very likely that the type of nonlinear asymtotic stability one should obtain is conditional, and subcritical instabilities are therefore not excluded.

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