

# Stability of Liquid-vapor Flow Down an Inclined Channel With Phase Change

*by*

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## ***Abstract***

We study the stability of a two-phase flow between heated inclined plates. The temperature of the bottom plate is held below the vaporization temperature and the top plate is hotter than the vaporization temperature. A water film is on the cold wall and a vapor film on the hot wall. The flow is driven down the walls by gravity. We find a basic flow with steady distributions of temperature but no phase change. The linear stability of this basic state is studied in the frame of incompressible fluid dynamics, without convection, but allowing for phase change at the interface between the two phases. An ambiguity in the choice of the conditions to be required of the temperature at the phase change boundary is identified and discussed. Three different instabilities were found, one due to the Reynolds stress when Reynolds number is large, one due to interfacial friction which is associated with the viscosity difference of the two phases, and another due to phase changes at the interface.

## 1. ***Introduction***

The problem we are considering is related to the problem of stability of laminar film condensation (Nusselt's solution) on an inclined cool plate (see Unsal & Thomas [1978] for a fairly thorough review of this literature) and the problem of a falling film of liquid down an inclined plate which was decisively analyzed by Yih [1963]. Yih's problem was generalized to two-phase Poiseuille flow between parallel plates (Blennerhassett [1980], Renardy [1987]).

It is necessary that we explain how our problem which is being considered here differs from those mentioned in the foregoing paragraph and allied studies. Problems of the Yih, Renardy type have fully developed basic flows but no phase change. In the present study we allow phase changes but cannot accommodate applied pressure gradients; Poiseuille flow cannot be treated rigorously in our frame. Our problem differs from the ones on laminar film condensation because the second wall and the vapor are active and the basic flow is fully developed and not of boundary layer type. The second wall allows the system to attain a steady fully developed temperature profile in the basic state which cannot exist in a semi-infinite region. The fully-developed basic flow which we study is particularly convenient for a computational study of stability since approximations are not required, and the flow could conceivably be attained in experiments. We are not in the frame of laminar film condensation because phase changes do not occur in the basic flow.

We could not obtain a steady fully developed flow for core-annular flow because there is a pressure jump across the cylindrical interface due to interfacial tension which interdicts the existence of a common saturation temperature there. For a similar reason, there are no developed steady flows with phase change which are driven by pressure gradients. On the other hand, developed steady flows in free fall between heated inclined plates can be considered if the change of density with temperature is neglected. This neglecting is usually done in the film condensation problems, and we follow this path, assuming in all that follows

that the densities of the water and vapor are fixed constants independent of pressure and temperature.

## 2. **Governing equations**

We are going to assume that the density of the vapor and the density of the water are constants independent of variation of temperature or pressure across the channel. This means that the pressure will be a dynamical variable uncoupled from thermodynamics and that convective currents set up by the working of gravity on thermally induced variations of temperature will be ignored. It is conceivable that thermally induced buoyancy could produce some important effects, but Spindler [1982] did allow for these effects and they were not important.

The velocities of both phases satisfy the incompressible Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1)$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}. \quad (2.2)$$

In the energy equation we shall neglect the dissipation term:

$$\rho C_p \frac{dT}{dt} = k \nabla^2 T. \quad (2.3)$$

At the interface  $l(x,t) = y - (R - \delta(x,t)) = 0$  we have mass, momentum and energy balances:

$$\text{Mass:} \quad -\dot{m}(\mathbf{x},t) = \rho_1 (\mathbf{u}_1 - \mathbf{u}_\Sigma) \cdot \mathbf{n}_{12} = \rho_2 (\mathbf{u}_2 - \mathbf{u}_\Sigma) \cdot \mathbf{n}_{12}. \quad (2.4)$$

$$\text{Momentum:} \quad \dot{m} \hat{\mathbf{u}} \hat{\mathbf{u}} - \hat{\mathbf{u}} p \hat{\mathbf{n}}_{12} + 2 \hat{\mathbf{u}} \mu \mathbf{D}[\mathbf{u}] \cdot \mathbf{n}_{12} \hat{\mathbf{u}} = 2H\sigma \mathbf{n}_{12}, \quad (2.5)$$

$$\text{Energy:} \quad -\hat{\mathbf{u}} k \nabla T \hat{\mathbf{u}} \cdot \mathbf{n}_{12} = 2 \hat{\mathbf{u}} \mu (\mathbf{u} - \mathbf{u}_\Sigma) \cdot \mathbf{D}[\mathbf{u}] \cdot \mathbf{n}_{12} \hat{\mathbf{u}} + \dot{m} \hat{\mathbf{u}} h + \frac{1}{2} |\mathbf{u} - \mathbf{u}_\Sigma|^2 \hat{\mathbf{u}}, \quad (2.6)$$

with kinematic conditions:

$$\mathbf{u}_\Sigma \cdot \mathbf{n}_{12} = -\frac{\partial l / \partial t}{|\mathbf{t}|}, \quad \hat{\mathbf{u}} \hat{\mathbf{u}} \cdot \mathbf{t} = 0, \quad (2.7)$$

where  $\mathbf{n}_{12}$  is the normal of the interface directed from vapor to water, and  $\mathbf{t}$  is the tangent. The saturation temperature at equilibrium is determined as given by the Clapeyron equation as a function of pressure

$$T = T_s = \tilde{T}(p) . \quad (2.8)$$

By equilibrium we mean the condition in which there is no evaporation or condensation,  $\dot{m} = 0$ . This means that the pressure across a phase change boundary must be continuous. Obviously, if the pressure is different on the two sides of the interface,  $T_{s1} = \tilde{T}(p_{s1}) \neq \tilde{T}(p_{s2}) = T_{s2}$ , the temperature cannot be the same in the water and water vapor. There will be a temperature discontinuity

$$T_1 = \tilde{T}(p_v) , \quad T_2 = \tilde{T}(p_w) . \quad (2.9)$$

This shows that thermodynamic equilibrium means that the water and its vapor are not in thermal equilibrium. The existence of a temperature discontinuity evidently cannot be eliminated by rigorous application of first principles. Schrage [1953] says "... There is no reason why the temperature of the gas phase should necessarily be the same as that of the liquid or solid surface in all cases." Indeed, classic kinetic theory calculations (Pao[1971], Sone and Onishi[1978], Aoki and Cercignani[1983], Onishi [1984] and Cercignani, Fiszdon and Frezzotti [1985]) indicate that for monatomic vapor large temperature jumps exist at interfaces. Shankar and Deshpande [1990] have measured the temperature distribution in the vapor between an evaporating liquid surface and a cooler condensing surface in water, Freon 113 and mercury. The temperature profiles obtained in mercury showed large jumps at the interface as large as almost 50% of the applied temperature difference.

The usual approximation made in the study of phase changes of liquid and vapor is to require thermal equilibrium

$$\hat{u}T\hat{o} = T_1 - T_2 = 0, \quad (2.10)_1$$

together with thermodynamic equilibrium in the vapor

$$T_1 = \tilde{T}(p_v) , \quad (2.10)_2$$

(See Plesset and Zwick [1954], Gebhardt [1961], Ishii [1975]). In general,  $p_w \neq p_v$ , so that the water is not in thermodynamic equilibrium.

The choice of temperature conditions at the phase change boundary appears to be an unresolved question of physics. Two of the possibilities are expressed as (2.9) and (2.10). There are many other possibilities.

The enthalpy  $h$  of water and water vapor are determined by the temperature and density and the difference between them is called the heat of vaporization

$$h_{fg} = \hat{u}h\hat{o} . \quad (2.11)$$

At  $T_s=100^\circ$  and atmosphere pressure,  $h_{fg}$  is  $2.257 \times 10^6$  joules per kilogram.

The values of the parameters which shall be taken as constants in the analysis to follow are their table values at  $T = T_s = 100^\circ \text{ C}$ ,

$$\begin{aligned} \rho_1 &= 0.585 \text{ g/m}^3, \\ \rho_2 &= 0.965 \text{ g/cm}^3, \\ \mu_1 &= 0.0125 \text{ cp}, \\ \mu_2 &= 0.28 \text{ cp}, \\ k_1 &= 2.5 \times 10^{-4} \text{ joule/cm sec } ^\circ\text{C}, \\ k_2 &= 6.8 \times 10^{-3} \text{ joule/cm sec } ^\circ\text{C}, \\ C_{p1} &= 1.96 \text{ joule/gm } ^\circ\text{C}, \\ C_{p2} &= 4.18 \text{ joule/gm } ^\circ\text{C}, \\ h_{fg} &= 2.257 \times 10^3 \text{ joule/gm}, \\ \sigma &= 64.4 \text{ dynes/cm}. \end{aligned} \quad (2.12)$$

### 3. Governing equations for the inclined channel

We now write the governing equations for flow down an inclined channel, figure 1, restricting the analysis to two space dimensions.

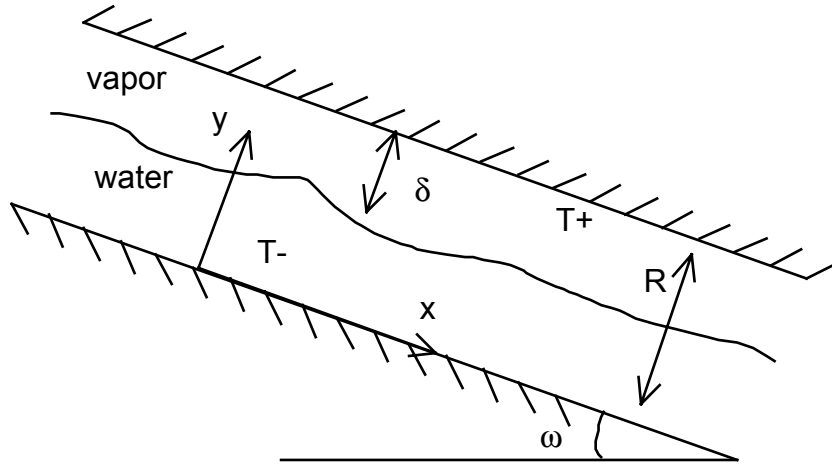


Figure1. Schematic of free fall of water(2) and vapor(1) down an inclined channel of height  $R$ . The interface is at  $y=R-\delta(x,t)$ .  $\omega$  is the angle of inclination from horizontal.

The interface is represented by

$$I(x,y,t) = y - (R - \delta(x,t)) = 0 . \quad (3.1)$$

Then

$$-n_{12} = n_{21} = \frac{-1}{|-1|} = \frac{\mathbf{e}_y + \mathbf{e}_x d_x}{\sqrt{1+d_x^2}} , \quad (3.2)$$

$$-t_{12} = t_{21} = \frac{-\mathbf{e}_y d_x + \mathbf{e}_x}{\sqrt{1+d_x^2}} . \quad (3.3)$$

The equations of motion and energy are resolved in the usual way after noting that

$$\mathbf{g} = -\mathbf{e}_z g = -g[(\mathbf{e}_x \cdot \mathbf{e}_z)\mathbf{e}_x + (\mathbf{e}_y \cdot \mathbf{e}_z)\mathbf{e}_y] = -g[\sin\omega \mathbf{e}_x - \cos\omega \mathbf{e}_y] . \quad (3.4)$$

Then

$$\rho \frac{du}{dt} = \rho g \sin\omega - \frac{\partial f}{\partial x} + \mu \nabla^2 u , \quad (3.5)$$

$$\rho \frac{dv}{dt} = - \frac{\partial f}{\partial y} + \mu \nabla^2 v, \quad (3.6)$$

where

$$\phi = p + \rho g y \cos \omega, \quad (3.7)$$

and (2.3) governs  $T$ . These equations hold both in the water and vapor phase with appropriate values for the constants.

The interface conditions (2.4) through (2.10) may be expressed in coordinate form using (3.7) and

$$2H = - \frac{d_{xx}}{(1+d_x^2)^{3/2}}. \quad (3.8)$$

The continuity of the tangential component of velocity (2.7) implies that on  $y=R - \delta$

$$\hat{u} - v\delta_x + u\hat{o} = 0. \quad (3.9)$$

The normal and tangential component of (2.5) can be written, using (3.7), as

$$-\hat{u}\phi\hat{o} + \hat{u}\rho\hat{g}(R-\delta)\cos\omega - \dot{m} \frac{\hat{u}v+d_xu\hat{o}}{(1+d_x^2)^{1/2}} + \frac{2}{1+d_x^2} \\ \hat{\mathbf{U}}\mu \left( \frac{\partial v}{\partial y} + d_x \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + d_x^2 \frac{\partial u}{\partial x} \right) \hat{\mathbf{O}} = - \frac{sd_{xx}}{(1+d_x^2)^{3/2}}, \quad (3.10)$$

and

$$(1-\delta_x^2) \hat{\mathbf{U}}\mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \hat{\mathbf{O}} - 2\delta_x \hat{\mathbf{U}}\mu \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \hat{\mathbf{O}} = 0. \quad (3.11)$$

Then, since the tangential component of velocity is continuous across  $y=R - \delta$

$$(\mathbf{u}-\mathbf{u}_\Sigma) = [(\mathbf{u}-\mathbf{u}_\Sigma) \cdot \mathbf{n}_{12}] \mathbf{n}_{12} = [(\mathbf{u}-\mathbf{u}_\Sigma) \cdot \mathbf{n}_{21}] \mathbf{n}_{21},$$

and

$$(\mathbf{u}-\mathbf{u}_\Sigma) \cdot \mathbf{n}_{21} = (v+\delta_xu+\delta_t) / (1+\delta_x^2)^{1/2}. \quad (3.12)$$

It follows that

$$\dot{m} = \rho(v + \delta_x u + \delta_t) / (1 + \delta_x^2)^{1/2}, \quad (3.13)$$

and the energy balance (2.6) reduces to

$$\begin{aligned} \frac{\hat{U} k \left( -d_x \frac{\partial T}{\partial x} - \frac{\partial T}{\partial y} \right) \hat{O}}{(1 + d_x^2)^{1/2}} &= \dot{m} \hat{u} h \hat{o} + \frac{\dot{m} \hat{u} (v + d_x u + d_t)^2 \hat{o}}{(1 + d_x^2)} \\ &+ \frac{2}{(1 + d_x^2)^{3/2}} \hat{U} \mu (v + d_x u + d_t) \left\{ \frac{\partial u}{\partial y} + d_x \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + d_x^2 \frac{\partial u}{\partial y} \right\} \hat{O}. \end{aligned} \quad (3.14)$$

We have already mentioned that we do not know what temperature conditions to apply at the phase change boundary. We could demand that the temperature be at the saturation values in the water and water vapor. If the pressure is not continuous, then the temperature will not be continuous. We could also require that either water or its vapor be at saturation and enforce the continuity of temperature. Then the second phase will not be at saturation.

#### 4. **Basic flow**

There is a steady, developed, solenoidal solution

$$(u, v, \phi, T, \delta) = (U(y), 0, 0, T, \delta_0), \quad (4.1)$$

in the form

$$\left. \begin{aligned} U_1(y) &= -\frac{\rho_1 g \sin \omega}{2\mu_1} (R-y)^2 + C_1 (R-y) \quad \forall y \in (R-d_0, R) \\ U_2(y) &= -\frac{\rho_2 g \sin \omega}{2\mu_2} (R-y)^2 + C_2 (R-y) + C_3 \quad \forall y \in (0, R-d_0) \end{aligned} \right\}, \quad (4.2)$$

where

$$\delta_0 = \frac{(T_+ - T_s) k_1 R}{(T_+ - T_s) k_1 + (T_s - T_-) k_2} \stackrel{\text{def}}{=} \beta R, \quad (4.3)$$

defines  $\beta$ , the vapor volume fraction or relative thickness of the vapor layer and



$$\begin{aligned}
 C_1 &= \frac{g \sin \omega R [b^2 (r_1 \mu_2 - 2r_1 \mu_1 + r_2 \mu_1) + 2b \mu_1 (r_1 - r_2) + \mu_1 r_2]}{2\mu_1 (\mu_1 + b(\mu_2 - \mu_1))} , \\
 C_2 &= \frac{g \sin \omega R [b^2 (2r_2 \mu_2 - r_2 \mu_1 + r_1 \mu_2) + r_2 \mu_1]}{2\mu_2 (\mu_2 + b(\mu_2 - \mu_1))} , \\
 C_3 &= \frac{g \sin \omega R^2 [r_2 b (\mu_2 - \mu_1) - b^2 (2r_2 \mu_2 - r_2 \mu_1 - r_1 \mu_2)]}{2\mu_2 (\mu_1 + b(\mu_2 - \mu_1))} .
 \end{aligned} \tag{4.4}$$

The temperature is given by

$$\left. \begin{aligned}
 T_1(y) &= T_+ - (T_+ - T_s) \frac{R-y}{d_0} \quad \forall y \in (R-d_0, R) \\
 T_2(y) &= T_s - (T_s - T_-) \frac{R-y-d_0}{R-d_0} \quad \forall y \in (0, R-d_0)
 \end{aligned} \right\} , \tag{4.5}$$

At  $y=R - \delta_0$ ,  $T_1=T_2=T_s$ , and

$$\left. \begin{aligned}
 P_1 &= -r_1 g y \cos \omega + A_1 \quad \forall y \in (R-d_0, R) \\
 P_2 &= -r_2 g y \cos \omega + A_2 \quad \forall y \in (0, R-d_0)
 \end{aligned} \right\} , \tag{4.6}$$

with  $A_1$  and  $A_2$  selected so that

$$\hat{u} \rho \hat{\omega} = - \hat{u} \rho \hat{g} (R - \delta_0) \cos \omega + \hat{u} A \hat{\omega} = 0 . \tag{4.7}$$

i.e. the pressure at the vapor-water interface is continuous, the basic flow is in thermodynamic equilibrium with

$$T_s = T(P(R-\delta_0)) , \quad \dot{m} = 0 . \tag{4.8}$$

### 5. *Linearized equations*

Let  $u, v, p, \theta, h$  be perturbations of  $U, 0, 0, T, \delta_0$ . The linearized equations for the perturbations are

$$u_x + v_y = 0 , \tag{5.1}$$

$$u_t + U u_x + v U' = - \frac{1}{r} \phi_x + v \nabla^2 u , \tag{5.2}$$

$$v_t + Uv_x = -\frac{1}{r} \phi_y + v \nabla^2 v, \quad (5.3)$$

$$\rho c_p (\theta_t + U\theta_x + vT') = k \nabla^2 \theta. \quad (5.4)$$

$u, v$  and  $\theta$  vanish on  $y=0$  and  $y=R$ . The interface conditions are evaluated on  $y=R - \delta_0$ .

$$\dot{m} = \rho_1(v_1 - U_1 h_x - h_t) = \rho_2(v_2 - U_2 h_x - h_t), \quad (5.5)$$

$$\hat{u}u + U_y h \hat{\theta} = 0, \quad (5.6)$$

$$-\hat{u} \phi \hat{\theta} + \hat{u} \rho \hat{\theta} g h \cos \omega + 2\hat{u} \mu v_y \hat{\theta} = \sigma h_{xx}, \quad (5.7)$$

$$\hat{u} \mu (u_y + v_x + U_{yy} h) \hat{\theta} = 0, \quad (5.8)$$

$$\hat{u} k \theta_y \hat{\theta} + \dot{m} h_{fg} = 0. \quad (5.9)$$

We have to choose a temperature condition at the phase change boundary.

We require saturation in vapor or side 1,

$$\theta_1 = \tilde{T}'(p_s) p_1 + [-T_1' + \tilde{T}'(p_s) P_1'] h. \quad (5.10)$$

Then either

$$\theta_2 = \tilde{T}'(p_s) p_2 + [-T_2' + \tilde{T}'(p_s) P_2'] h, \quad (5.11)_1$$

or

$$[\theta] + \left[ \frac{dT}{dy} \right] h = 0. \quad (5.11)_2$$

We do not know which of the two choices (5.11)<sub>1</sub> or (5.11)<sub>2</sub> better represents conditions to be described at a phase change boundary.

## 6. *Perturbation equations and normal modes*

To make our equations dimensionless we use following scales:

$$\text{length:} \quad H_0 \stackrel{\text{def}}{=} R - \delta_0,$$

$$\begin{aligned}
 \text{velocity:} & \quad V_0 = \frac{(r_2 + b(r_1 - r_2))g \sin w}{2(\mu_1 + b(\mu_2 - \mu_1))} \delta_0 H_0, \\
 \text{time:} & \quad \frac{H_0}{V_0}, \\
 \text{pressure:} & \quad \rho_2 V_0^2, \\
 \text{temperature:} & \quad T_s - T_-,
 \end{aligned}$$

where  $V_0$  is just the velocity at the interface.

We shall use the same letters for dimensional and dimensionless variables. The dimensionless parameters listed below appear in the dimensionless equations and  $i=(1,2)$  = (vapor, water).

$$\begin{aligned}
 R_i &= \frac{r_i V_0 H_0}{\mu_i}, & \text{Reynolds number} \\
 Pr_i &= \frac{\mu_i c_i}{k_i}, & \text{Prandtl number} \\
 Pe_i &= R_i Pr_i, & \text{Peclet number} \\
 W &= \frac{s}{r_1 H_0 V_0^2}, & \text{Weber number} \\
 \xi &= \frac{k_2}{k_1}, & \text{Thermal conductivity ratio} \\
 \Gamma &= \frac{k_1 (T_s - T_-)}{h_{fg} \mu_1}, & \text{Phase change number} \\
 \Pi_i &= \frac{r_i T_s V_0^2}{h_{fg} (T_s - T_-)} \left( \frac{1}{r_1} - \frac{1}{r_2} \right), & \text{Dynamic pressure over latent heat} \\
 \tilde{G} &= \frac{\cos w g H_0}{V_0^2}, & \text{Gravity number} \\
 \zeta &= \frac{r_2}{r_1}, & \text{Density ratio} \\
 m &= \frac{\mu_2}{\mu_1}, & \text{Viscosity ratio}
 \end{aligned}$$

$$\tau_1 = \frac{T_+}{T_s}, \quad \text{Hot wall temperature ratio}$$

$$\tau_2 = \frac{T_-}{T_s}, \quad \text{Cold wall temperature ratio}$$

$$r = \frac{R}{H_0}, \quad \text{Relative distance between the two walls.}$$

If we specify the fluids as water and its vapor then all their material parameters are determined.

We are left with four independent parameters  $R_2$ ,  $r$ ,  $\tau_2$ ,  $\omega$  and

$$\zeta = 1.6502 \times 10^3,$$

$$m = 22.47,$$

$$\xi = 27.25.$$

The basic flow in dimensionless form is given by

$$U_1(y) = a_1(r-y)^2 + b_1(r-y), \quad (6.1)$$

$$U_2(y) = a_2(r-y)^2 + b_2(r-y) + c_2, \quad (6.2)$$

$$T_1(y) = \frac{t_1}{1-t_2} - \frac{t_1-1}{1-t_2} \frac{r-y}{r-1}, \quad (6.3)$$

$$T_2(y) = \frac{1}{1-t_2} - (1-y), \quad (6.4)$$

$$\text{where } a_1 = -\frac{(1+\beta(m-1))(1-\beta)}{(\zeta + \beta(1-\zeta))\beta}, \quad a_2 = \frac{z}{m} a_1,$$

$$b_1 = \frac{(b^2(z+m-2)+2b(1-z)+z)}{b(z+b(1-z))},$$

$$b_2 = \frac{(b^2(2zm-z-m)+z)}{bm(z+b(1-z))},$$

$$c_2 = \frac{(z(m-1)-b(2zm-z-m)+z)}{m(1-b)(z+b(1-z))}.$$

Let  $u, v, p, \theta, h$  be perturbations of  $U, 0, 0, T, H_0$ . In terms of normal modes

$$\mathbf{v}(x,y,t) = \{u(y),iv(y)\}\exp(i\alpha(x-ct)),$$

$$p(x,y,t) = p(y)\exp(i\alpha(x-ct)),$$

$$\theta(x,y,t) = \theta(y)\exp(i\alpha(x-ct)),$$

$$h(x,t) = h \exp(i\alpha(x-ct)),$$

we get equations for  $u(y),v(y), p(y),\theta(y)$  and  $h$ :

$$\alpha u + v' = 0, \quad (6.5)$$

$$\alpha(U - c)u + v \frac{dU}{dy} = -\alpha p + \frac{i}{R} (a^2 u - u''), \quad (6.6)$$

$$\alpha(U - c)v = p' + \frac{i}{R} (a^2 v - v''), \quad (6.7)$$

$$\alpha(U - c)\theta + v \frac{dT}{dy} = \frac{i}{Pe} (a^2 \theta - \theta''), \quad (6.8)$$

together with boundary conditions:

$$u_1(r) = v_1(r) = \theta_1(r) = 0, \quad (6.9)$$

$$u_2(0) = v_2(0) = \theta_2(0) = 0. \quad (6.10)$$

After  $p(y)$  and  $h$  have been eliminated, the jump conditions at the unperturbed interface  $y=1$  can be expressed as

$$\alpha[u] (U_1 - \zeta U_2) + (v_1 - \zeta v_2) \left[ \frac{dU}{dy} \right] = \alpha c(1 - \zeta)[u], \quad (6.11)$$

$$\begin{aligned} & \left( \frac{d^2 U_1}{dy^2} - m \frac{d^2 U_2}{dy^2} \right) u_1 - \left( \frac{d^2 U_1}{dy^2} - m \frac{d^2 U_2}{dy^2} \right) u_2 + u_1 \left[ \frac{dU}{dy} \right] - m u_2 \left[ \frac{dU}{dy} \right] \\ & - \alpha v_1 \left[ \frac{dU}{dy} \right] + \alpha m v_2 \left[ \frac{dU}{dy} \right] = 0, \end{aligned} \quad (6.12)$$

$$\begin{aligned} & \frac{i}{aR_1} u_1' \left[ \frac{dU}{dy} \right] - \frac{iz}{aR_2} u_2' \left[ \frac{dU}{dy} \right] - \left\{ \left( \frac{ia}{R_1} - U_1 \right) \left[ \frac{dU}{dy} \right] + Wa^2 + (1-z)\tilde{G} \right\} u_1 \\ & + \left\{ z \left( \frac{ia}{R_2} - U_2 \right) \left[ \frac{dU}{dy} \right] + Wa^2 + (1-z)\tilde{G} \right\} u_2 + 2 \left[ \frac{dU}{dy} \right] \left( \frac{i}{R_1} v_1' - \frac{zi}{R_2} v_2' \right) + \\ & \left( \frac{dU_1}{dy} \frac{v_1}{a} - z \frac{dU_2}{dy} \frac{v_2}{a} \right) \left[ \frac{dU}{dy} \right] = (u_1 - \zeta u_2) c \left[ \frac{dU}{dy} \right], \end{aligned} \quad (6.13)$$

$$-(\alpha u_2 - \alpha u_1)U_1 + v_1 \left[ \frac{dU}{dy} \right] - \frac{iG}{R_2} \theta_1' \left[ \frac{dU}{dy} \right] + \frac{iG}{R_2} \zeta \theta_2' \left[ \frac{dU}{dy} \right] = \alpha(u_1 - u_2)c. \quad (6.14)$$

The condition (5.10) that the temperature on the vapor side of the interface is at saturation can be written as

$$\Pi_1 \left\{ iu_1'' - (ia^2 - aR_1 U_1)u_1 + R_1 v_1 \frac{dU_1}{dy} \right\} + \alpha R_1 (\theta_1 - (T_1' + \Pi_1 \tilde{G})) \frac{[u]}{[U']} = \alpha \Pi_1 R_1 u_1 c. \quad (6.15)$$

Similarly for water side

$$\Pi_2 \left\{ iu_2'' - (ia^2 - aR_2 U_2)u_2 + R_2 v_2 \frac{dU_2}{dy} \right\} + \alpha R_2 (\theta_2 - (T_2' + \Pi_2 \tilde{G})) \frac{[u]}{[U']} = \alpha \Pi_2 R_2 u_2 c. \quad (6.16)$$

The continuity of temperature requires that

$$\left[ \frac{dU}{dy} \right] [\theta] - \left[ \frac{dT}{dy} \right] [u] = 0. \quad (6.17)$$

(6.15), (6.16) and (6.17) can not be enforced simultaneously. A choice must be made.

### 7. *Two different Interfacial Temperature Conditions*

When (6.15) and (6.16) are adopted, the thermodynamic equilibrium is required at the interface, consequently the temperature continuity is not guaranteed. This closes the mathematical formation of the eigenvalue problem. We shall designate this problem as case I. Alternatively we can satisfy thermal equilibrium, namely (6.17) and require that one of the phases, say, the vapor phase be at saturation temperature. We designate this problem as case II.

These two different systems of equations differ only on one equation, although generally it should be expected that they will give different results, for the water-vapor case, the differences are insignificant ( table 1). Hence, only the results of case II will be reported.

Table1. A comparison of the results from different choices of interfacial temperature conditions for typical parameters in the vertical case,  $\omega = 90^\circ$ .

parameters	eigenvalue with the maximum growth rate	
	case I	case II

$R_2 = 10$ $r = 1.5$ $\tau_2 = 0.97$	$\tilde{a} = 0.0130,$ $\tilde{a} c = 1.040555, 0.152185i$	$\tilde{a} = 0.0131,$ $\tilde{a} c = 1.040536, 0.152197i$
$R_2 = 1000$ $r = 2.0$ $\tau_2 = 0.97$	$\tilde{a} = 0.250,$ $\tilde{a} c = 0.266496, 0.0072672i$	$\tilde{a} = 0.249,$ $\tilde{a} c = 0.265408, 0.0072667i$

## 8. Energy Analysis

The temperature or thermal energy equation is coupled to mechanical energy through the mechanism of phase change. In this case we may get two energy identities, one for mechanical energy designated with a subscript M and another for thermal energy designated with subscript T. The mechanical energy equation is obtained as follows. Suppose  $(u, v, \theta)$  are the components of an eigenvector associated with the maximum growth rate of one of the problems satisfying the equations (6.5) to (6.15) and one of the equations of (6.16) or (6.17). To get the equation governing the evolution of the mechanical energy of the disturbance, we multiply (6.6) and (6.7) with  $u^*$  and  $v^*$ , the complex conjugates of  $u$  and  $v$  respectively, then integrate the sum of them over both the liquid and vapor regions

$$\begin{aligned} \langle \alpha(U-c)(u^2+v^2) \rangle = & \frac{1}{R} \langle \alpha^2(u^2+v^2) + \left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2 \rangle - \langle \zeta v u^* \frac{dU}{dy} \rangle \\ & - \left[ \frac{Z_i}{R} (u' u^* + v' v^*) \right] + [\zeta p v^*] \end{aligned} \quad (8.1)$$

where  $\langle \int_1^2 \dots \rangle = \int_1^2 \dots$ ,  $u^2 = uu^*$ ,  $v^2 = vv^*$  and we used the boundary and interfacial conditions

to evaluate the integrands. (8.1) is the energy balance for the small disturbance, every term in it can be interpreted as some kind of energy. The imaginary part of the right hand side represents the growth of the energy of the disturbance and the left side may be split into three parts

$$\dot{E}_M = I_M - D_M + B_M \quad (8.2)$$

where  $\dot{E}_M = \alpha c_i \langle (u^2+v^2) \rangle$ ,

$$I_M = \text{Im} \left\langle \zeta v u^* \frac{dU}{dy} \right\rangle, \quad \text{energy production from the basic flow}$$

$$D_M = \frac{1}{R} \left\langle \alpha^2 (u^2 + v^2) + \left( \frac{du}{dy} \right)^2 + \left( \frac{dv}{dy} \right)^2 \right\rangle, \quad \text{viscous dissipation rate}$$

$$B_M = \text{Im} \left\{ \left[ \frac{z_i}{R} (u' u^* + v' v^*) \right] - [\zeta p v^*] \right\}, \quad \text{energy production from interface.}$$

We may transform the last term of  $B_M$  as follows

$$[\zeta p v^*] = [\zeta p] v_2^* + [\zeta v^*] p_1, \quad (8.3)$$

where  $[\zeta p]$  can be evaluated from the jump condition as

$$2 \left[ \frac{z_i}{R} v' \right] - (\alpha^2 W + (1 - \zeta) \tilde{G}) \frac{[u]}{[dy]}.$$

Then  $B_M$  can be further decomposed into four parts

$$B_M = B_1 + P + B_2 + G \quad (8.4)$$

where

$$B_1 = \alpha^2 W \text{Im} \left( \frac{[u] v_2^*}{[dy]} \right)$$

can be regarded as the energy supply rate due to the surface tension;

$$P = - \text{Im} ([\zeta v^*] p_1)$$

arises from the change of phase;

$$B_2 = \text{Im} \left\{ \left[ \frac{z_i}{R} (u' u^* + v' v^*) \right] - 2 \left[ \frac{z_i}{R} v' \right] v_2^* \right\}$$

represents interfacial friction due to the viscosity difference;

and

$$G = \tilde{G} (1 - z) \text{Im} \left( \frac{[u] v_2^*}{[dy]} \right)$$

is the gravity term.

To get the equation governing the production of thermal energy we multiply (6.8) by  $\theta^*$ , and integrate over both vapor and water regions

$$\langle \alpha (U - c) \theta^2 \rangle = \left\langle \frac{i}{Pe} (a^2 q^2 + \left( \frac{dq}{dy} \right)^2) \right\rangle - \langle v \theta^* \frac{dT}{dy} \rangle$$



$$- \left[ \frac{i}{Pe} \theta' \theta^* \right]. \quad (8.5)$$

(8.5) can be written as follows

$$\dot{E}_T = I_T - D_T + B_T \quad (8.6)$$

where

$$\begin{aligned} \dot{E}_T &= \alpha c_i \langle \theta^2 \rangle, \\ I_T &= \left\langle \frac{1}{Pe} (a^2 q^2 + \left(\frac{dq}{dy}\right)^2) \right\rangle, \\ D_T &= \text{Im} \left\langle v \theta^* \frac{dT}{dy} \right\rangle, \\ B_T &= R \left( \left[ \frac{1}{Pe} \theta' \theta^* \right] \right). \end{aligned}$$

## 9. *Horizontal Case*

The horizontal case deserves special treatment because in this case gravity does not drive the flow. The basic state is motionless with a linear temperature profile and zero velocity. Since there is no prescribed velocity a different unit of velocity scale,  $\frac{m_1 z}{r_1 H_0 m}$  is used to make

the equations dimensionless. The governing equations then become:

$$v^{(4)} - 2\alpha^2 v'' + \alpha^4 v = i\alpha c R (\alpha^2 v - v''), \quad (9.1)$$

$$\theta'' - \alpha^2 \theta - iPe v T' = -i\alpha c Pe \theta. \quad (9.2)$$

The general solution of this system can be expressed as

$$v(y) = Ae^{\alpha y} + Be^{-\alpha y} + Ce^{\hat{a} y} + De^{-\hat{a} y}, \quad (9.3)$$

$$\theta(y) = Ee^{\hat{g} y} + Fe^{-\hat{g} y} + \tilde{q}(y), \quad (9.4)$$

where  $\tilde{q}(y) = \frac{T'}{ac} \left[ Ae^{\alpha y} + Be^{-\alpha y} + \frac{Pe}{Pe-R} (Ce^{\hat{a} y} + De^{-\hat{a} y}) \right]$ , and  $\hat{a}^2 = \alpha^2 - i\alpha c R$ ,

$$\hat{g}^2 = \alpha^2 - i\alpha c Pe.$$

Substituting (9.3) and (9.4) into the boundary and interface conditions, we are lead to an eigenvalue problem for a 13X13 matrix acting on a vector whose 13 components are the

coefficients A,B,C,D,E,F for both vapor and water phase and the interface position variable  $h$ . This system can not be solved analytically. However for each set of parameters, we can find the eigenvalues numerically. There are three independent parameters  $H_0$ ,  $r$  and  $\tau_2$ . Figure 2 displays the neutral curve when  $H_0$  and  $r$  are fixed,  $\tau_2$  varies. It shows that when  $1-\tau_2 = (T_s-T_\infty)/T_s$  increases at a fixed value of  $\alpha$  the basic state becomes unstable. The case in which  $\tau_2$  and  $H_0$  are fixed and  $r$  changes is represented in figure 3 which shows that the basic state is more stable when vapor layer is thicker.

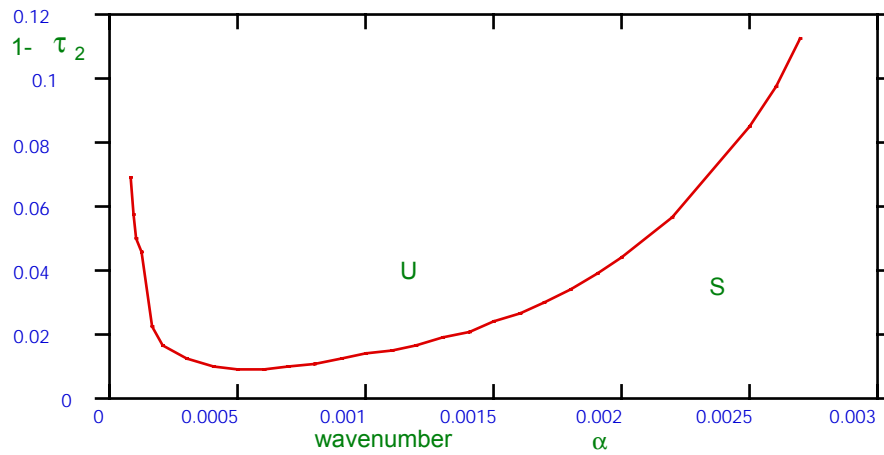


Figure 2. Neutral curve when  $H_0=0.001\text{m}$ ,  $r=1.3$ ,  $\tilde{G}=0.00434$ ,  $W=48.4$ ,  $\Gamma$ ,  $\Pi_1$ , and  $\Pi_2$  change with  $(1-\tau_2)$ .

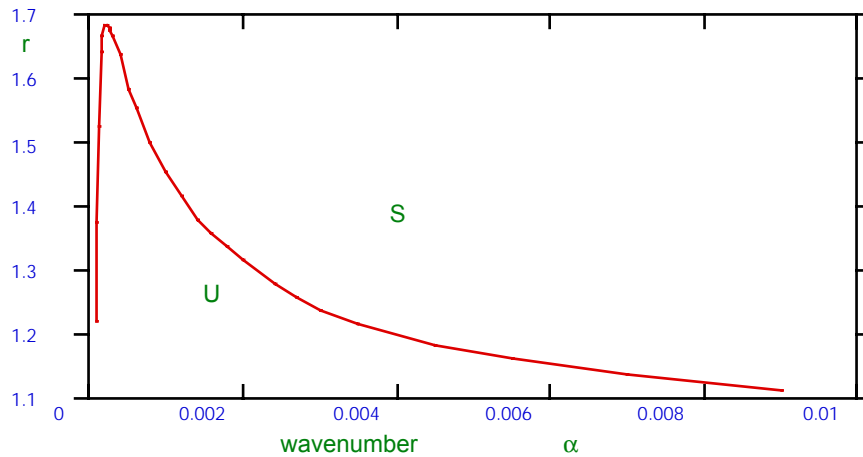


Figure3. Neutral curve when  $\tau_2$  (i.e. the temperature of the lower plane) is fixed (0.95):  
 $H_0=0.001\text{m}$ ,  $\tilde{G} =0.00434$  ,  $W = 484$ ,  $\Gamma=0.231\text{e}^{-3}$ ,  $\Pi_1= 0.2\text{e}^{-4}$ ,  $\Pi_2 = 0.0338$ .

The energy analysis (table 2) shows that phase change term is the dominant destabilizing term in the energy balance, surface tension and gravity are stabilizing but of small magnitude. The instability is solely caused by the change of phase.

Table 2. Energy analysis :  $H_0=0.001\text{m}$ ,  $\tilde{G} =0.00434$  ,  $W = 48.4$  and  $\tau_1 =1.2$ .

$1-\tau_2$	$\tilde{a}$	$\tilde{a} c_i$	$\dot{E}_M$	$B_1$	P	$B_2$	G	$\dot{E}_T$	$B_T$	$I_T - D_T$
0.017	3.6e-4	1.6e-6	1.2e-3	-4.6e-13	1.00	2.0e-5	-5.5e-7	7.1e-6	1.0	-9.99e-1
0.070	5.2e-3	4.0e-3	3.8e-1	-1.0e-8	1.38	3.5e-6	-5.6e-4	1.0e-3	1.0	-9.99e-1

10. **Vertical Case**

In the general case an analytic form for the eigenfunction cannot be found and the finite element code of Hu & Joseph [1989] was used to solve the eigenvalue problem. Figure 4 displays neutral curves in the vertical case when  $r=2.0, \tau_2 =0.97$ .

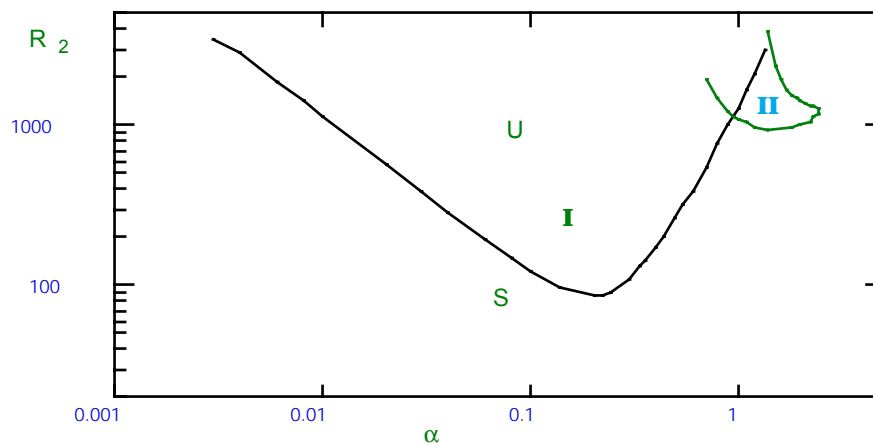


Figure 4. Neutral curve when  $r=2.0, \tau_2 =0.97$ .

The neutral curve consists of two branches, each of which represents a different mechanism of instability. The energy analysis of this case (table 3, where the viscous dissipation term  $D_M$  is used to normalize (8.4) and  $D_T$  is used to normalize (8.6)) shows that in region I the instability is caused by the interfacial friction  $B_2$  together with the phase change term  $P$  when Reynolds number is small. On the other hand the instability in region II arises from the Reynolds stress in the water layer. There is a overlapping region where two unstable modes exist , and a mode jump is observed at  $R_2$  about 1130.

Table 3. Energy analysis for the mode of maximum growth rate in the vertical case:  
 $(r , \tau_2 ) = (2.0,0.97)$ .

$R_2$	$\tilde{a}$	$\tilde{a} c_i$	$\dot{E}_M$	$I_M-D_M$	$B_1$	$P$	$B_2$	$\dot{E}_T$	$B_T$	$I_T - D_T$
86.7	0.198	1.40e-5	8.2e-5	-0.997	-2.1e-3	0.546	0.427	5.7e-6	0.72	-0.720
900	0.243	7.43e-3	1.2e-1	-0.998	-1.0e-1	0.072	1.130	3.2e-2	0.57	-0.537
1200	1.491	2.07e-2	2.16	3.26	-0.873	-0.002	-0.225	4.2e-2	0.71	-0.672
2000	1.176	3.50e-2	2.96	3.29	-0.314	0.002	-0.012	1.4e-2	0.72	-0.683

When the vapor layer is thinner (figure 5, where  $r =1.5$ ,  $\tau_2=0.97$  ), another instability appears in the longwave range. Energy analysis (table 4) shows that this instability arises from phase change. We can also see that the maximum growthrate decreases as Reynolds number increases and at  $R_2$  about 670, a mode jump occurs. The new mode of instability is associated with the Reynolds stress with phase change playing only a minor role.

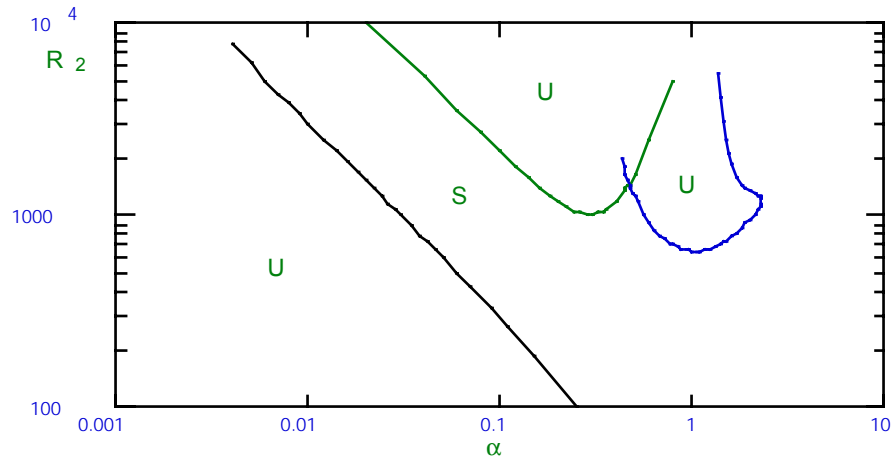


Figure 5. Neutral curve for vertical case  $\omega = 90^\circ$ :  $(r, \tau_2) = (1.5, 0.97)$ .

Table 4. Energy analysis for the mode of maximum growth rate in the vertical case (figure 5) :  $(r, \tau_2) = (1.5, 0.97)$ .

$\mathbb{R}_2$	$\tilde{a}$	$\tilde{a} c_i$	$\dot{E}_M$	$I_M - D_M$	$B_1$	$P$	$B_2$	$\dot{E}_T$	$B_T$	$I_T - D_T$
100	0.0060	1.52e-2	6.9e-2	-1.000	-6.8e-8	1.07	3.5e-3	1.7e-3	0.96	-0.962
300	0.0041	5.12e-3	6.9e-2	-0.999	-2.2e-8	1.06	7.9e-3	1.7e-3	0.96	-0.962
650	0.0031	2.28e-3	6.9e-2	-0.998	-9.4e-9	1.05	1.4e-2	1.7e-3	0.96	-0.962
700	1.041	5.67e-3	6.7e-1	1.09	-3.1e-1	0.11	-0.21	3.4e-3	1.24	-1.239
2000	1.065	8.11e-2	5.61	6.06	-4.8e-1	0.07	2.6e-2	0.121	1.40	-1.280

Figure 6 displays the neutral curves when Reynolds number and  $r$  are fixed ( $\mathbb{R}_2=1000$ ,  $r=1.5$ ) and  $\tau_2$  varies. Three different instabilities can be identified. For long waves there is an unstable mode associated with phase change, consistent with the results for the horizontal case, this mode goes unstable when  $1-\tau_2 = (T_s - T_\infty)/T_s$  increases at a fixed  $\alpha$ . However in this case the mode with the maximum growthrate is associated with the Reynolds stress, which is always unstable (right most region of figure 6), while the unstable region resulted from the interfacial friction is split into two parts.

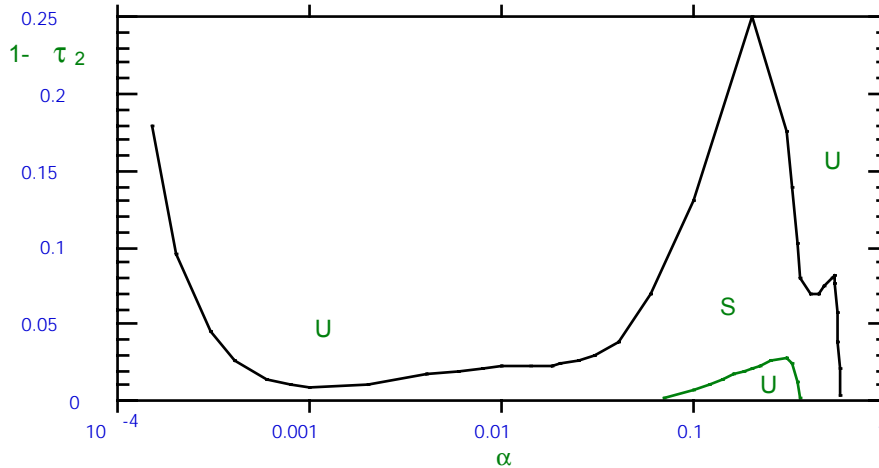


Figure 6. Neutral curve when Reynolds number and  $r$  are fixed:  $R_2 = 1000$ ,  $r=1.5$ .

Finally we briefly discuss the effect of the inclination angle  $\omega$ . Notice that in our dimensionless form  $\omega$  only appears in the gravity parameter, and from the energy analysis we see, as can be expected when the vapor lies above the water, that gravity is stabilizing, however when phase change is the dominant destabilizing factor, the influence of the gravity is almost negligible. In the inclined cases, there is an unstable mode caused by the interfacial friction, in these cases the gravity term is larger but not large enough to stabilize the flow (table 5).

Table 5. Critical points in the case when  $r=2.0, \tau_2 = 0.97$  with different inclination angles.

$\omega$	critical Reynolds number, $\tilde{R}_2$	critical wave number, $\tilde{\alpha}$	wavespeed, $\text{real}(\tilde{\alpha} c)$
90°	86.7	0.198	0.305
30°	93.4	0.186	0.296
10°	106.1	0.169	0.272

## 11. **Conclusions**

The problem of stability of fully developed flow of liquid and its vapor under gravity between heated and cold parallel plates was considered. The water and its vapor were assumed to be incompressible, but vaporization and condensation at the water-vapor interface could occur. In the fully developed case there is no phase change at the flat interface but the perturbation induces phase changes under the condition that the total volume of vapor and the total volume of liquid is conserved. Three kinds of instability can arise, an instability due to the Reynolds stress at higher Reynolds numbers, an instability due to interfacial friction which is associated with the viscosity difference and will arise even in the absence of phase change and a strong phase change instability at the interface which can occur even between horizontal plates heated above with no basic motion. All these instabilities arise as overstability so that Hopf bifurcation into periodic solutions is expected. Analysis of bifurcations, presently underway, should reveal whether periodic solutions are stable, can be observed, and decide when the waves propagate.

The issue of temperature conditions at a phase change interface is an important one. The problem arises whenever a pressure jump across the interface is allowed even when the phases are solid or liquid. If the pressures on each side of the interface are different and the temperature is at saturation, then there must be a discontinuity of temperature. Different choices of thermal interface conditions are possible. Although in special cases treated here, water and its vapor, the stability results do not depend sensitively on the choice of conditions of the temperature at the interface we do not expect weak dependence in general especially when the dependence of the saturation temperature on the pressure is not small. The choice of temperature conditions at a phase change boundary goes beyond continuum thermomechanics and appears to require some form of molecular theory.

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