Lubricated pipelining: stability of core-annular flow
Part IV: Ginzburg-Landau equations

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Nonlinear stability of core-annular flow near points of the neutral curves at which perfect core-annular flow loses stability is studied using the Ginzburg-Landau equations. Most of the core-annular flows are always unstable. Therefore the set of core-annular flows having critical Reynolds numbers is small, so that the set of flows for which our analysis applies is small. An efficient and accurate algorithm for computing all the coefficients of the Ginzburg-Landau equation is implemented. Over 100 sets of coefficients were calculated (see Chen [1990]). The nonlinear flows seen in the experiments do not appear to be modulations of monochromatic waves, and we see no evidence for soliton-like structures. We explore the bifurcation structure of finite amplitude monochromatic waves at criticality. The bifurcation theory is consistent with observations in some of the flow cases to which it applies and is not inconsistent in the other cases to which it applies.

1. Introduction

This paper is a continuation of study of the stability of water lubricated core-annular flows. In the previous studies, Joseph, Renardy and Renardy [1984], Preziosi, Chen and Joseph [1989] (PCJ), Hu and Joseph [1989] (HJ), Chen, Bai and Joseph [1990] (CBJ), and more recently Bai, Chen and Joseph [1990] (BCJ), calculations from the linear theory of stability were reported and compared with experiments. Surprisingly, the linear theory of stability turned out to be good for predicting wave lengths, wave speeds and flow types in flows which are far from the perfect core-annular flow (which the linear theory is supposed to perturb only slightly). However, there are some situations for which the linear theory fails and it is of interest here to see what understanding can be achieved from nonlinear theory. One such situation mentioned by BCJ is a regime in which oil seizes the pipe wall. Efforts are also made here to correlate the "bamboo" waves, shown in Figure 1, which are the dominant flow regime in up-flow, to the weakly nonlinear analysis. Unfortunately, it is found that these waves cannot be obtained from this theory.

(insert Figure 1 near here)

The first type of nonlinear analysis we might try is bifurcation theory. This theory, however, is restricted in applications to those cases in which there is a threshold for instability. In our situation this means cases in which stable PCAF (perfect core-annular flow) is possible (the neutral curves are separated as in Figures 3 and 4). In these cases we may go beyond bifurcation into monochromatic waves and derive amplitude equations which allow for slow modulations of wavy flow in space and time. This amplitude equation is called the Ginzburg-Landau equation*. There are many regimes of flow which give rise to separated neutral curves for which the Ginzburg-Landau equation may be applied. There are even more regimes in which PCAF is not possible and analytical approaches to the nonlinear problem in these cases seem to be unknown. The neutral curves shown in Figures 10, 14 of PCJ and Figure 8.4 of CBJ where the upper and the lower branches have merged to form left and right branches,

* The so-called "Ginzburg-Landau" equation which we derive actually follows the work of Newell [1974] and Stewartson and Stuart [1971] who extended the work of Newell and Whitehead [1969) and Segel [1969) to the unsteady case in which the marginally stable eigenvalue at criticality is purely imaginary, as in Hopf bifurcation. Ginzburg and Landau [1950] did write down, but did not derive, a differential amplitude equation with slow modulation for the theory of superconductivity.

will not allow for bifurcation analysis. Unfortunately, the case $m \ll 1$, which is typical of applications in which the oil is very viscous, is one of these cases for which we have no nonlinear analysis (see Hu, Lundgren and Joseph [1990]).

We have computed the coefficients of the Ginzburg-Landau equations for many different cases. We have adapted an efficient numerical method, the singular value decomposition (SVD), to problems of bifurcation. As in many control and statistics problems, we think it is the method of choice for the computation of the coefficients of the amplitude equations and normal forms. This method is described in Section 4.

Amplitude equations are derived under restricted conditions. For example, the Ginzburg-Landau equation presumably applies only to small amplitude waves which modulate a basic wave with a steady component and at most two harmonics, as in (3.5). Once derived, the amplitude equations take on a life of their own and are applied in all sorts of situations for which they were never intended.

To know when and where amplitude equations apply, it is necessary ultimately to check the predictions of the theory against the results of the experiments. This requires that one go beyond the qualitative arguments based on the form of the equation. These types of qualitative predications are basically worthless for evaluating when and where a given amplitude equation may by relevant. What is needed is explicit coefficients for the amplitude equations based on the linear theory and control over experiments.

In this paper we present coefficients of the Ginzburg-Landau equations appropriate for different situations of interest and make some comparisons with experiments. We look mainly at those coefficients that control bifurcation. Modulational effects are briefly discussed in Section 8 and Section 9. The real challenge is to see whether exotic effects which are known to be generated by some solutions of the Ginzburg-Landau equations persist when the coefficients computed are correct for experimental situations for which they are meant to apply. The formation of solitons and chaos are two such effects which have been examined in a qualitative way in works by Moon, Huerre and Redekopp [1983] and Bretherton and Spiegel [1983] which we might like to test in future works.

2. Nonlinear evolution of axisymmetric disturbances in core-annular flows

Two immiscible fluids are flowing inside a pipe of radius R_2. The interface between the two fluids is perfect cylindrical, $r=R_1$. Fluid 1 is located in the core and fluid 2 in the annulus. We are interested in the stability of this core-annular flow.

It was shown by PCJ and CBJ that there are five independent controlling parameters: a, m, ζ_2, J^* and $\boldsymbol{R}_1$ for horizontal core-annular flow and six for vertical core-annular flow: a, m, ζ_2, J^*, $\boldsymbol{R}_g$ and F (defined below). Although a multi-parameter bifurcation analysis is possible, we here restrict our attention to the simplest case in which a single parameter is varied for fixed values of other five. We prefer a parameter that we control in our experiments once the working fluids and the pipe are chosen. For horizontal flow, the Reynolds number $\boldsymbol{R}_1$ defined in PCJ can be used as the bifurcation parameter. For vertical flow, however, the Reynolds number defined in CBJ is based on gravity and is more like a geometrical than a dynamical parameter. A better parameter is the forcing ratio $F = \frac{f}{\rho_1 g}$, where $f = -\frac{d\hat{P}_1}{dx} = -\frac{d\hat{P}_2}{dx}$ is the applied pressure gradient. In this paper we use a different equivalent set of parameters incorporating both horizontal and vertical flows.

We shall choose the magnitude of the center-line velocity $|W(0)|$ as the velocity scale, R_1 as the length scale, and $\frac{R_1}{|W(0)|}$ as the time scale. We define the following parameters:

$$a = \frac{R_2}{R_1} ,$$

$$(m_1, m_2) = (1, m) = (1, \frac{\mu_2}{\mu_1}),$$

$$(\zeta_1, \zeta_2) = (1, \zeta) = (1, \frac{\rho_2}{\rho_1}) ,$$

$$\boldsymbol{R} \overset{\text{def.}}{=} \boldsymbol{R}_1 = \frac{|W(0)|\, \rho_1\, R_1}{\mu_1} , \quad \text{(Reynolds number)}$$

$$\boldsymbol{R}_2 = \frac{|W(0)|\, \rho_2\, R_1}{\mu_2} = \frac{\zeta_2}{m} \boldsymbol{R} ,$$

$$\boldsymbol{R}_g = \frac{g\, R_2^3}{\nu_1^2} = \frac{g\, R_1^3\, a^3}{\nu_1^2} ,$$

(Reynolds number based on gravity. This is different from $\boldsymbol{R}_g = \sqrt{\frac{g\, R_1^3}{\nu_1^2}}$ used by CBJ)

$$J^* = \frac{T\, R_2}{\rho_1\, \nu_1^2} ,$$

$$K = \frac{f + \rho_1\, g}{f + \rho_2\, g} \quad \text{(ratio of driving forces in core and annulus)}$$

where

$$f = -\frac{d\hat{P}}{dx}$$

is the applied pressure gradient and f is one and the same constant for both core and annulus for concentric basic flows considered. The cylindrical polar coordinate system is chosen such that gravity is acting in the positive x direction. We choose the Reynolds number $\boldsymbol{R}$ as our bifurcation parameter. When the density is matched, K=1 and gravity does not enter the problem. $\boldsymbol{R}_g$, J^* are known constants once the working fluids and pipe radius are given, independent of flow conditions.

The basic flow in dimensional form is given in (2.2) of CBJ. The velocity at the centerline of the pipe is

$$W(0) = \frac{f + \rho_1 g}{4\mu_1} R_1^2 + \frac{f + \rho_2 g}{4\mu_2} (R_2^2 - R_1^2) + \frac{\text{û}\rho\text{ô}\, g R_1^2}{2\mu_2} \ln\frac{R_2}{R_1} .$$

Using this relation we can show that the parameter K can be expressed in terms of $\boldsymbol{R}_g$ and $\boldsymbol{R}$. To do this we need to distinguish between the case $W(0)>0$ and $W(0)<0$. For convenience, we will loosely refer to flows with $W(0)>0$ as down-flows and $W(0)<0$ as up-flows, although mixed flows are also possible for both cases, depending on the magnitude of $W(0)$ or f, as shown in CBJ. Then the dimensionless basic flow can be expressed as:

(a) *down-flow:* $W(0) > 0$

$$K(\boldsymbol{R}) = \frac{4ma^3\boldsymbol{R} + \text{û}\zeta\text{ô}\, \boldsymbol{R}_g (a^2-1-2\ln a)}{4ma^3\boldsymbol{R} - \text{û}\zeta\text{ô}\, \boldsymbol{R}_g (m+2\ln a)} , \tag{2.1$_a$}$$

$$W_1(r, \boldsymbol{R}) = 1 - \frac{mK(\boldsymbol{R})r^2}{mK(\boldsymbol{R})+a^2-1+2(K(\boldsymbol{R})-1)\ln a} , \quad 0 \le r \le 1 \tag{2.1$_b$}$$

$$W_2(r, \boldsymbol{R}) = \frac{a^2-r^2-2(K(\boldsymbol{R})-1)\ln\frac{r}{a}}{mK(\boldsymbol{R})+a^2-1+2(K(\boldsymbol{R})-1)\ln a} \quad 1 \le r \le a \tag{2.1$_c$}$$

(b) *up-flow:* $W(0) < 0$

$$K(\boldsymbol{R}) = \frac{4ma^3\boldsymbol{R} - \text{û}\zeta\text{ô}\, \boldsymbol{R}_g (a^2-1-2\ln a)}{4ma^3\boldsymbol{R} + \text{û}\zeta\text{ô}\, \boldsymbol{R}_g (m+2\ln a)} , \tag{2.2$_a$}$$

$$W_1(r, \boldsymbol{R}) = -1 + \frac{mK(\boldsymbol{R})r^2}{mK(\boldsymbol{R})+a^2-1+2(K(\boldsymbol{R})-1)\ln a} , \quad 0 \le r \le 1 \tag{2.2$_b$}$$

$$W_2(r, \boldsymbol{R}) = -\frac{a^2-r^2-2(K(\boldsymbol{R})-1)\ln\frac{r}{a}}{mK(\boldsymbol{R}) + a^2-1+2(K(\boldsymbol{R})-1)\ln a} . \quad 1 \le r \le a \tag{2.2$_c$}$$

In the above formulas, the jump û•ô is defined as

$\hat{u} \bullet \hat{o} = (\bullet)_1 - (\bullet)_2$.

It is easy to see from these expressions that the up-flow velocity is formally the negative of the down-flow velocity except that the parameter K (***R***) is different. However, down-flow and up-flow can be treated uniformly, using the velocity profile (2.1), with $\boldsymbol{R}_g>0$ for down-flows and $\boldsymbol{R}_g<0$ for up-flows since any up-flow can be obtained from a down-flow by simply reversing the direction of gravity.

The basic flow (2.1) depends on the Reynolds number ***R*** through the parameter K (***R***) and enter the basic amplitude (3.11) which is to be derived only through d_1.

Perfect core-annular flow (2.1) can be realized only if the controlling parameters fall in a certain range, as in the case of a vertical pipe studied by CBJ. It is also possible to realize perfect core-annular flow in a horizontal pipe if the densities of oil and water are matched. The experiment 2 of Charles *et al* [1961], called "oil in water concentric", can be regarded as an example of perfect core-annular flow in a horizontal pipe.

Numerical experiments using linear theory have shown that, without exception, axisymmetric disturbances are most dangerous (see PCJ, HJ, CBJ). Therefore we restrict our analysis to axisymmetric disturbances. Nevertheless nonaxisymmetric waves arise in practice. The photograph of "corkscrew" waves exhibited in Figure 2 is a good example. These "corkscrew" waves can result from the instabilities due to finite nonaxisymmetric disturbances.

(Insert Figure 2 near here)

For axisymmetric finite disturbances, the disturbance velocity is of the form $\mathbf{u}$=(u,0,w) in the cylindrical coordinates (r,θ,x) and $\frac{\partial(\bullet)}{\partial\theta} = 0$. The full nonlinear evolution equations for $\mathbf{u}$ in dimensionless form are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial x} = 0, \tag{2.3$_a$}$$

$$\frac{\partial u}{\partial t} + W\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial r} + \frac{1}{\boldsymbol{R}_\ell}\left[\nabla^2 u - \frac{u}{r^2}\right], \tag{2.3$_b$}$$

$$\frac{\partial w}{\partial t} + W\frac{\partial w}{\partial x} + W'u + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial x} + \frac{1}{\boldsymbol{R}_\ell}\nabla^2 w \tag{2.3$_c$}$$

where

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial^2 f}{\partial x^2},$$

and ℓ=1 when $0 \le r \le 1+\delta(x,t)$ and ℓ=2 when $1+\delta(x,t) \le r \le a$. $\delta(x,t)$ is the dimensionless deviation of the interface from the perfect cylindrical one r=1. The primes indicate derivatives with respect to r. On the pipe wall r=a , we have the no-slip condition

$$u = w = 0, \tag{2.4}$$

and at the center of the pipe, r = 0, u, w, p must be bounded.

At the interface, $r=1+\delta(x,t)$, we have the kinematic condition

$$u = \frac{\partial \delta}{\partial t} + (W_1 + w_1)\frac{\partial \delta}{\partial x} = \frac{\partial \delta}{\partial t} + (W_2 + w_2)\frac{\partial \delta}{\partial x}, \tag{2.5}$$

and the continuity of velocity

$$[\![u]\!]_\delta = [\![W + w]\!]_\delta = 0, \tag{2.6}$$

where the subscript δ refers to the deformed interface $r=1+\delta(x,t)$.

The shear stress and normal stress balances on the interface are

$$[\![m\{(1-\delta_x^2)(W' + u_x + w_r) + 2\delta_x(u_r - w_x)\}]\!]_\delta = 0, \tag{2.7}$$

$$- \hat{u}\zeta p \hat{o}_{\delta} + \frac{1}{\boldsymbol{R}_1} \frac{2}{1+\delta_x^2} \hat{u} m\{ u_r - \delta_x (W' + u_x + w_r) + \delta_x^2 w_x \} \hat{o}_{\delta}$$

$$= \frac{J^*}{a\boldsymbol{R}_1^2} \left\{ \frac{\delta_{xx}}{(1+\delta_x^2)^{3/2}} - \frac{1}{(1+\delta)(1+\delta_x^2)^{1/2}} + 1 \right\} , \qquad (2.8)$$

where the subscripts r,x indicate differentiations with regard to r and x respectively.

To simplify these equations further, we introduce a perturbation stream function ψ in each region:

$$u = -\frac{\psi_x}{r} ,$$

$$w = \frac{\psi_r}{r} .$$

Then the field equations can be reduced to a single equation for the stream function ψ by eliminating pressure p:

$$(L\psi)_t - \left(W'' - \frac{W'}{r}\right) \psi_x + \left(W + \frac{1}{r}\psi_r\right)(L\psi)_x - \frac{1}{r}\psi_x (L\psi)_r + \frac{2}{r^2}\psi_x L\psi = \frac{1}{\boldsymbol{R}_1} L^2\psi \qquad (2.9)$$

where the operator L is defined as

$$L = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} .$$

$$\text{At } r = a : \ \psi = \psi_r = 0. \qquad (2.10)$$

$$\text{At } r = 0 : \ \psi = \psi_r = 0. \qquad (2.11)$$

All the interface conditions can be expressed in terms of perturbation stream function ψ, resulting in a system of differential equations for $\psi_1(r,x,t)$, $\psi_2(r,x,t)$ and $\delta(x,t)$.

To study weakly nonlinear stability, we expand the interfacial conditions around the unperturbed interface r=1 and truncate the Taylor series at order $O(\delta^3)$. For this purpose we notice that from the linear theory, we have

$$u \sim w \sim d. \tag{2.12}$$

The resulting interface conditions up to the third order can be summarized as:

Kinematic condition:

$$L_{i1}(\psi_1 , \delta) = Q_{i1}(\psi_1 , \delta) + C_{i1}(\psi_1 , \delta), \qquad (2.13)_a$$

Continuity of velocity:

$$û L_{i2}(\psi) ô = Q_{i2}(\psi_1, \psi_2, \delta) + C_{i2}(\psi_1, \psi_2, \delta) , \qquad (2.13)_b$$

$$û L_{i3}(\psi,\delta) ô = Q_{i3}(\psi_1, \psi_2, \delta) + C_{i3}(\psi_1, \psi_2, \delta) , \qquad (2.13)_c$$

Shear stress balance:

$$û m L_{i4}(\psi,\delta) ô = Q_{i4}(\psi_1, \psi_2, \delta) + C_{i4}(\psi_1, \psi_2, \delta) , \qquad (2.13)_d$$

Normal stress balance:

$$û L_{ii5}(\psi,\delta) ô - \frac{J^*}{a\boldsymbol{R}_1^2} (\delta_{xxx} + \delta_x) = Q_{i5}(\psi_1, \psi_2, \delta) + C_{i5}(\psi_1, \psi_2, \delta). \qquad (2.13)_e$$

In the above expressions, the jump û(•)ô without subscript δ refers to the jump evaluated at the undeformed interface r=1 and all the quantities are evaluated at r=1 as well. The symbols L, Q, C refer to linear, quadratic and cubic differential operators respectively. The subscripts i indicate that all these operators are defined on the interface r=1 only. These interfacial operators are listed in Chen [1990].

The reduced system (2.9), (2.10), (2.11) and (2.13) is used to derive the amplitude equation.

3. Multiple scales, wave packets and the Ginzburg-Landau equation

The derivation of the amplitude equation near criticality, using the techniques of multiple scales, is now well-known and the details can be found in Newell [1974] or Stewartson and Stuart [1971]. We introduce a small perturbation parameter ε, defined by

$$\varepsilon^2 = \left| d_{1r} (\boldsymbol{R} - \boldsymbol{R}_c) \right| , \tag{3.1}$$

where we have adopted the notation of Stewartson and Stuart [1971] for d_{1r}

$$d_{1r} = \text{Real}\, \{d_1\},$$

$$d_1 = -i \left\{ \frac{\square(\alpha_c)}{\square \boldsymbol{R}} \right\} (\alpha_c, \boldsymbol{R}_c) \ . \tag{3.2}$$

Here $-i\alpha_c$ is the complex growth rate for linear stability of the basic flow and $(\alpha_c, \boldsymbol{R}_c)$ is the point at the nose of the neutral curve. This critical point is a minimum of $\boldsymbol{R}$ with respect to K on the upper branch of the neutral curve and a maximum on the lower branch. Here, "upper" and "lower" refer to the bifurcation parameter $\boldsymbol{R}$, not the wave number α as traditionally assigned. The basic flow loses stability as $\boldsymbol{R}$ is increased past $\boldsymbol{R}_c$ on the upper branch. Here, $d_{1r}>0$ on the upper branch, and $d_{1r}<0$ on the lower branch. We may consider the first case $d_{1r}>0$, $\boldsymbol{R} > \boldsymbol{R}_c$ and then generalize to cover the three other possibilities.

Introduce the slow spatial and time scales

$$\xi = \varepsilon (x - c_g t),$$

$$\tau = \varepsilon^2 t , \tag{3.3$_1$}$$

where c_g is the group velocity at criticality. These scales are appropriate for a wave packet centered at the nose of the neutral curve, and the long time behavior of this wave train is examined in the frame moving with its group velocity c_g. The perturbation stream function ψ and the interface perturbation δ are assumed to be slowly varying functions of ξ,τ :

$$\psi \varnothing \psi(\xi, \tau; r, x, t),$$

$$\delta \varnothing \delta(\xi, \tau; x, t),$$

$$\frac{\partial}{\partial t} \varnothing \frac{\partial}{\partial t} - \varepsilon\, c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau} ,$$

$$\frac{\partial}{\partial x} \varnothing \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi} . \qquad (3.3)_2$$

We then define the traveling wave factor of the amplitude

$$E \overset{\text{def}}{=} \exp\left[i\alpha_c (x - c_r t)\right] , \qquad (3.4)$$

where c_r is the phase velocity at criticality. For a wave packet centered around the critical state, we can assume that ψ and δ have the following forms

$$\psi = \psi_0(r, \xi, \tau) + \{ \psi_1(r, \xi, \tau)\, E + \text{c.c} \} + \{ \psi_2(r, \xi, \tau)\, E^2 + \text{c.c} \} + \text{h.h.}$$

$$\delta = \delta_0(\xi, \tau) + \{ \delta_1(\xi, \tau)\, E + \text{c.c} \} + \{ \delta_2(\xi, \tau)\, E^2 + \text{c.c} \} + \text{h.h.} , \qquad (3.5)$$

where c.c stands for complex conjugate and h.h. for higher harmonics. We assume that the fundamental wave $\psi_1(r, \xi, \tau)\, E$ is of order ε and expansions in ε yield

$$\psi_1 = \varepsilon\, \psi_{11}(r, \xi, \tau) + \varepsilon^2\, \psi_{12}(r, \xi, \tau) + \varepsilon^3\, \psi_{13}(r, \xi, \tau) + O(\varepsilon^4),$$

$$\psi_2 = \varepsilon^2\, \psi_{22}(r, \xi, \tau) + O(\varepsilon^4),$$

$$\psi_0 = \varepsilon^2\, \psi_{02}(r, \xi, \tau) + O(\varepsilon^4), \qquad (3.6)$$

and similarly,

$$\delta_1 = \varepsilon\, \delta_{11}(\xi, \tau) + \varepsilon^2\, \delta_{12}(\xi, \tau) + \varepsilon^3\, \delta_{13}(\xi, \tau) + O\,(\varepsilon^4),$$

$$\delta_2 = \varepsilon^2\, \delta_{22}(\xi, \tau) + O\,(\varepsilon^4),$$

$$\delta_0 = \varepsilon^2\, \delta_{02}(\xi, \tau) + O\,(\varepsilon^4). \tag{3.7}$$

Substitute the above expansions into the nonlinear systems of equations and identify different orders (k, n) ⁄ (E^k, ε^n) to obtain a sequence of differential equations. To obtain the amplitude equation at the lowest order, we only need to consider k=0, 1, 2 exponentials (3.4) and n=1,2,3 powers of the small parameter ε.

At order (1,1) we have the linear eigenvalue problem at criticality and if we denote the eigenfunction at criticality to be $\varphi(r)$, then

$$\psi_{11}(r, \xi, \tau) = A(\xi, \tau)\, \varphi(r),$$

$$\delta_{11}(\xi, \tau) = A(\xi, \tau)\, \eta_{11}\,, \tag{3.8}$$

where η_{11} is a constant which can be expressed in terms of the value of φ at r = 1 and $A(\xi, \tau)$ is the slowly varying amplitude of the fundamental wave. The equations which arise at orders (0, 2), (2, 2), (1, 2) support separated product solutions of the following type

$$\psi_{02}(r, \xi, \tau) = |A(\xi, \tau)|^2\, F(r)\,,$$

$$\delta_{02}(\xi, \tau) = |A(\xi, \tau)|^2\, \eta_{02}\,;$$

$$\psi_{22}(r, \xi, \tau) = A^2(\xi, \tau)\, G(r)\,,$$

$$\delta_{22}(\xi, \tau) = A^2(\xi, \tau)\eta_{22}\,;$$

$$\psi_{12}(r, \xi, \tau) = \frac{\partial A(\xi, \tau)}{\partial \xi} H(r) + A_2(\xi, \tau)\varphi(r) ,$$

$$\delta_{12}(\xi, \tau) = \frac{\partial A(\xi, \tau)}{\partial \xi} \eta_{12} + A_2(\xi, \tau)\, \eta_{11} . \tag{3.9}$$

Then at orders (1, 2) and (1, 3), we have

$$L_1(H, \eta_{12}) = F\,(\varphi(r), c_g),$$

$$L_1(\psi_{13}, \delta_{13}) = J_1 \frac{\partial A}{\partial \tau} + J_2 \frac{\partial^2 A}{\partial \xi^2} + J_3\, A + J_4 \,|\, A\,|^2 A + J_5 \frac{\partial A_2}{\partial \xi} \tag{3.10}$$

where L_1 is the linear Orr-Sommerfeld operator at criticality and J_i, i=1,…,5 are functions of $\varphi(r)$, F(r), G(r) and H(r). Applying the Fredholm alternative at order (1,2), we can obtain a formula determining the group velocity c_g. At order (1,3), the application of the Fredholm alternative yields the Ginzburg-Landau equation governing the amplitude $A(\xi,\tau)$ of the fundamental wave,

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \frac{d_1}{d_{1r}} A - l \,|\, A\,|^2 A\,. \tag{3.11}$$

The term $\frac{\partial A_2}{\partial \xi}$ does not appear because its coefficient vanishes when the formula for the group velocity c_g arising from the Fredholm alternative is used. The complementary part of the solution of the singular problem at order (1,2) has no effect on the final amplitude equation. The coefficient of the cubic term, l, is called the first Landau constant and it depends on all the lower order solutions. The coefficients a_2 , d_1 and l are complex in general and can be computed using the Fredholm alternative. For the upper branch of the neutral curve $d_{1r}>0$ and for the lower branch $d_{1r}<0$. For non-degenerate cases, the real part of a_2 is always positive for both the upper and the lower branch because the growth rate reaches a maximum at the critical point, the nose of the neutral curve ($a_{2r}=0$ if the neutral curve has a higher order (>2) contact with $\boldsymbol{R} = \boldsymbol{R}_c$).

We may write a uniform form of the Ginzburg-Landau equation, valid for both the upper and lower branch of the neutral curves

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \mathrm{sgn}(d_{1r})\, \mathrm{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) \frac{d_1}{d_{1r}} A - l \mid A \mid^2 A\,, \tag{3.12}$$

by taking proper account of the various sign possibilities offered by (3.1). Here the parameter $\mathrm{sgn}(d_{1r})\ \mathrm{sgn}(\boldsymbol{R}–\boldsymbol{R}_c)$ measures the distance from the bifurcation threshold (linear growth or damping), $\mathrm{sgn}(d_{1r})\ \mathrm{sgn}(\boldsymbol{R}–\boldsymbol{R}_c)\frac{d_{1i}}{d_{1r}}$ corresponds to the frequency shift due to the linear dispersion, a_{2r}, a_{2i}, l_r, l_i are associated with diffusion(a_{2r}>0), dispersion, nonlinear saturation(l_r) and nonlinear renormalization of the frequency respectively.

The Landau constant l depends on the normalization of the eigenvector $\varphi(r)$ of the spectral problem, but is independent of the normalization of the adjoint eigenvector. If we use a different normalization for the eigenvector $\varphi(r)$ such that

$$\varphi(r) \varnothing q\, \varphi(r),\ A(\xi, \tau) \varnothing q\, A(\xi, \tau)$$

where q is any non-zero constant, we find, using (3.12), that

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \mathrm{sgn}(d_{1r})\, \mathrm{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) \frac{d_1}{d_{1r}} A - l\, |q|^2\, |A|^2 A\,. \tag{3.13}$$

The Landau constant will become unique if a well-defined amplitude is introduced. This is especially important when pursuing higher order Landau constants (Joseph and Sattinger [1972), Herbert [1980], Sen and Venkateswarlu [1983]). In the lowest order case, the Ginzburg-Landau equation (3.12), we can simply rescale the amplitude function $A(\xi, \tau)$

$$A(\xi, \tau) \varnothing \frac{A(\xi, \tau)}{|q| \sqrt{|l_r|}} \tag{3.14}$$

where $l = l_r + i l_i$, to get a Ginzburg-Landau equation with coefficients independent of q:

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \text{sgn}(d_{1r})\ \text{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) \frac{d_1}{d_{1r}} A - \left(\text{sgn}(l_r) + iC_n\right) |A|^2 A\ , \tag{3.15}$$

where $C_n = \dfrac{l_i}{|l_r|}$ is a parameter independent of the normalization condition for $\varphi(r)$.

Another useful rescaled form of (3.13) can be obtained by introducing the following transformations:

$$A(\xi, \tau) = \frac{\hat{A}(\xi, \tau)}{|q| \sqrt{|l_r|}}\ \exp\left[i\ \text{sgn}(d_{1r})\ \text{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) \frac{d_{1i}}{d_{1r}} \right]\ ,$$

$$\hat{\xi} = \frac{\xi}{\sqrt{|a_{2r}|}}\ , \quad C_d = \frac{a_{2i}}{|a_{2r}|}\ , \quad C_n = \frac{l_i}{|l_r|}\ . \tag{3.16}$$

After dropping the roofs, we get

$$\frac{\partial A}{\partial \tau} - \left(\text{sgn}(a_{2r}) + iC_d\right) \frac{\partial^2 A}{\partial \xi^2} = \text{sgn}(d_{1r})\ \text{sgn}(\boldsymbol{R} - \boldsymbol{R}_c)\ A - \left(\text{sgn}(l_r) + i\ C_n\right) |A|^2 A\ . \tag{3.17}$$

The form (3.17) was first introduced by Moon, Huerre and Redekopp [1983] in their study of transition to chaos in solutions of the Ginzburg-Landau equation. Since $a_{2r}>0$ we can replace $\text{sgn}(a_{2r})$ by +1. $\text{sgn}(d_{1r})= +1$ for the upper branch and $\text{sgn}(d_{1r})= -1$ for the lower branch. Equation (3.17) can be regarded as the canonical form of the Ginzburg-Landau equation.

The spectral problem (1, 1) and the boundary value problems at orders (0, 2), (2, 2), (1, 2) which are needed to compute the coefficients of the Ginzburg-Landau equation (3.12) are listed in Chen [1990]. We note that at each order the interface parameter η can be eliminated. All the algebraic operations are carried out by the symbolic manipulator REDUCE2 and independently checked by hand. An efficient method for computing the coefficients of the Ginzburg-Landau equation is presented in the next section.

4. Singular value decomposition and its application to the numerical computation of the coefficients of amplitude equations and normal forms

There are many equations used as model equations for the study of various physical process. These equations arise as an asymptotic solvability condition, which is a condition on the leading order approximation to the solution of a more complicated set of equations, which ensures that the later iterates of the approximation remain uniformly bounded. Examples of these equations are the Korteweg-de Vries equation and its generalizations, the Ginzburg-Landau equation and its generalizations, and the Davey-Stewartson equations (Craik [1983], Newell [1985]). For parallel shear flows, the coefficients of these model equations are in general given by very lengthy domain integrals expressing solvability conditions, commonly known as the Fredholm alternative.

The Fredholm alternative requires that the inhomogeneous terms in the underlying system of differential equations, which contain the unknown coefficients, be orthogonal to the independent eigenvector spanning the null space of the adjoint system of differential equations. Typically the underlying system of the inhomogeneous differential equation is discretized and solved as an inhomogeneous matrix-valued problem. We find that the solvability conditions which lead to values of the unknown coefficients are conveniently and economically computed by application of the singular value decomposition directly to the matrix formulation.

The singular value decomposition (SVD) is one of the most important decompositions in matrix algebra and is widely used for statistics and for solving least squares problems (see Golub and Van Loan [1983]). The decomposition theorem can be stated as follows: each and every $M \infty N$ complex valued matrix **T** can be reduced to diagonal form by unitary transformations **U** and **V**,

$$\mathbf{T} = \mathbf{U}\,\mathrm{diag}[\sigma_1, \sigma_2, \cdots\cdots, \sigma_N]\,\mathbf{V}^H , \tag{4.1}$$

where $\sigma_1 \varepsilon \sigma_2 \varepsilon \ldots \varepsilon \sigma_N \varepsilon 0$ are real-valued scalars, called the singular values of **T**. Here **U** is an $M \infty N$ column orthonormal matrix, **V** an $N \infty N$ unitary matrix and $\mathbf{V}^H$ is the Hermitian transpose of **V**. The columns of **U** and **V** are called the left and right singular vectors of **T**, respectively.

When M=N, **T** is a square matrix and

$$\mathbf{U}\,\mathbf{U}^H = \mathbf{U}^H\,\mathbf{U} = \mathbf{I} \tag{4.2}$$

$$\mathbf{V}\,\mathbf{V}^H = \mathbf{V}^H\,\mathbf{V} = \mathbf{I}\,. \tag{4.3}$$

Consider the generalized matrix eigenvalue problem

$$(\mathbf{A} - c\mathbf{B})\,\mathbf{x} = 0, \tag{4.4}$$

where **A**, **B** are both square $N \infty N$ complex matrices. Assume that c is an semi-simple eigenvalue of (4.4) with algebraic and geometric multiplicity K. Then, applying SVD to the matrix **A**–c**B**, we get

$$\mathbf{A} - c\mathbf{B} = \mathbf{U}\,\mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_{N-K}, 0, 0, \cdots, 0]\,\mathbf{V}^H , \tag{4.5}$$

where $\sigma_1 \varepsilon \sigma_2 \varepsilon \cdots \varepsilon \sigma_{N\text{-}K} > 0$ are real constants (see Wilkinson [1977)).

Let

$$\mathbf{U} = [\,\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{N\text{-}K}, \mathbf{u}_{N\text{-}K+1}, \cdots, \mathbf{u}_N\,], \tag{4.6}$$

$$\mathbf{V} = [\,\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{N\text{-}K}, \mathbf{v}_{N\text{-}K+1}, \cdots, \mathbf{v}_N\,], \tag{4.7}$$

where $\mathbf{u}_j$, $\mathbf{v}_j$ (j=1, ⋯ , N) are the column vectors of matrices **U** and **V** respectively. From (4.4) and (4.5) we see that diag[σ_1, σ_2, ⋯ , σ_{N-K}, 0, 0, ⋯ , 0]**y**=0, where $\mathbf{V}^H\mathbf{x}=\mathbf{y}$ and **x** is the eigenvector corresponding to the eigenvalue c. Therefore we have

$$\mathbf{V}^H \mathbf{x} = \mathbf{y} = [\, 0, 0, \cdots , 0, y_{N-K+1}, \ldots , y_N \,], \tag{4.8}$$

where y_{N-K+1}, ⋯ , y_N are K arbitrary constants. Then **x**=**Vy** is an eigenvector of **A**–c**B**. We find, in this way, that the column vectors $\mathbf{v}_j$, j=N–K+1, ⋯ , N, are the K independent eigenvectors corresponding to c, normalized with

$$\mathbf{v}_j^* \mathbf{v}_j^T = 1, \quad j = N–K+1, \cdots , N,$$

where star * denotes the complex conjugate and superscript T for transpose. Similarly the column vectors $\mathbf{u}_j$, j=N–K+1, ⋯ , N, are the K independent eigenvectors of the problem adjoint to (4.4):

$$(\mathbf{A} - c\,\mathbf{B})^H \mathbf{x} = 0\ . \tag{4.9}$$

They are the corresponding adjoint eigenvectors, normalized with

$$\mathbf{u}_j^* \mathbf{u}_j^T = 1, \quad j = N–K+1, \cdots , N.$$

The application of SVD to solve the inhomogeneous system of algebraic equations

$$(\mathbf{A} - c\mathbf{B})\, \mathbf{x} = \mathbf{f} \tag{4.10}$$

is straightforward. Suppose c is an semi-simple eigenvalue of (4.4) of multiplicity K. We use SVD to decompose **A**–c**B** in the form (4.5). We then compute

$$\mathrm{diag}[\sigma_1, \sigma_2, \ldots , \sigma_{N-K}, 0, 0, \ldots , 0]\ \mathbf{V}^H \mathbf{x} = \mathbf{U}^H \mathbf{f}\ . \tag{4.11}$$

The last K components of the vector on the left of (4.11) are identically zero and so must be those on the right. This defines the Fredholm alternative, the solvability conditions

$$\mathbf{u}_j^{*}\mathbf{f}^{T} = 0, \quad j = N–K+1, \ldots, N, \tag{4.12}$$

for the inhomogeneous matrix problem (4.10). The conditions (4.12) are necessary and sufficient for solvability of the inhomogeneous problem (4.10) in $\mathbf{C}$ when c is an eigenvalue of **A** relative to **B**.

The solution to the inhomogeneous equation (4.10) is given by

$$\mathbf{x} = \mathbf{V}_S\,\mathbf{g} + \sum_{j=N-K+1}^{N} \beta_j \mathbf{v}_j \ , \tag{4.13}$$

where the N ∞ (N –K) matrix $\mathbf{V}_S$ is given by

$$\mathbf{V}_S = [\ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{N-K}\ ] \qquad ,$$

with $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{N-K}$ given by (4.7) and the vector **g** has N–K components given by

$$\mathbf{g} = [\sigma_1^{-1}\,\mathbf{u}_1^{*}\mathbf{f}^{T}, \sigma_2^{-1}\,\mathbf{u}_2^{*}\mathbf{f}^{T}, \ldots, \sigma_{N-K}^{-1}\,\mathbf{u}_{N-K}^{*}\mathbf{f}^{T}],$$

where the $\mathbf{u}_j$' s are those given by (4.6). The β_j' s are constants and can be determined by K normalization conditions.

The solvability conditions (4.12) are easy to compute. We also notice that the SVD is a very stable and reliable algorithm.

The applications of the above SVD algorithm to bifurcation theory is studied in detail by Chen and Joseph [1990]. Independently, Newell, Passot and Souli [1989] applied the same algorithm to the bifurcation study of convection at finite Rayleigh numbers in large containers. The algorithm takes advantage of the matrix formulations of the perturbation problems stated in Section 3. Specifically, the problems (0, 2), (2, 2) are invertible and (1, 2), (1, 3) are singular. For these singular problems, a singular system of algebraic equations of the form (4.10) arises after discretization and the techniques described above is readily applicable. For the spectral problem, we have

$$(\mathbf{A} - c_r \mathbf{B})\, \boldsymbol{\varphi} = 0 ,$$

where the matrix $\mathbf{A}-c_r\mathbf{B}$ and the vector $\boldsymbol{\varphi}$ result from the discretization of the Orr-Sommerfeld operator at criticality and the eigenfunction $\varphi(r)$ respectively. At orders (1, 2) and (1, 3), we have the following singular algebraic equations

$$(\mathbf{A} - c_r\mathbf{B})\, \mathbf{h} = \mathbf{f}\,(\boldsymbol{\varphi} , c_g), \tag{4.14}$$

$$(\mathbf{A} - c_r\mathbf{B})\, \psi_{13} = \frac{\partial A}{\partial \tau}\, \mathbf{f}_1 + \frac{\partial^2 A}{\partial \xi^2}\, \mathbf{f}_2 + \frac{A}{d_{1r}}\, \mathbf{f}_3 + |A|^2 A\, \mathbf{f}_4 . \tag{4.15}$$

Assume at criticality c_r is semi-simple with multiplicity K=1. Then (4.14) can be solved by first using the solvability condition (4.12) to evaluate the group velocity c_g and then using the formula (4.13) without the complementary part (β_j=0) because of the fact that the complementary part has no contribution to the final amplitude equation. Application of the solvability condition (4.12) to (4.15) generates the coefficients of the Ginzburg-Landau equation (3.12):

$$a_2 = - \overset{*}{\mathbf{u}}_N\, \mathbf{f}_2^T \,/\, \overset{*}{\mathbf{u}}_N\, \mathbf{f}_1^T ,$$

$$d_1 = - \overset{*}{\mathbf{u}}_N\, \mathbf{f}_3^T \,/\, \overset{*}{\mathbf{u}}_N\, \mathbf{f}_1^T ,$$

$$l = \overset{*}{\mathbf{u}}_N\, \mathbf{f}_4^T \,/\, \overset{*}{\mathbf{u}}_N\, \mathbf{f}_1^T ,$$

and

$$\overset{*}{\mathbf{u}}_N\, \mathbf{f}_1^T \neq 0.$$

The above procedure is applied to one-fluid plane Poiseuille flow and compared with values obtained by Reynolds and Potter [1967] (RP) and Davey, Hocking and Stewartson [1974] (DHS) using analytical formulas. In order to compare the accuracy of the present scheme, the same normalization condition for the eigenvector as in RP and DHS is used to make the Landau constant unique. The results are presented in Table 1. It can be seen that the present algorithm gives accurate and reliable results and should find a wide range of applications in similar situations. The same algorithm is applied to the bifurcation analysis of core-annular flows and a Chebychev psuedo-spectral method is used for the discretizations of the differential equations.

Table 1 Comparison of the coefficients of the Ginzburg-Landau equation for one fluid plane Poiseuille flow, α_c = 1.02055, $\boldsymbol{R}_c$ = 5772.22

	RP, DHS	Present
c_g	0.383	0.383098912
d_1	(0.168 + i 0.811) 10^{-5}	(0.1682592671 + i 0.8112758976) 10^{-5}
a_2	0.187 + i 0.0275	0.186701701 + i 0.02746400111
l	–30.8 + i 173	–30.9434352 + i 172.826053

For all the calculations we performed for nonlinear stability of core-annular flows, we normalize the eigen-streamfunctions φ_1 and φ_2 with discrete L_2 norms such that $||\varphi_1||^2 + ||\varphi_2||^{2=}1$. Most of our results are summarized in Tables 2 through 10.

(Insert Tables 2–10 near here)

5. Nonlinear stability of core-annular flows

The nature of the bifurcation of core-annular flows is determined by the real part of the Landau constants l in (3.12). If l_r>0, the bifurcation is supercritical and a finite amplitude equilibrium solution exists. On the other hand, if l_r<0, the bifurcation is subcritical, the bifurcating solution of (3.12) will burst in finite time and higher order theory is needed (Davey, Stewartson and Stuart [1974]).

The coefficients of the Ginzburg-Landau equations for different parameters are listed in Tables 2–10. Since we are mainly interested in the direction of the bifurcations, we have only listed the values of the critical states (α_c, $\boldsymbol{R}_c(\alpha_c)$), sgn($l_r$), C_d and C_n in these tables, corresponding to the canonical form (3.17). The values of c_r, c_g, d_1, a_2 and l are documented in Chen [1990]. The first thing to look at in these tables is the next to the last column labeled sgn(l_r). A plus sign here means that the bifurcation is supercritical, subcritical for the minus sign.

The cases studied in Tables 2 –10 explore the general features of bifurcation of core-annular flow. The cases with parameters corresponding to some of the experiments of Charles *et al* [1961] and BCJ are discussed in Section 10. As mentioned early, our bifurcation analysis is valid only near the nose of the neutral curves. This means that such analysis is applicable only when the upper branch and the lower branch of the neutral curve are separated, i.e., there exists a Reynolds number window within which core-annular flow is linearly stable, as in Figures 3 and 4. In other words, we can only study those cases where linearly stable core-annular flow is possible. The reader may understand the disposition of subcritical and supercritical solutions in the presence of upper and lower branches of neutral curves by study of the sketch and caption comprising Figure 5. PCJ, BCJ have shown that only when the parameters a, m, ζ, J^* fall in certain subspace of the parameter space that such stable core-annular flow is possible. Typically, there is a “thin layer effect”: a thin lubricating layer, i.e. small value of $a-1=R_2/R_1-1$, has stabilization effect on core-annular flow. It is also shown by Hu, Lundgren and Joseph [1990], that if the oil is too viscous, $m=\mu_2/\mu_1 \ll 1$, stable core-annular flow is very difficult to achieve. We have thus restricted our studies in Tables 2

–10 to those values of a and m, typically small values of a –1 and values of m of order 10^{-1}, with which linearly stable core-annular flows are possible.

(Insert Figures 3, 4 and 5 near here)

The parameter $|\boldsymbol{R}_g|$=0.5 is used for all the cases considered in Tables 2–10. This parameter enters the equations only as a product $(\zeta_2 - 1)\ \boldsymbol{R}_g$, hence plays no role when the densities of the two fluids are the same, ζ_2=1. We can vary the effect of effective gravity $(\zeta_2 - 1)\ \boldsymbol{R}_g$ by varying the value of ζ_2 for a fixed value of $\boldsymbol{R}_g$.

We are going to divide the tables into two groups according to the value of capillary number J^*. The first group is for J^*=1, corresponding to weak capillary effects typical for our experiments. The results for J^*=1 are summarized in Tables 2–7. The second group is for J^*=2000, corresponding to strong capillary effects. This case is of interest for low viscosity cores for which the capillary number is large.

There is an important difference in the lower branch of the neutral curves when J^*=1 and J^*=2000 that is evident from a comparison of Figures 3 and 4. When J^*=1, the maximum value of $\boldsymbol{R}(\alpha)$ on the lower branch of the neutral curve occurs near α=0. When J^*=2000, the maximum value of $\boldsymbol{R}(\alpha)$ on the lower branch of the neutral curve occurs at a finite value near 0.6.

The lower branch of the neutral curve for J^*=1 has a region in the neighborhood of $(\alpha,\boldsymbol{R}(\alpha))=(0,\boldsymbol{R}(0))$ in which the analysis of long waves may be relevant. In the case of very long waves it may be impossible to obtain an amplitude equation of the Ginzburg-Landau type. The critical wave number at the nose of the neutral curve tends to zero so that the wave you are supposed to modulate is already hugely long. (We are indebted to A. Frenkel for this remark. He noted that to have a Ginzburg-Landau equation, the sideband width $\Delta\alpha$ ought to be small relative to the wave number α on which it is centered.) There are other types of lubrication-type approximations describing waves of slow variation rather than the slowly varying envelope of modulated waves as in the Ginzburg-Landau equation.

In fact Frenkel *et al* [1987) derived an amplitude equation of the Kuramoto-Sivashinsky type from an analysis of long waves. Frenkel [1988] found a condition in which the motions of the core and of the annulus can be coupled in the Kuramoto-Sivashinsky system leading to an additional linear term which disperses waves. A clear exposition of this work based on a systematic expansion in powers of a small parameter together with numerical solutions of Kuramoto-Sivashinsky-Frenkel equation has been given by Papageorgiou, Maldarelli and Rumschitzki [1990].

We are concerned that the neglect of inertia p**u**• **u** in nonlinear theories for long waves can lead to large errors in the case when wave number $\tilde{\alpha}$ of maximum is bounded strictly away from zero. A monochromatic linear wave proportional to $\exp[i\alpha(x-ct)]$ undergoes repeated multiplication leading to rapid growth of higher harmonics, which eventually may be strapped by dispersion and dissipation. This type of effect is removed from nonlinear long wave theories by assumption.

6. Small capillary numbers

When the capillary parameter J^* is small, the maximum growth rate for capillary instability cannot be greatly different than the value α=0 which maximizes $\boldsymbol{R}(\alpha)$ on the lower branch of the neutral curve shown in Figure 3. We have selected the value J^*=1 to represent weak surface tension. Our results are contained in the coefficients displayed in Tables 2–7 and in Figure 3. Tables 2 through 5 are for a=1.25, corresponding to water to oil volume ratio $V_w/V_o=a^2-1=0.5025$. The coefficients of the Ginzburg-Landau equations in the case ζ_2=1 are exhibited in Table 2. We know from energy analysis of the linear problem that PCAF in the region above the upper branch of the neutral curve is unstable to interfacial friction (see HJ, BCJ). We expect that wavy core flow will arise from this instability. The entries in Table 2 show that the bifurcating waves are supercritical when $m\varepsilon 0.85$ and subcritical when $m \leq 0.8$. Stable small amplitude shear waves are expected in the supercritical case and something else far from PCAF, perhaps large waves, in the subcritical case. Small viscosity differences can lead to stable wavy flow at the interface.

Turning next to the lower branch of the neutral curve which is prey to modified capillary instability, we note that the bifurcation is supercritical when $m\varepsilon 0.7$ and subcritical when $m \leq 0.5$. We expect to see small amplitude capillary waves in the supercritical case. This can be interpreted to mean that linear capillary instability is nonlinearly shear stabilized when $m \geq 0.7$. The flows which bifurcate subcritically ($m \leq 0.5$) should be far from PCAF. These more viscous cores probably break into slugs or bubbles far from PCAF connected by thin threads.

The results in Table 2 indicate that when the densities of the fluids are matched, a large viscosity difference leads to subcritical bifurcation while a small viscosity difference can result in supercritical bifurcation.

We study next the effects of changing the density of the lubricant for the same fixed $a=1.25$. If the lubricant is heavier than the core, say water and oil, $\zeta_2>1$. Can we modify the nature of the bifurcations, i.e., change the dynamics of lubrication, by varying ζ_2? Table 3 shows that for fluids with $m=0.7$, the bifurcation of the upper branch for $\zeta_2=1$ can be changed from sub to supercritical by increasing the density ratio to $\zeta_2=1.2$. We can stabilize small amplitude bifurcating waves driven by interfacial friction by increasing the density of the lubricant. The change of density does not destabilize the supercritical bifurcating solution on the lower branch. A result of the same general nature is shown in Table 4 and 5 for fluids with larger viscosity difference, $m=0.5$ and m=0.2, the only difference is that the wavy solutions below the lower branch of the neutral curve are all stable when m=0.7 and all unstable when m=0.5, m=0.2. These results suggest that for given surface tension, the bifurcation of the upper branch is sensitive to changes in density ratio, while the lower branch is only sensitive to the viscosity ratio.

In Table 6 we look at fluids with m=0.5 for the effects of varying ζ_2. The difference here is that there is much less water: a=1.1 and water to oil volume ratio $V_w/V_o=a^2-1=0.21$, there is five times more oil than water. This is a thin lubricating layer. The bifurcation of the periodic solution from the lower branch of the neutral curve is subcritical for ζ_2 between 0.5 and 1.6, as in the case a=1.25. The bifurcation of a periodic solution from the upper branch of the neutral curve can be changed from sub to supercritical by increasing the density of the lubricant. However, the transition density ratio for a=1.1 occurs between ζ_2=1.4 and ζ_2=1.5, a larger transition ratio than for a=1.25, m=0.5, which is between ζ_2=1.0 and ζ_2=1.2 (see Table 4). Suppose $1.2\le\zeta_2\le1.5$, m=0.5. If the lubricating layer is relatively thick, a=1.25, then the upper branch will bifurcate supercritically. However, if the lubricating layer is thin, say a=1.1, then the upper branch will bifurcate subcritically. This indicates some kind of nonlinear breakdown of the "thin layer effect" : in order to achieve a linearly stable core-annular flow, we need to have a thin lubricating layer. However, if the layer is too thin, the bifurcation of the upper branch will become subcritical. The exact physical implication of this subcritical bifurcation is not clear to us. However, in our experiments (BCJ), in a region where the superficial oil velocity is large and superficial water velocity is relatively small, corresponding to very small values of a, oil sticks to the wall.

The role that density differences play in the stability of core-annular flow is interesting. The effects of the density difference on the neutral stability curves of core-annular flow were studied in CBJ. The calculations of CBJ as well as the weakly nonlinear ones presented here show that the upper branches of the neutral curves are more sensitive to the changes of density ratio ζ_2 than the lower branch. For the upper branches, the Reynolds number is large and the effect of the effective gravity ûζô$\boldsymbol{R}_g$ is negligible. The only place that the density ratio ζ_2 enters the equations is through the jump in the perturbation pressure in the normal stress balance equation at the interface, (2.8). Relatively small changes in ζ_2 can cause a large perturbation of the pressure jump when the Reynolds number is large. This changes the stability of the upper branch considerably. When the Reynolds number is large, the pressure jump is basically equal to the jump in the inertia of the fluids which is large in this case. On the other hand, when the Reynolds number is small, density stratification manifests itself mainly through the effective gravity term ûζô$\boldsymbol{R}_g$ in the basic flow. This term is not too large for the small pipes we have considered and the change of the lower branch is relatively small for the moderate changes in ζ_2.

We have also computed a few cases of up-flow, for a=1.1, with $\boldsymbol{R}_g= -0.5$. These results are listed in Table 7. After comparing this table with relevant entries in the previous tables, we found that there are only slight changes in the values of coefficients and the type of bifurcations remain the same for both upper branch and lower branch. This is expected for the case of not too large $|\boldsymbol{R}_g|$ and *fixed value of* a. Effective gravity ûζô$\boldsymbol{R}_g$ has little effect on the stability, and particularly for the upper branch, there is almost no difference between up- and down-flows for both the neutral curves and bifurcation. For the lower branch, there is a slight shift of the neutral curves between the up- and down-flows, but the type of bifurcation is not affected. We do see big differences in up and down flow in experiment but this difference is due to the accumulation of oil in down flow and its depletion in up flow due to gravity, giving *different* values of a to up and down flows.

7. Large capillary numbers

As the surface tension parameter J^* is increased, the wave number corresponding to the most unstable mode of the lower branch tends to the capillary limit α=0.69. CBJ showed that linearly stable CAF is possible only when ζ_2 is large enough in case of large J^*. The neutral curve for a=1.3, m=0.5, J^*=2000, $\boldsymbol{R}_g$=0.5 and ζ_2=1.2 is shown in Figure 4. For this set of parameters a, m, J^*, $\boldsymbol{R}_g$, linearly stable CAF is only possible when $\zeta_2 \geq 1.2$. A heavy lubricant will stabilize capillary instability, the critical Reynolds number below which the flow is unstable to capillarity is decreased as ζ_2 is increased. However, increasing the density of the lubricant does not change the nature of the bifurcation from the capillary branch, which is always subcritical (Table 8) when a=1.3, m=0.5 and J^*=2000. The subcritical bifurcation here may lead to the capillary break-up of the oil core and the formation of oil slugs and bubbles. For the set of parameters in Table 8, the bifurcations from the upper branch are always supercritical, leading to the finite travelling waves at the interface.

The second example of large capillary number is exhibited in Table 9 for the parameters a=1.1, m=0.5, J^*=2000 and $\boldsymbol{R}_g$ =0.5. Linearly stable CAF is possible for a much wider range of ζ_2 because of the stabilization effect of thin lubricating layer. However, the bifurcation of the capillary branch remains subcritical for all the density ratios considered. These results for J^*=2000 and those for J^*=1 show that the bifurcation of the lower branch is insensitive to the changes in density difference and water fraction. For the upper branch, from Table 9 (a), we see that there is a range of density ratios within which the bifurcation of the upper branch is subcritical. Outside this range, i.e. for small and large density ratios, the bifurcation becomes supercritical. This result also holds when J^*=1, as shown in Section 6, but the subcritical range is different. The breakdown of “thin layer effect” for J*=1 also occurs for J*=2000.

How do changes in the viscosity ratio m change the bifurcations of core-annular flows when J^* is large? Table 10 gives results for a=1.1, m=0.9, J^*=2000, $\boldsymbol{R}_g$=0.5 and down-flow. The remarkable difference between m=0.9 and m=0.5 is that now for m=0.9, the lower capillary branch bifurcates supercritically for all the density ratios considered, even for $\zeta_2<1$. This means that finite amplitude capillary waves are saturated nonlinearly by the action of a small viscosity difference, m near one. This nonlinear saturation by small viscosity difference also occurs when J^* is small (see Section 6). For the upper branch, there is still a range of density ratios within which the bifurcation is subcritical, as in the case of weak capillarity.

8. Modulational instability

The Ginzburg-Landau equation can be used to analyze the stability of finite amplitude solutions which bifurcate from perfect core-annular flow. It is well-known that the Ginzburg-Landau equation (3.17) admits plane travelling wave solution

$$A(\xi, \tau) = A_0 \exp\left[\, i\,(\beta_0\, \xi - \gamma_0\, \tau)\right] , \qquad (8.1)_a$$

where A_0, β_0, γ_0 are all real constants and given by

$$|A_0|^2 = \mathrm{sgn}(l_r)\left[\, \mathrm{sgn}(d_{1r})\, \mathrm{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) - \beta_0^2 \,\right] \geq 0 , \qquad (8.1)_b$$

$$\gamma_0 = \beta_0^2\, C_d + |A_0|^2\, C_n . \qquad (8.1)_c$$

The travelling wave solution (8.1) has typical nonlinear properties; the amplitude depends on the wave number β_0 and the frequency depends on the wave amplitude.

The stability of the equilibrium solution (8.1) was studied by Newell [1974], Stuart and DiPrima [1978] and Moon [1982]. Their analysis provides an unified treatment of the well-known Eckhaus instability and Benjamin-Feir instability. Their results can be efficiently expressed in terms of parameters appearing in (3.17). Assume a perturbation of (8.1) in the form

$$A(\xi, \tau) = (1 + a(\xi, \tau)\,)\, A_0 \exp\left[\, i\,(\beta_0\, \xi - \gamma_0\, \tau)\right]$$

where the perturbation function $a(\xi, \tau)$ is given by

$$a(\xi, \tau) = B_1 \exp\left[\, p\, \tau + i\, q\, \xi\,\right] + B_2^* \exp\left[\, p^*\, \tau - i\, q\, \xi\,\right] ,$$

where star * stands for complex conjugate, q is a real wave number and B_1, B_2 , p are all complex numbers. The eigenvalue p is given by

$$p = \frac{1}{2}\left\{- (C_+ + C_-^*\,) \pm \sqrt{(C_+ + C_-^*)^2 - 4\,(C_+\, C_-^* - |C_0|^2)}\;\right\} \tag{8.2}$$

where

$$C_0 = [\mathrm{sgn}(l_r) + i\, C_n]\; |A_0|^2 ,$$

$$C_+ = -\mathrm{sgn}(d_{1r})\, \mathrm{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) - i\gamma_0 + \beta_0^2\,(1 + iC_d) + 2\,|A_0|^2\,[\mathrm{sgn}(l_r) + iC_n]$$
$$+\, 2\,\beta_0\,(1 + iC_d\,)\, q + (1 + iC_d\,)\, q^2 ,$$

$$C_- = -\mathrm{sgn}(d_{1r})\, \mathrm{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) - i\,\gamma_0 + \beta_0^2\,(1 + iC_d) + 2\,|A_0|^2\,[\mathrm{sgn}(l_r) + iC_n] +$$
$$-2\,\beta_0\,(1 + i\,C_d\,)\, q + (1 + i\,C_d\,)\, q^2 .$$

The travelling wave solution (8.1) is linearly stable if the real part of the eigenvalue p is negative Re(p)<0 and is unstable if Re(p)>0. The most general situation can be studied by evaluating p in the parameter space $\{C_d, C_n, \beta_0, q\}$, following procedures used by Stuart and DiPrima [1978]. Of particular interest is Newell's stability criterion for the spatially uniform plane wave solution. The uniform solution is given by

$$\beta_0 = 0 ,$$

$$|A_0|^2 = \text{sgn}(l_r)\ \text{sgn}(d_{1r})\ \text{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) = 1\ , \tag{8.3}$$

$$\gamma_0 = C_n\ .$$

The stability criterion can be expressed as

$$1 + C_d\, C_n > 0, \quad \text{for sgn}(l_r) = 1,\ \text{sgn}(d_{1r})\ \text{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) = 1\ ,$$

$$-1 + C_d\, C_n > 0, \quad \text{for sgn}(l_r) = -1,\ \text{sgn}(d_{1r})\ \text{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) = -1\ .$$

This stability criterion can be easily checked for the cases we computed for core-annular flow and we will not discuss these issues further. More general results on the linear modulational instabilities are studied in the C_d–C_n plane by Moon [1982]. In general, the modulation instabilities lead to a contraction of the wave numbers for which the bifurcated solutions can be stable by the removal of "side bands". We have not been able to identify modulated waves in experiments on core-annular flows.

9. Soliton-like solutions of the Ginzburg-Landau equation

When the real part of the Landau constant l_r is positive and other parameters are in appropriate range, there are soliton-like solutions of the Ginzburg-Landau equation which have been discussed by Hocking and Stewartson [1972]. These solutions have been called "breathers" by Holmes [1986]. Consider equation (3.17). A time periodic, spatially decaying solution of the Ginzburg-Landau equation can be expressed as

$$A(\xi, \tau) = \lambda\, L \exp\,[\, i\, \omega\, \tau\,]\ [\, \text{sech}(\lambda\xi)\,]^{1 + i\, M}\ , \tag{9.1}$$

where, λ, L, ω, M are all real constants. The formulas determining these constants are given by (assuming that C_d–C_n≠0)

$$M_{1,2} = \frac{3(1 + C_d\, C_n) \pm \sqrt{9(1 + C_d\, C_n)^2 + 8(C_d - C_n)^2}}{2(C_d - C_n)}$$

$$L^2 = M^2 + 3\, C_d\, M - 2\,, \tag{9.2}$$

$$\lambda^2 = \frac{\mathrm{sgn}(d_{1r})\, \mathrm{sgn}(\boldsymbol{R} - \boldsymbol{R}_c)}{M^2 + 2\, C_d\, M - 1}\,,$$

$$\omega = -\lambda^2 \left(C_d\, M^2 - 2\, M - C_d\right).$$

We have two possible M values and they are both real. To ensure that L and λ are both real, we require that

$$L^2 > 0 \text{ and } \lambda^2 > 0. \tag{9.3}$$

It can be easily shown that (9.3) is not possible when $\mathrm{sgn}(d_{1r})\, \mathrm{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) = -1$. This means that the soliton-like solution does not exist when PCAF is stable. In the unstable case $\mathrm{sgn}(d_{1r})\, \mathrm{sgn}(\boldsymbol{R} - \boldsymbol{R}_c) = +1$ and (9.3) is satisfied if $M_{1,2}$ satisfies

$$M_{1,2} > M_{L2} = \frac{-3C_d + \sqrt{9\, C_d^2 + 8}}{2}\,, \tag{9.4}$$

or

$$M_{1,2} < M_{L1} = \frac{-3C_d - \sqrt{9\, C_d^2 + 8}}{2}\,.$$

When (9.4) is satisfied soliton-like solutions of the Ginzburg-Landau equation (3.17) exist. If $C_d = C_n$, then $M=0$, $L^2 = -2 < 0$ and no soliton-like solution exists. We have not seen any soliton-like solutions which could arise from modulation of monochromatic waves in experiments on core-annular flows.

10. Experiments

In the somewhat restricted situation of Hopf bifurcation of strictly periodic waves at a simple eigenvalue, we could say that the supercritical waves are stable whilst the subcritical waves are unstable. To compare bifurcation analysis with experiments, we must first identify a flow with a critical Reynolds number $\boldsymbol{R}_c$. There are then upper and lower critical values, $\boldsymbol{R}_{cU}$ and $\boldsymbol{R}_{cL}$ (see Figure 5). If the operating $\boldsymbol{R}$ is in a region of instability of PCAF near criticality, then supercritical bifurcating solutions are in this same region of instability of PCAF. Under restrictive hypotheses, the supercritical bifurcating solution is stable. We say that bifurcation theory is consistent with experiments when the observed supercritical solution is just a small perturbation of PCAF (we would need to compare details of the bifurcated solution with experiments to test the theory, and we have not done this). On the other hand, if the bifurcation is subcritical when $\boldsymbol{R}$ is in the unstable region for PCAF (shaded region in Figure 5), then the bifurcating solution is unstable when its amplitude is small, but may recover stability for large amplitudes. In this case the observed flows would be far from PCAF. If $\boldsymbol{R}$ is in a region of stability of PCAF, $\boldsymbol{R}_{cL}<\boldsymbol{R}<\boldsymbol{R}_{cU}$, and both bifurcations are supercritical, we might expect to see stable PCAF, stable both to small and finite disturbances. If, on the other hand, one or both bifurcations are subcritical and $\boldsymbol{R}_{cL}<\boldsymbol{R}<\boldsymbol{R}_{cU}$, then the conclusion of bifurcation theory is ambiguous. Without ambiguity, we may conclude that if a flow different than stable PCAF is seen in the linear stable range, then one or both bifurcations should be subcritical with large deviations (with an ambiguous “large”) from PCAF.

(Insert Figure 5 near here)

In Table 11, we have compared bifurcation theory with experiments. The comparisons exhibited in rows one and two can be said to show agreements between theory and experiments. Less can be concluded from rows three through seven, but in all cases, there is no obvious inconsistency between experiments and bifurcation theory.

Table 11 Comparison of bifurcation theory and experiments. Among many experiments, there are only a small number to which the Ginzburg-Landau theory might apply. They are listed here. sgn(l_r)= +1 for supercritical, sgn(l_r)= –1 for subcritical bifurcations. Subscripts U and L are referred to the upper and lower branch respectively.

Experiment	Operating $\boldsymbol{R}$	sgn $(l_r)_U$	sgn $(l_r)_L$	Observations and comments
Charles et al #2 (1961)	$\boldsymbol{R}>\boldsymbol{R}_{cU}$	+1		The bifurcation is supercritical and the flow is near to PCAF.
CBJ (1990) Free fall	$\boldsymbol{R}_{cL}<\boldsymbol{R}<\boldsymbol{R}_{cU}$	+1	+1	PCAF is observed. There is agreement between experiments and linear and nonlinear theory.
BCJ (1990) E2, up-flow	$\boldsymbol{R}>\boldsymbol{R}_{cU}$	–1		Bamboo waves, a structure far from PCAF.
BCJ (1990) F1, up-flow	$\boldsymbol{R}>\boldsymbol{R}_{cU}$	–1		Oil sticks to the pipe wall.
BCJ (1990) #2, down-flow	$\boldsymbol{R}>\boldsymbol{R}_{cU}$	–1	–1	Intermittent corkscrew waves are observed.
BCJ (1990) #3, down-flow	$\boldsymbol{R}_{cL}<\boldsymbol{R}<\boldsymbol{R}_{cU}$	–1	–1	Intermittent corkscrew waves are observed.
BCJ (1990) #4, down-flow	$\boldsymbol{R}\gg\boldsymbol{R}_{cU}$	+1		Disturbed bamboo waves, perhaps not too far from PCAF, are observed.

We wish to draw attention to a possible interpretation of bamboo waves as a structure far from PCAF. When the waves are large, as in Figure 1, this is obvious. In one interpretation, bamboo waves arise from shear stabilization of capillary instability which in pure form leads to spheres of oil, far from PCAF. There is no stable capillary figure close to a cylinder.

We turn next to a discussion of theory and experiments in which the entries in Table 11 are explained in a wider context.

We first consider the experiment labeled as 2 in Figure 3 of the paper by Charles *et al* [1961] (reproduced as Figure 1 in PCJ and Figure 6 in HJ), one sees a slightly perturbed PCAF which is labeled as “oil in water concentric”. The neutral curve for this case is shown as Figure 13 in PCJ. The operating condition, a=1.21, J^*=2102, m=0.0532, $\boldsymbol{R}_1$=138.6, is just above the nose (α_c, $\boldsymbol{R}_c$)=(2.24, 138.2) of the upper neutral curve, hence PCAF is linerarly unstable. For this case the coefficients of the amplitude equation (3.17) are

$C_d = -1.3967$,

$C_n = -0.02393$,

$\text{sign}(l_r) = +1$.

Since l_r>0, the bifurcation is supercritical and only a small perturbation of PCAF is expected and is realized.

The remaining ten cases corresponding to Figure 3 of Charles *et al* [1961], are either always unstable or with operating conditions (operating Reynolds numbers) far away from the critical conditions (see PCJ and Chen [1990] for the neutral curves). Thus bifurcation analysis cannot be applied.

The analysis of Section 6, 7 may be well applied to situations when the two fluids involved have similar physical properties. CBJ designed an experiment to test the validity of linear theory. The fluids and the size of pipe were chosen with the guidance of the linear theory. They have sucessfully realized globally stable perfect core-annular flow in the free-fall apparatus and shown perfect agreement between linear theory and their experiment. The neutral curve for this experiment is given in Figure 20 of CBJ. The experiment falls in the linearly stable region and is not too far away from the critical condition of the lower branch. After converting the parameters to the ones used in the present paper, we have, for the experiment,

$$a = 1.86, \quad m = 0.33, \quad \zeta_2 = 1.4, \quad J^* = 2.26, \quad \boldsymbol{R}_g = 21.31, \quad \boldsymbol{R} = 8.22 .$$

The bifurcation near the critical state of the lower branch, $(\alpha_c, \boldsymbol{R}_c) = (0.04, 6.31)$, is supercritical:

$$C_d = 89.65,$$

$$C_n = 21.31,$$

$$\text{sign}(l_r) = +1.$$

The upper branch, although far away with $(\alpha_c, \boldsymbol{R}_c) = (0.5, 153.0)$, also bifurcates supercritically. Perfect core-annular flow was observed in the experiment and it agrees with both the linear and nonlinear theory.

The difficulties one encounters when applying the above bifurcation theory to the practical situations of lubricated pipelining lie in the fact that, when the viscosity ratio of water to oil, $m=\mu_2/\mu_1$, is very small, say m of order 10^{-5} which is typical for crude oil and water, PCAF is always linearly unstable and thus there is no critical states for bifurcation analysis. This restriction severely limited the parameter ranges that we could apply such analysis.

The experiments of BCJ revealed many interesting features of nonlinear waves in lubricated pipelining. For these experiments, we have

$$m = 1.66 \times 10^{-3}, \quad \zeta_2 = 1.0994, \quad J^* = 0.1019, \quad \boldsymbol{R}_g = 2.4 .$$

For this value of m, linearly stable CAF can be obtained only when a is very small (say $a<1.15$). The flow charts in BCJ show how different flow regime changes with respect to the superficial velocities of water and oil, V_w, V_o. In the up-flow chart, there is a region in the V_w, V_o plane called “wavy CAF”, which corresponds to the bamboo waves observed (see Figure 1). For the points marked #1 through #9, and D2, E3 in the bamboo wave regime, the upper and lower neutral curves are connected and they are linearly unstable at all Reynolds numbers. This is because of the large values of a for these points. For point E2, a=1.12, $\boldsymbol{R}_1$=1.2283. The upper and lower branches of the neutral curves are separated. The experimental line $\boldsymbol{R}_1$=1.2283 is cutting through the upper branch, linearly unstable and is not too far away from the nose of the neutral curve (α_c, $\boldsymbol{R}_c$)=(2.41, 0.501). The bifurcation at this point is found to be subcritical with

$C_d = 0.9623$,

$C_n = -0.9657$,

$\mathrm{sign}(l_r) = -1$.

This subcritical bifurcation indicates that in order to achieve the experimentally observed stable bamboo waves at E2, higher order theory is needed.

The examples discussed above show that the current lowest order bifurcation theory is hopeless for the prediction of bamboo waves observed by BCJ, either because of the corresponding linear theory predicts linear instability for all Reynolds numbers, or because of the subcriticality of the bifurcation. For the later case, we may supplement a higher order theory. Nevertheless, the above examples suggest that bamboo waves are flows far from PCAF and fully numerical simulations may be required for their characterizations.

The flow regime called “oil sticking on the wall” in the up-flow chart is a region where V_w is small and V_o is large. In this region, there is little water in the pipe and thus the value of a is small. We found that for these small values of a, the upper and lower branch of the neutral curves are separated, due to the strong stabilization of thin lubricating layer ("thin layer effect"). The experimental lines $\boldsymbol{R}_1$=$\boldsymbol{R}_E$ cut through the upper branch. However, the bifurcations of the upper branch are found to be all subcritical. This is the nonlinear break-down of the "thin layer effect" discussed in Section 6. An example in this region is the point F1, where a=1.03, $\boldsymbol{R}_1$=2.2838. The nose of the upper branch is (α_c, $\boldsymbol{R}_c$)=(12.0, 2.04) and we have a subcritical bifurcation with

C_d = –0.1479,

C_n = –0.7556,

sign(l_r) = –1.

Whether these subcritical bifurcations correlate to the losses of lubrication observed in the experiments remains to be resolved, either by higher order theory or numerical solution. Obviously the phenomena observed in the experiments are very nonlinear.

Turning now to the down-flow chart of BCJ. The points #2, #3 fall in a region called "disturbed CAF", corresponding to "corkscrew" waves as in Figure 2. The linear theory predicts that #2 , #3 are linearly stable to infinitesimal disturbances, axisymmetric and nonaxisymmetric. The bifurcations of the upper and lower branches are all subcritical. It is obvious that these "corkscrew" waves are nonaxisymmetric and due to finite nonaxisymmetric disturbances which are not considered in this paper.

The point marked #4 in the down-flow chart falls in a region called "Disturbed CAF", which corresponds to axisymmetric, very short stem bamboo waves. For point #4, a=1.09, $\boldsymbol{R}_1$=9.59, the upper and lower branch are separated, and $\boldsymbol{R}_1$=9.59 cuts through the upper branch. In this case, the bifurcation at the nose of upper branch, (α_c, $\boldsymbol{R}_c$)=(3.6, 1.36) is supercritical with

$C_d = 0.6590,$

$C_n = -0.5718,$

$\text{sign}(\lambda_r) = +1.$

However, the experimental point $\boldsymbol{R}_1$=9.59 is far away from the nose and the information on the bifurcation at the nose may be not relevant to the observed equilibrium waves.

A summary of the results testing bifurcation theory are in Table 11. It is evident from the above discussions that the usefulness of the bifurcation analysis is very restricted. For the situations of practical interest, $m \ll 1$, the bifurcation analysis is either not applicable or failed to provide useful information relevant to the experimentally observed phenomena. On the other hand, in all these cases we have obtained useful information from the study of the linear theory of stability. It seems to us that, unlike linear theory, weakly nonlinear theory is valid only in a too narrowly defined set of conditions to be of much use in our problem. Perhaps direct numerical approaches have more to offer.

11. Summary and discussion

(1) There are regions of parameter space in which PCAF is possible. For these parameters we can write two Ginzburg-Landau equations, one near the minimum point of the upper branch and another near the maximum point of the lower branch. At the upper branch, PCAF loses stability to waves generated by interfacial friction. At the lower branch PCAF loses stability to capillary waves.

(2) There are yet more regions of parameter space in which PCAF is not possible. For these cases, bifurcation analysis is not applicable.

(3) The singular value decomposition is a useful numerical method for computing all the coefficients of the Ginzburg-Landau equation.

(4) The stability of wavy flows near the upper branch of the neutral curve can be controlled by varying the density ratio ζ_2. For example, when the other parameters are fixed we can choose a best $\zeta_2=\hat{\zeta}_2$ to maximize the minimum critical value $\boldsymbol{R}_U(\zeta_2)$ of the linear theory on the upper branch of the neutral curve (CBJ). The bifurcation of waves from the upper branch $\boldsymbol{R}_U(\zeta_2)$ of the neutral curve will be supercritical if ζ_2 is large enough. For smaller values of ζ_2 the bifurcation is subcritical.

(5) The lower branch is less sensitive to changes in ζ_2 than the upper branch. The critical Reynolds number above which down-flow is linearly stable decreases with increasing ζ_2. In up-flow, smaller values of ζ_2 lead to larger regions of linear stability (CBJ). The bifurcation of the lower branch is controlled by the viscosity difference and surface tension. Changes of ζ_2 do not change the directions of bifurcation of the lower branch.

(6) The viscosity ratio, m, plays a key role in determining both the linear and nonlinear stability of core-annular flows. When m is small, linear stable PCAF cannot be achieved (PCJ, HLJ). Stable PCAF can be achieved only when the viscosity difference 1–m is small.

(7) Other things being equal, the linear theory tells us that we will get larger intervals of the Reynolds number in which PCAF is stable if the lubricating layer is thin, a∅1. We can say that this stability will be realized practically even when PCAF is unstable, if the bifurcating solutions of small amplitude are stable. This means that a robust form of lubricated pipelining with thin lubricating films is expected when the bifurcations are supercritical, but nonlinear failures may occur when the bifurcations are subcritical. We have in fact found that subcritical bifurcations for thin film solutions which bifurcate from the upper branch and these solutions lie in a region of thin film parameter space in which a failure of lubrication does occur (see BCJ). On the other hand, there are cases for which increasing the thickness of the lubricating layer can change subcritical to supercritical bifurcation. We may verify this by comparing the entries in Table 4(a) with Table 6(a) and the entries in Table 8(a) with Table 9(a) for ζ_2=1.2 and ζ_2=1.4.

(8) When the flows are slight perturbations of PCAF, experiments agree with both linear and nonlinear theories. One example is the experiment #2 of Charles *et al* [1961] where nearly perfect core-annular flow is observed. The operating Reynolds number of the experiment is slightly above the nose of the upper branch of the neutral curve where the bifurcation is found to be supercritical. An even more convincing example is the free-fall experiment of CBJ in which PCAF is predicted and observed.

(9) The finite amplitude bamboo waves observed by BCJ are evidently too far from PCAF to be described by our Ginzburg-Landau equation. In most cases encountered in the experiments of BCJ, the bifurcation theory can not be applied because the corresponding PCAF is linearly unstable at all Reynolds numbers. In other cases, the experimental Reynolds numbers cut through the upper branch of the neutral curves, but the bifurcations near the nose of the upper branch are subcritical. These results suggest that bamboo waves and other flows far from PCAF perhaps may be best treated by direct numerical methods.

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