Viscous and Viscoelastic Potential Flow

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Abstract

Potential flows of incompressible fluids admit a pressure (Bernoulli) equation when the divergence of the stress is a gradient as in inviscid fluids, viscous fluids, linear viscoelastic fluids and second-order fluids. We show that the equation balancing drag and acceleration is the same for all these fluids independent of the viscosity or any viscoelastic parameter and that the drag is zero in steady flow. The unsteady drag on bubbles in a viscous (and possibly in a viscoelastic) fluid may be approximated by evaluating the dissipation integral of the approximating potential flow because the neglected dissipation in the vorticity layer at the traction-free boundary of the bubble gets smaller as the Reynolds number is increased. Using the potential flow approximation, the drag D on a spherical gas bubble of radius a rising with velocity $U(t)$ in a linear viscoelastic liquid of density ρ and shear modules $G(s)$ is given by

$$D = \frac{2}{3}\pi a^3 \rho \dot{U} + 12\pi a \int_{-\infty}^{t} G(t-\tau) U(\tau) d\tau$$

and in a second-order fluid by

$$D = \pi a \left(\frac{2}{3} a^2 \rho + 12\alpha_1 \right) \dot{U} + 12\pi a \mu U$$

where $\alpha_1 < 0$ is the coefficient of the first normal stress and μ is the viscosity of the fluid. Because α_1 is negative, we see from this formula that the unsteady normal stresses oppose inertia; that is, oppose the acceleration reaction. When $U(t)$ is slowly varying, the two formulas coincide. For steady flow, we obtain $D = 12\pi a \mu U$ for both viscous and viscoelastic fluids. In the case where the dynamic contribution of the interior flow of the bubble cannot be ignored as in the case of liquid bubbles, the dissipation method gives an estimation of the rate of total kinetic energy of the flows instead of the drag. When the dynamic effect of the interior flow is negligible but the density is important, this formula for the rate of total kinetic energy leads to $D = (\rho_a - \rho) V_B \mathbf{g} \bullet \mathbf{e_x} - \rho_a V_B \dot{U}$ where ρ_a is the density of the fluid (or air) inside the bubble and V_B is the volume of the bubble.

Classical theorems of vorticity for potential flow of ideal fluids hold equally for viscous and viscoelastic fluids. The drag and lift on two-dimensional bodies of arbitrary cross section in viscoelastic potential flow are the same as in potential flow of an inviscid fluid but the moment M in a linear viscoelastic fluid is given by

$$M = M_I + 2\int_{-\infty}^{t} [G(t-\tau)\Gamma(\tau)] d\tau$$

where M_I is the inviscid moment and $\Gamma(t)$ is the circulation, and

$$M = M_I + 2\mu\Gamma + 2\alpha_1 \frac{\partial \Gamma}{\partial t}$$

in a second-order fluid. When $\Gamma(t)$ is slowly varying, the two formulas for M coincide. For steady flow, they reduce to

$$M = M_I + 2\mu\Gamma$$

which is also the expression for M in both steady and unsteady potential flow of a viscous fluid.

Potential flows of models of a viscoelastic fluid like Maxwell's are studied. These models do not admit potential flows unless the curl of the divergence of the extra-stress vanishes. This leads to an over-determined system of equations for the components of the stress. Special potential flow solutions like uniform flow and simple extension satisfy these extra conditions automatically but other special solutions like the potential vortex can satisfy the equations for some models and not for others.

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References

1. Introduction

Potential flows arise from the kinematic assumption that the curl of the velocity vanishes identically in some region of space, $\boldsymbol{\omega} \overset{\text{def}}{=} \text{curl}\,\mathbf{u} = 0$. In this case, the velocity is given by the gradient of a potential, $\mathbf{u} = \nabla\phi$. If, in addition, the material is incompressible, then $\text{div}\,\mathbf{u} = 0$ and $\nabla^2\phi = 0$. None of this depends on the constitutive equation of the fluid. In fact most constitutive equations are not compatible with the assumption that $\text{curl}\,\mathbf{u} = 0$, in general. For example, if the viscosity μ of a Newtonian fluid varies from point to point, then

$$\rho\,\text{curl}\left(\frac{d\mathbf{u}}{dt}\right) = \text{curl}\,\left[-\nabla p + \text{div}(\mu\mathbf{A})\right] = \text{curl}\,\left(\mu\nabla^2\mathbf{u}\right) + \text{curl}\,\left(\mathbf{A}\nabla\mu\right) \tag{1.1}$$

where $\mathbf{A} \overset{\text{def}}{=} \mathbf{L} + \mathbf{L}^T$ and $\mathbf{L} \overset{\text{def}}{=} \nabla\mathbf{u}$, ρ is the density which only depends on time and, p is a to-be-determined scalar field called the pressure. All the terms except the last vanish when $\mathbf{u} = \nabla\phi$. This term amounts to a "torque" which generates vorticity. Most constitutive equations will generate vorticity because the curl of the divergence of the stress produces such a torque.

There are special irrotational motions which satisfy the equations of motion even for fluids which will not generally accommodate potential flows. For example, since the stress must be Galilean invariant, uniform motion is a potential flow which satisfies the equations of motion independent of the constitutive equation. Another such potential flow, greatly loved by rheologists, is pure extensional or elongational flow which leads to the concept of extensional viscosity.

In general, potential flows will not satisfy the boundary conditions at solid walls or free surfaces. This is why potential flows are almost impossible to achieve exactly in practice. In particular, this feature is probably at the bottom of the apparent disagreement of the different instruments which claim to measure extensional viscosity. None of them achieve the irrotational flows necessary for backing out the rheology. However, we know how to use potential flows in viscous fluid mechanics, where we were instructed by Prandtl. Perhaps we may also learn how to use potential flow to study the fluid dynamics of viscoelastic liquids.

There are some special constitutive equations which are compatible with the assumption that $\text{curl}\,\mathbf{u} = 0$, in general. Among these are inviscid fluids, viscous fluids with constant viscosity (Joseph, Liao and Hu [1993]), second-order fluids (Joseph [1992]) and linear viscoelastic fluids which perturb rest or uniform flow (Liao and Joseph [1992]). The second-order fluid arises asymptotically from the class of simple fluids by the slowing of histories which Coleman and Noll [1960] called a retardation. The retardation can be said to arise on slow and slowly varying motions, where slow variations mean spatial gradients are small when the velocity is small and time derivatives of order n scale with $|\mathbf{u}|^n$. In viscous fluid mechanics we generally associate potential flows with high Reynolds numbers, i.e. fast flows. The constitutive equation (6.1) for a linear viscoelastic liquid is the appropriate asymptotic form for simple fluids in motions which perturb uniform flows which need not be slow. We should show that second-order fluids also arise as perturbations of fast uniform flows when the perturbations are slowly varying (cf. (6.2) and (6.7)). We have worked out the consequences of the second-order theory in a mathematically rigorous way without considering the domain of deformations in which second-order fluids are valid. In fact this theory should not be expected to give good results for rapidly varying flows or in other motions outside of its domain of applicability.

In sections 2 and 9 we consider the possibility that more general models of a viscoelastic fluid could support special irrotational flows even if they do not have a pressure function in general. Such special solutions can be found; some are universal and others work for some models and not for others. The conclusion that viscoelastic liquids will not admit potential flows is too sweeping, but if you are not within the class of deformations that give rise to second-order or linear viscoelastic fluids the chances that a special potential flow can be achieved are slight.

2. Drag and dissipation in potential flow

The stress in an incompressible liquid can be written as

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \tag{2.1}$$

where p is a to-be-determined scalar field and $\mathbf{S}$ is the part of the stress which is related to the deformation by a constitutive equation. We are going to write $\mathbf{S} = \mathbf{S}[\mathbf{u}]$, meaning $\mathbf{S}$ is functional of the history of $\mathbf{u}$. The formulas relating drag and dissipation do not require that we choose a constitutive equation.

Consider the motion of a solid body or bubble in a liquid in three dimensions. Suppose that the body B moves forward with a velocity $U\mathbf{e_x}$ and that it neither rotates nor changes shape or volume. The absolute velocity $\mathbf{u}$ and the relative velocity $\mathbf{v}$ of the fluid are then related by

$$\mathbf{u} = U\,\mathbf{e_x} + \mathbf{v} \tag{2.2}$$

with

$$\mathbf{v} \bullet \mathbf{n}\big|_{\partial B} = 0 \tag{2.3}$$

where $\mathbf{n}$ is the inward normal on the boundary ∂B of B. The fluid outside B is unbounded. We assume that the flow is irrotational far from the body, since the volume V of the fluid outside B is a material volume (because no mass crosses ∂B), we apply the Reynolds transport theorem to the kinetic energy E of the fluid in V to obtain

$$\frac{dE}{dt} = \frac{d}{dt}\int_{V(t)} \rho\frac{|\mathbf{u}|^2}{2}\,dV = \int_{V(t)} \rho\,\mathbf{u}\bullet\frac{d\mathbf{u}}{dt}\,dV$$

where the integrals converge because $\mathbf{u} = O(r^{-3})$ as $r\to\infty$ in irrotational flow. For the types of constitutive equations to be considered in this paper, it can be shown that, as $r\to\infty$, $\mathbf{T} = O(r^{-2})$. Therefore, using $\operatorname{div}\mathbf{u} = 0$ and $\rho\, d\mathbf{u}/dt = \operatorname{div}\mathbf{T}+\rho\mathbf{g}$, we obtain

$$\frac{dE}{dt} = \int_{V(t)} \mathbf{u}\bullet(\operatorname{div}\mathbf{T}+\rho\mathbf{g})\,dV = \int_{\partial B} \mathbf{u}\bullet(\mathbf{T}\bullet\mathbf{n}+(\rho\mathbf{g}\bullet\mathbf{x})\mathbf{n})dS - \int_{V(t)} \mathbf{L}[\mathbf{u}]\mathbf{:}\mathbf{S}[\mathbf{u}]\,dV$$

where $\mathbf{T}\bullet\mathbf{n}$ is the negative of the traction vector expressing the force exerted by the fluid on the body and $\mathbf{x}$ is the position vector. We may rewrite this, using (2.2) and (2.3), as

$$\frac{dE}{dt} = UD + \int_{\partial B} \mathbf{v}\bullet(\mathbf{T}\bullet\mathbf{n})dS - \int_{V(t)} \mathbf{L}[\mathbf{u}]\mathbf{:}\mathbf{S}[\mathbf{u}]\,dV \tag{2.4}$$

where

$$D \overset{\text{def}}{=} \int_{\partial B} \mathbf{e_x}\bullet(\mathbf{T}\bullet\mathbf{n})dS - \rho V_B\mathbf{g}\bullet\mathbf{e_x} \tag{2.5}$$

is the drag exerted on the fluid by the body and V_B is the volume of the body. According to (2.3), because $p\,\mathbf{v}\bullet\mathbf{n} = 0$ at each point of ∂B, we have

$$\int_{\partial B} \mathbf{v} \bullet (\mathbf{T} \bullet \mathbf{n}) dS = \int_{\partial B} \mathbf{v} \bullet (\mathbf{S}[\mathbf{u}] \bullet \mathbf{n}) \, dS .$$

There are two standard situations, neither of which holds in potential flow, in which

$$\mathbf{v} \bullet (\mathbf{T} \bullet \mathbf{n}) \big|_{\partial B} + 0. \tag{2.6}$$

If B is a rigid solid, $\mathbf{v}|_{\partial B} + 0$. If B is a bubble, the tangential component of the traction vector vanishes, i.e.,

$$\boldsymbol{\tau} \bullet (\mathbf{T} \bullet \mathbf{n}) \big|_{\partial B} = 0, \text{ for all } \boldsymbol{\tau} \perp \mathbf{n}. \tag{2.7}$$

Since $\mathbf{v} \perp \mathbf{n}$ by (2.3), we obtain (2.6). Thus, for steady flows, (2.4) becomes

$$D = \frac{1}{U} \int_{V(t)} \mathbf{L}[\mathbf{u}] \mathbf{:} \mathbf{S}[\mathbf{u}] \, dV \tag{2.8}$$

In general, potential flow fails to satisfy (2.6) or (2.7) and therefore (2.8). However, many flows are approximately irrotational outside thin vorticity layers, so that (2.8) might be used to obtain approximate values of the drag. In fact, Levich [1949] used (2.8) and $\mathbf{u} = \nabla \phi$ to approximate the drag in the steady ascent of a bubble in a viscous fluid, obtaining good agreement with experiments.

For strict potential flows, the momentum equation may be written as

$$\text{div} \mathbf{S}[\nabla \phi] = \nabla \psi, \tag{2.9}$$

provided that the body force field is conservative. Obviously (2.9) holds for Newtonian fluids of constant viscosity with $\psi = 0$; it also holds for linear viscoelastic fluids with $\psi = 0$ (see (6.5)); and less trivially for second-order fluids with $\psi = \hat{\beta}\gamma^2 / 2$, where $\hat{\beta}$ is the climbing constant and $\gamma^2 = \frac{1}{2} \text{tr}(\mathbf{A}^2)$ (see Joseph [1992]). For models like Jeffreys', $\mathbf{S} = \mathbf{S}_{\text{N}} + \mathbf{S}_{\text{E}}$, where $\mathbf{S}_{\text{N}} = \mu \mathbf{A}[\mathbf{u}]$, (2.9) need only be checked for $\mathbf{S}_E$. Generally, (2.9) and the constitutive equations lead to an over-determined system of differential equations for the components of $\mathbf{S}$. Special solutions of this over-determined system can be found even for models that do not admit potential flow generally (see section 9). Using (2.9), we obtain

$$\int_{\partial B} \mathbf{v} \bullet \mathbf{S}[\mathbf{u}] \bullet \mathbf{n} dS = \int_{V(t)} \mathrm{div}(\mathbf{v} \bullet \mathbf{S}[\mathbf{u}]) dV = \int_{V(t)} \mathbf{v} \bullet \nabla \psi dV + \int_{V(t)} \mathbf{L}[\mathbf{u}]\mathbf{:}\mathbf{S}[\mathbf{u}] dV .$$

The first integral on the right-hand side vanishes when $\mathrm{div}\,\mathbf{v} = 0$ and $\mathbf{v}\bullet\mathbf{n}\big|_{\partial B} = 0$, and (2.4) reduces to

$$\frac{dE}{dt} = UD. \tag{2.10}$$

In a potential flow of the type under consideration, one has (see Batchelor [1967], page 403)

$$E = \frac{1}{2} e\rho V_B U^2 \tag{2.11}$$

where e is a constant depending only on the shape of the body. For spheres, it can be shown that $e = \frac{1}{2}$. Using (2.11), (2.10) becomes

$$e\rho V_B \frac{dU}{dt} = D .$$

This equation shows that the drag in potential flow is independent of the constitutive equation of liquids satisfying (2.9) and vanishes when the flow is steady.

3. Potential flow approximations for the terminal velocity of rising bubbles

The idea that viscous forces in regions of potential flow may actually dominate the dissipation of energy seems to have been first advanced by Lamb [1924] who showed that in some cases of wave motion the rate of dissipation can be calculated with sufficient accuracy by regarding the motion as irrotational. The computation of the drag D on a sphere in potential flow using the dissipation method seems to have been given first by Bateman [1932] and repeated Ackeret [1952]. They found that $D = 12\pi a\mu U$ where μ is the viscosity, a the radius of the sphere and U its velocity. This drag is twice the Stokes drag and is in better agreement with the measured drag for Reynolds numbers in excess of about 8.

The same calculation for a rising spherical gas bubble was given by Levich [1949]. Measured values of the drag on spherical gas bubbles are close to $12\pi a\mu U$ for

Reynolds numbers larger than about 20. The reasons for the success of the dissipation method in predicting the drag on gas bubbles have to do with the fact that vorticity is confined to thin layers and the contribution of this vorticity to the drag is smaller in the case of gas bubbles, where the shear traction rather than the relative velocity must vanish on the surface of the sphere. A good explanation was given by Levich [1962] and by Moore [1959, 1963]; a convenient reference is Batchelor [1967]. Brabston and Keller [1975] did a direct numerical simulation of the drag on a gas spherical bubble in steady ascent at terminal velocity U in a Newtonian fluid and found the same kind of agreement with experiments. In fact, the agreement between experiments and potential flow calculations using the dissipation method are fairly good for Reynolds numbers as small as 5 and improves (rather than deteriorates) as the Reynolds number increases.

The idea that viscosity may act strongly in the regions in which vorticity is effectively zero appears to contradict explanations of boundary layers which have appeared repeatedly since Prandtl. For example, Glauert [1943] says (p.142) that

> "...Prandtl's conception of the problem is that the effect of the viscosity is important only in a narrow boundary layer surrounding the surface of the body and that the viscosity may be ignored in the free fluid outside this layer."

According to Harper [1972], this view of boundary layers is correct for solid spheres but not for spherical bubbles. He says that

> " For $R \gg 1$, the theories of motion past solid spheres and tangentially stress-free bubbles are quite different. It is easy to see why this must be so. In either case vorticity must be generated at the surface because irrotational flow does not satisfy all the boundary conditions. The vorticity remains within a boundary layer of thickness $\delta = O(aR^{-1/2})$, for it is convected around the surface in a time t of order a/U, during which viscosity can diffuse it away to a distance δ if $\delta^2 = O(\nu t) = O(a^2/R)$. But for a solid sphere the fluid velocity must change by $O(U)$ across the layer, because it vanishes on the sphere, whereas for a gas bubble the normal derivative of velocity must change by $O(U/a)$ in order that the shear stress be zero. That implies that the velocity itself changes by $O(U\delta/a) = O(UR^{-1/2}) = o(U)$...
>
> In the boundary layer on the bubble, therefore, the fluid velocity is only slightly perturbed from that of the irrotational flow, and velocity derivatives are of the same order as in the irrotational flow. Then the viscous dissipation integral has the same value as in the irrotational flow, to the first order, because the total volume of the boundary layer, of order $a^2\delta$, is much less than the volume, of order a^3, of the region in which the velocity derivatives are of order U/a. The volume of the wake is not small, but the velocity derivatives in it are, and it contributes to the dissipation only in higher order terms..."

For flows in which the vorticity is confined to narrow layers the kinetic energy E should be well approximated by potential flow (even if the dissipation in not). Then using (2.3), (2.7), and (2.11), (2.4) becomes

$$e\rho V_B \frac{dU}{dt} \cong D - \frac{1}{U} \int_{V(t)} \mathbf{L}[\ \phi]\mathbf{:S}[\ \phi]\, dV\,. \tag{3.1}$$

In the problem of the rising bubble where the contribution from the flow inside the bubble cannot be neglected we get

$$\frac{dE}{dt} = \frac{dE_1}{dt} + \frac{dE_2}{dt} = \int_{\partial B} (\mathbf{u}_2 \bullet \mathbf{T}_2 - \mathbf{u}_1 \bullet \mathbf{T}_1) \bullet \mathbf{n} dS + \int_{\partial B} \mathbf{g} \bullet \mathbf{x} (\rho_2 \mathbf{u}_2 - \rho_1 \mathbf{u}_1) \bullet \mathbf{n} dS - \Phi(\mathbf{x}, t) \tag{3.2}$$

where the region 1 is inside the bubble and 2 is outside, **n** is the normal vector on the surface which points into the bubble and

$$\Phi(\mathbf{x},t) \overset{\text{def}}{=} \int_B \mathbf{L}[\mathbf{u}_1]\mathbf{:S}[\mathbf{u}_1] dV + \int_{V(t)} \mathbf{L}[\mathbf{u}_2]\mathbf{:S}[\mathbf{u}_2] dV$$

is the total rate of energy dissipation. On the surface of the bubble the normal velocity and the shear stress are continuous; that is,

$$(\mathbf{u}_2 - \mathbf{u}_1) \bullet \mathbf{n} = 0 \text{ on } \partial B$$

and

$$\boldsymbol{\tau} \bullet (\mathbf{T}_2 - \mathbf{T}_1) \bullet \mathbf{n} = 0 \text{ on } \partial B \text{ for all } \boldsymbol{\tau} \perp \mathbf{n}. \tag{3.3}$$

Since the bubble is neither rotating nor deforming, we can decompose the velocity as in (2.2) and (2.3). Then inserting (2.2) into (3.2), we find, after using a recent result of Hesla, Huang and Joseph [1993] which says the mean value of the jump of the traction vector vanishes on the closed surface of a drop

$$\int_{\partial B} \mathbf{e_x} \bullet (\mathbf{T}_2 - \mathbf{T}_1) \bullet \mathbf{n} dV = 0, \tag{3.4}$$

that

$$\frac{dE}{dt} = -U[(\rho_2 - \rho_1) V_B]\mathbf{g} \bullet \mathbf{e_x} + \int_{\partial B} (\mathbf{v}_2 \bullet \mathbf{T}_2 - \mathbf{v}_1 \bullet \mathbf{T}_1) \bullet \mathbf{n} dS - \Phi(\mathbf{x}, t). \tag{3.5}$$

Moreover, if $\mathbf{v}_1 = \mathbf{v}_2$, then, after applying (2.3) and (3.3), (3.5) reduces to

$$\frac{dE}{dt} = -U[(\rho_2 - \rho_1) V_B]\mathbf{g} \bullet \mathbf{e_x} - \Phi(\mathbf{x}, t). \tag{3.6}$$

Equation (3.6) can be used to form an unsteady extension of the drag formula introduced by Levich [1949]. We first assume that the air bubble does not exert a shear traction on the liquid outside. This implies that a vorticity layer is required in the liquid to adjust the potential flow stress to its zero shear traction value on the free surface. This vorticity layer is much weaker than the layer required on a moving solid, or on a viscous bubble, in which the velocity of the potential flow rather than its derivative must be adjusted to its no-slip value. If the dissipation in the bubble is neglected, then the kinetic energy of the gas becomes

$$\rho_1 V_B U\dot{U} = \frac{dE_1}{dt} = -U\int_{\partial B}(\mathbf{e_x}\bullet\mathbf{T}_1)\bullet\mathbf{n}\,dS - U\int_{\partial B}(\rho_1\mathbf{g}\bullet\mathbf{x})\mathbf{e_x}\bullet\mathbf{n}\,dS$$

where $\frac{\partial}{\partial t}$ is denoted by a superposed dot. This implies that

$$\int_{\partial B}(\mathbf{e_x}\bullet\mathbf{T}_1)\bullet\mathbf{n}\,dS = \rho_1 V_B(\mathbf{g}\bullet\mathbf{e_x} - \dot{U}).$$

Using this equation, (3.4), and (2.5), we find that the drag induced by the flow outside the body is

$$D = (\rho_1 - \rho_2)V_B\mathbf{g}\bullet\mathbf{e_x} - \rho_1 V_B\dot{U}.$$

This approximate formula for drag is independent of the constitutive equation of the fluid.

4. Momentum, circulation, and vorticity equations for a second-order fluid

The constitutive equation of a second-order fluid is

$$\mathbf{S} = \mu\mathbf{A} + \alpha_1\mathbf{B} + \alpha_2\mathbf{A}^2 \tag{4.1}$$

where $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is twice the rate-of-strain tensor **D** which is the symmetric part of the velocity-gradient tensor $\mathbf{L} = \nabla\mathbf{u}$, $\mathbf{B} \overset{\text{def}}{=} \overset{\Delta}{\mathbf{A}} = \partial\mathbf{A}/\partial t + (\mathbf{u}\bullet\nabla)\mathbf{A} + \mathbf{A}\mathbf{L} + \mathbf{L}^T\mathbf{A}$ is the lower convected invariant derivative of **A**, μ is the zero-shear viscosity, $\alpha_1 = -n_1/2$ and $\alpha_2 = n_1 + n_2$, where $n_i = \lim_{\kappa\to 0} N_i(\kappa)/\kappa^2$ for i = 1 and 2 are constants obtained from the first and second normal stress differences. In Appendix A we show that

$$\nabla \bullet \mathbf{S} = \mu \nabla^2 \mathbf{u} + \alpha_1 \left[\frac{d\nabla^2 \mathbf{u}}{dt} + \mathbf{L}^T \bullet (\nabla^2 \mathbf{u}) \right] + (\alpha_1 + \alpha_2)\left[\mathbf{A} \bullet (\nabla^2 \mathbf{u}) + \nabla \mathbf{\Omega} \bullet \mathbf{A} \right] + \frac{\hat{\beta}}{2} \nabla \gamma^2 \quad (4.2)$$

where $\hat{\beta} = 3\alpha_1 + 2\alpha_2$ is the climbing constant, $\gamma^2 = \frac{1}{2}\,\mathrm{tr}(\mathbf{A}^2)$, and $\mathbf{\Omega}$ **Error!**

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g} + \alpha_1 \left[\frac{d\nabla^2 \mathbf{u}}{dt} + \mathbf{L}^T \bullet (\nabla^2 \mathbf{u}) \right]$$
$$+ (\alpha_1 + \alpha_2)\left[\mathbf{A} \bullet (\nabla^2 \mathbf{u}) + \nabla \mathbf{\Omega} \bullet \mathbf{A} \right] + \frac{\hat{\beta}}{2} \nabla \gamma^2 . \quad (4.3)$$

For potential flow, $\mathbf{u} = \nabla \phi$, $\nabla^2 \mathbf{u}$ and $\mathbf{\Omega}$ vanish and $d\mathbf{u}/dt = \nabla(\partial \phi / \partial t + |\mathbf{u}|^2 / 2)$, so that (4.3) may be written as

$$\nabla \left[p + \rho \frac{\partial \phi}{\partial t} + \rho \frac{|\mathbf{u}|^2}{2} - \frac{\hat{\beta}}{2} \gamma^2 - \rho \mathbf{g} \bullet \mathbf{x} \right] = 0 .$$

Hence

$$p = -\rho \frac{\partial \phi}{\partial t} - \rho \frac{|\mathbf{u}|^2}{2} + \frac{\hat{\beta}}{2} \gamma^2 + \rho \mathbf{g} \bullet \mathbf{x} + C(t) . \quad (4.4)$$

Lumley [1972] derived a Bernoulli equation for a dilute polymer solution on the centerline of an axisymmetric contraction. He notes that

> "Recent measurements of cavitation in dilute polymer solutions indicate that observed differences from cavitation in Newtonian media may be due to local pressure differences resulting from the non-Newtonian constitutive relation governing these dilute solutions. No convenient means of estimating the departure of the pressure from the Newtonian (inertial) value presently exists, and, of course, no general expression is possible..."

Inserting (4.1) and (4.4) into (2.1), we obtain, using index notation, that

$$T_{ij} = \sigma_{ij} - \rho \mathbf{g} \bullet \mathbf{x} \delta_{ij}$$

where

$$\sigma_{ij} \overset{\text{def}}{=} -\left[C(t) + \hat{\beta} \phi_{,il} \phi_{,il} - \rho \phi_{,t} - \rho \frac{|\mathbf{u}|^2}{2} \right] \delta_{ij} + 2\left[\mu + \alpha_1 \left(\frac{\partial}{\partial t} + \mathbf{u} \bullet \nabla \right) \right] \phi_{,ij} + 4(\alpha_1 + \alpha_2) \phi_{,il} \phi_{,lj}$$

is the active dynamic stress.

Some criticisms of the notion of extensional viscosity follow easily from this analysis. The potential flow of a fluid near a point $(x_1, x_2, x_3) = (0, 0, 0)$ of stagnation is a purely extensional motion with

$$[u_1, u_2, u_3] = \frac{U\dot{S}}{L}[2x_1, -x_2, -x_3]$$

where $\dot{S}$ is the dimensionless rate of stretching. In this case,

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = \frac{\rho U^2}{2}\left[\frac{\dot{S}^2}{L^2}\left(4x_1^2 + x_2^2 + x_3^2\right) - 1\right]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+2\mu\frac{U\dot{S}}{L}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 2\left(\frac{U\dot{S}}{L}\right)^2\begin{bmatrix} -\alpha_1 + 2\alpha_2 & 0 & 0 \\ 0 & -7\alpha_1 - 4\alpha_2 & 0 \\ 0 & 0 & -7\alpha_1 - 4\alpha_2 \end{bmatrix}.$$

At the stagnation point the extensional stress is

$$\sigma_{11} = -\frac{\rho}{2}U^2 + 4\mu\frac{U\dot{S}}{L} + 2(2\alpha_2 - \alpha_1)\left(\frac{U\dot{S}}{L}\right)^2 \tag{4.5}$$

and the extensional stress difference is

$$\sigma_{11} - \sigma_{22} = 6\mu\frac{U\dot{S}}{L} + 12(\alpha_1 + \alpha_2)\left(\frac{U\dot{S}}{L}\right)^2 \overset{\text{def}}{=} 2\tilde{\eta}\frac{U\dot{S}}{L} \tag{4.6}$$

where $\tilde{\eta} = 3\mu + 6(\alpha_1 + \alpha_2)\frac{U}{L}\dot{S}$ is the extensional viscosity of a second-order fluid. Since $2\alpha_2 - \alpha_1 = \frac{5}{2}n_1 + n_2 > 0$ and $\alpha_1 + \alpha_2 = \frac{1}{2}n_1 + n_2 > 0$, both the normal stress term in (4.5) and the normal stress difference term in (4.6) are positive independent of the sign of $\dot{S}$. From (4.5) it follows that inertia and normal stresses are in competition. But you cannot see the effects of inertia in the formula (4.6) for the normal stress difference. Certainly this formula, or the associated extensional viscosity, could not be used to assess the force on bodies.

Let $\Gamma \overset{\text{def}}{=} \oint \mathbf{u} \bullet d\boldsymbol{l}$ be the circulation and suppose that $\rho\mathbf{g}$ is derivable from a potential as is true when $\mathbf{g}$ is gravity. Then, using (4.3) and $\nabla^2\mathbf{u} = -\nabla \wedge \boldsymbol{\omega}$, we obtain the circulation equation:

$$\frac{d\Gamma}{dt} = -\oint\left(\frac{\mu}{\rho}(\nabla\wedge\boldsymbol{\omega}) + \frac{\alpha_1}{\rho}\left[\frac{d(\nabla\wedge\boldsymbol{\omega})}{dt} + \mathbf{L}^T\bullet(\nabla\wedge\boldsymbol{\omega})\right]\right)\bullet d\boldsymbol{l}$$
$$+\oint\frac{(\alpha_1+\alpha_2)}{\rho}\left[-\mathbf{A}\bullet(\nabla\wedge\boldsymbol{\omega}) + \nabla\boldsymbol{\Omega}\bullet\mathbf{A}\right]\bullet d\boldsymbol{l}. \quad (4.7)$$

On the other hand, after taking curl of (4.3) and replacing $\nabla^2\mathbf{u}$ by $-\nabla\wedge\boldsymbol{\omega}$, we obtain the vorticity equation:

$$\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\omega}\bullet\nabla\mathbf{u} + \frac{\mu}{\rho}\nabla^2\boldsymbol{\omega} - \frac{\alpha_1}{\rho}\nabla\wedge\left[\frac{d(\nabla\wedge\boldsymbol{\omega})}{dt} + \mathbf{L}^T\bullet(\nabla\wedge\boldsymbol{\omega})\right]$$
$$+\frac{(\alpha_1+\alpha_2)}{\rho}\nabla\wedge\left[-\mathbf{A}\bullet(\nabla\wedge\boldsymbol{\omega}) + \nabla\boldsymbol{\Omega}\bullet\mathbf{A}\right]. \quad (4.8)$$

When α_1 and α_2 are zero, (4.7) and (4.8) reduce to

$$\frac{d\Gamma}{dt} = -\frac{\mu}{\rho}\oint(\nabla\wedge\boldsymbol{\omega})\bullet d\boldsymbol{l} \text{ and } \frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\omega}\bullet\nabla\mathbf{u} + \frac{\mu}{\rho}\nabla^2\boldsymbol{\omega}.$$

These equations govern the circulation and vorticity in a Newtonian fluid (see Batchelor [1967] page 267 and 269). When $\boldsymbol{\omega} + 0$,

$$\frac{d\Gamma}{dt} = 0 \text{ and } \frac{d\boldsymbol{\omega}}{dt} = 0. \quad (4.9)$$

This leads to the classical vorticity theorems, Kelvin's circulation theorem and the Cauchy-Lagrange theorem. The same conclusions (4.9) hold when $\boldsymbol{\omega} = 0$, and Γ and $\boldsymbol{\omega}$ satisfy the vorticity equations (4.7) and (4.8) for a second-order fluid. It follows that the classical theorems of vorticity hold for potential flow of a second-order fluid independent of the values of the material parameters μ, α_1 and α_2. Thus, the discussion of potential flow in no way requires us to turn to the theory of ideal fluids.

Since the boundary conditions at a solid or free surface cannot generally be satisfied by potential flow, potential flow cannot hold up to the boundary and at the very least a vorticity boundary layer will be required. Outside this boundary layer we get potential flow but the viscous and viscoelastic stresses are not zero. In the case of viscous fluids with $\alpha_1 = \alpha_2 = 0$, viscosity may or may not be important outside the vorticity layer. For solid bodies the dissipation in the vorticity layer will dominate the drag and the viscous stresses in the exterior potential flow will be negligible at high Reynolds numbers. But for rising bubbles where the vorticity layer is weak the viscous stresses in the exterior potential flow will dominate the drag and the dissipation of the vorticity layer will be negligible at high Reynolds numbers. We can hope that a similar result will hold for a second-order fluid.

5. Motion of a spherical gas bubble in a second-order fluid using the dissipation method

For a spherical bubble of radius a moving with speed U through a viscous fluid the flow outside the boundary layer and a narrow wake is given approximately by potential flow

$$\phi = -\frac{U}{2}\frac{a^3}{r^2}\cos\theta. \tag{5.1}$$

We can assume that this approximation is valid for a second-order fluid and see where it leads. The radial and tangential components of velocity are given by

$$u_r = \frac{\partial\phi}{\partial r} = U\frac{a^3}{r^3}\cos\theta \text{ and } u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{U}{2}\frac{a^3}{r^3}\sin\theta.$$

To complete the unsteady drag formula (3.1), we need **L** and **S**. Since in potential flow **A** = 2**L**, we have

$$\mathbf{B} = \overset{\Delta}{\mathbf{A}} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{u}\bullet\nabla)\mathbf{A} + \mathbf{A}^2 \ . \tag{5.2}$$

Using (5.1), we find, in spherical coordinates (r,θ,φ), that

$$\mathbf{A} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{r\varphi} \\ A_{r\theta} & A_{\theta\theta} & A_{\theta\varphi} \\ A_{r\varphi} & A_{\theta\varphi} & A_{\varphi\varphi} \end{pmatrix}$$

where $A_{r\varphi} = A_{\theta\varphi} = 0$, $A_{rr} = 2\frac{\partial u_r}{\partial r} = -6U\frac{a^3}{r^4}\cos\theta$, $A_{\theta\theta} = 2\left(\frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r}\right) = +3U\frac{a^3}{r^4}\cos\theta$, $A_{\varphi\varphi} = 2\left(\frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right) = +3U\frac{a^3}{r^4}\cos\theta$, and $A_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta} = -3U\frac{a^3}{r^4}\sin\theta$.

Hence (4.1) with (5.2) gives

$$\mathbf{S} = -3\left(\mu U + \alpha_1 \frac{\partial U}{\partial t}\right)\frac{a^3}{r^4}\begin{pmatrix} 2\cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -\cos\theta \end{pmatrix}$$

$$+3\alpha_1 U^2 \frac{a^3}{r^5}\begin{pmatrix} -12\cos^2\theta + 4 & -8\cos\theta\sin\theta & 0 \\ -8\cos\theta\sin\theta & 7\cos^2\theta - 3 & 0 \\ 0 & 0 & 5\cos^2\theta - 1 \end{pmatrix}$$

$$+3\alpha_1 U^2 \frac{a^6}{r^8}\begin{pmatrix} 15\cos^2\theta + 5 & 5\cos\theta\sin\theta & 0 \\ 5\cos\theta\sin\theta & \dfrac{3-5\cos^2\theta}{2} & 0 \\ 0 & 0 & -\dfrac{1+\cos^2\theta}{2} \end{pmatrix}$$

$$+9\alpha_2 U^2 \frac{a^6}{r^8}\begin{pmatrix} 3\cos^2\theta + 1 & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & 1 & 0 \\ 0 & 0 & \cos^2\theta \end{pmatrix}$$

and (4.4) becomes (see Appendix B)

$$p = \frac{\rho}{2}\frac{\partial U}{\partial t}\frac{a^3}{r^2}\cos\theta - \frac{\rho}{2}U^2\frac{a^3}{r^3}\left\{\left(1 - 3\cos^2\theta\right) + \frac{a^3}{4r^3}\left(1 + 3\cos^2\theta\right)\right\}$$

$$+ 9\hat{\beta}U^2\frac{a^6}{r^8}\left(\cos^2\theta + \frac{1}{2}\right) + \rho\mathbf{g}\bullet\mathbf{x}.$$

We may also write the dissipation integral as

$$\int_V \mathbf{L}[\nabla\phi]\mathbf{:S}[\nabla\phi]dV = \frac{1}{2}\int_V \mathbf{A:}\left[\mu\mathbf{A} + \alpha_1\frac{\partial\mathbf{A}}{\partial t}\right]dV + \frac{1}{2}\int_V \mathbf{A:}\left[\alpha_1(\mathbf{u}\bullet\nabla)\mathbf{A} + (\alpha_1 + \alpha_2)\mathbf{A}^2\right]dV$$

where $dV = 2\pi r^2 \sin\theta d\theta dr$, $0 \le \theta \le \pi$. The last integral vanishes after integrating over θ. Noting next that

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{\dot{U}}{U}\mathbf{A}$$

where $\dot{U} \equiv \partial U / \partial t$, we find that

$$\int_V \mathbf{L}\mathbf{:}\mathbf{S}\,dV = \frac{1}{2}\left(\mu + \alpha_1 \frac{\dot{U}}{U}\right)\int_V \mathbf{A}\mathbf{:}\mathbf{A}\,dV = 12\,\pi a U\left(\mu U + \alpha_1 \dot{U}\right). \tag{5.3}$$

After putting (5.3) into (3.1) with $e = \frac{1}{2}$ and $V_B = \frac{4}{3}\pi a^3$, we obtain

$$D = \pi a\left(\frac{2}{3}a^2\rho + 12\alpha_1\right)\dot{U} + 12\pi a \mu U. \tag{5.4}$$

The main result of this section is (5.4). Since α_1 is negative, we see that the elastic term has a different sign than the acceleration reaction (added mass) term. This then is yet another manifestation of the competition between elasticity and inertia. Elasticity will dominate when

$$-\frac{18\alpha_1}{\rho a^2} > 1.$$

In steady flow the drag on a spherical bubble rising in a second-order fluid is the same as that on a similar bubble rising in a viscous fluid at high Reynolds numbers, independent of the values of α_1 and α_2.

6. Motion of a spherical gas bubble rising in a linear viscoelastic fluid using the dissipation method

If a gas bubble rises through a linear viscoelastic fluid at velocity $U(t)\mathbf{e_x}$ which is nearly steady, the induced flow will be a small perturbation of that for the steady case, and the extra-stress is given by (see Joseph [1990], page 168)

$$\mathbf{S} = \int_{-\infty}^{t} G(t-\tau)\mathbf{A}\left[\mathbf{u}(\boldsymbol{\chi},\tau)\right]d\tau \tag{6.1}$$

where

$$\boldsymbol{\chi} = \mathbf{x} - \mathbf{e_x}\int_{\tau}^{t} U(s)\,ds = \begin{bmatrix} x - \int_{\tau}^{t} U(s)\,ds \\ y \\ z \end{bmatrix}$$

and $G(s) = \frac{\eta}{\lambda} e^{-s/\lambda}$ for the Maxwell model. Suppose that we represent the history of $\mathbf{u}(\boldsymbol{\chi},\tau)$, for $\tau < t$, as a Taylor series around the present value $\tau = t$. Then

$$\mathbf{u}(\boldsymbol{\chi},\tau) = \mathbf{u}(\mathbf{x},t) + \left[\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} + U(t)\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x}\right](\tau - t) + \mathrm{O}\left(|t-\tau|^2\right).$$

Hence,

$$\mathbf{S} = \mu \mathbf{A}[\mathbf{u}(\mathbf{x},t)] + \alpha_1 \left\{\frac{\partial \mathbf{A}[\mathbf{u}(\mathbf{x},t)]}{\partial t} + U(t)\frac{\partial \mathbf{A}[\mathbf{u}(\mathbf{x},t)]}{\partial x}\right\} + \mathrm{O}\left(|t-\tau|^2\right) \tag{6.2}$$

where

$$\mu \overset{\text{def}}{=} \int_{-\infty}^{t} G(t-\tau)\,d\tau \tag{6.3}$$

and

$$\alpha_1 \overset{\text{def}}{=} -\int_{-\infty}^{t} (t-\tau)G(t-\tau)\,d\tau. \tag{6.4}$$

If **u** = ϕ is a potential flow now and in the past, then, from (6.1), div**S** = ψ

where

$$\psi = \int_{-\infty}^{t} G(t-\tau)\nabla^2\phi(\boldsymbol{\chi},\tau)\,d\tau = 0, \tag{6.5}$$

and we get the same Bernoulli equation as in inviscid or viscous potential flow with

$$p = -\rho\frac{\partial \phi}{\partial t} - \rho\frac{|\mathbf{u}|^2}{2} + \rho\mathbf{g}\bullet\mathbf{x} + C(t) \tag{6.6}$$

where $C(t)$ is a constant of integration. Of course the pressure is not needed for the dissipation calculation. We have by (3.1) with $e = \frac{1}{2}$ and $V_B = \frac{4}{3}\pi a^3$ that

$$\frac{2}{3}\pi a^3 \rho \dot{U} = D - \frac{1}{2U}\int_V \mathbf{A}:\mathbf{S}\,dV$$

where **S** is given by (6.1) and

$$\mathbf{A}[\mathbf{u}(\chi,\tau)] = \frac{U(\tau)}{U(t)}\mathbf{A}[\mathbf{u}(\mathbf{x},t)].$$

Following now the procedure used in section 5, we find that

$$D = \frac{2}{3}\pi a^3 \rho \dot{U} + 12\pi a \int_{-\infty}^{t} G(t-\tau)U(\tau)d\tau. \tag{6.7}$$

Using (6.3) and (6.4), we can show that (6.7) reduces to (5.4) when $U(\tau)$ is slowly varying but not necessarily slow. We again get the Levich drag $D = 12\pi a\mu U$ for steady flow.

We intend to test the prediction that the rise velocity of bubbles in viscoelastic liquids, for modest rise velocities, is determined by a balance of weight and drag

$$12\pi a\mu U = \frac{4}{3}\pi a^3 \rho g$$

where ρ is the density of the liquid and g is gravity, independent of any viscoelastic parameter. High frequency back and forth motions of spherical bubbles in viscoelastic liquids might be well described by (6.7).

7. The irrotational motion of rigid bodies in viscoelastic liquids

It is of interest to consider the possibility that irrotational motions of viscoelastic liquids could have an application to real flows in the presence of a solid. Irrotational flows are important in viscous fluid mechanics at high Reynolds numbers, outside boundary layers and separated regions to which the vorticity is effectively confined. There is dissipation in the regions of potential flow, but at high Reynolds numbers this dissipation is usually a negligible part of the total dissipation (see Harper [1972]). There may be special cases of viscous and viscoelastic flows in the presence of solid bodies in which good results can be obtained from potential flow.

The flow of a viscous fluid, which is at rest at infinity, outside a long cylinder of radius a rotating with a steady angular velocity ω is an exact realization of viscous potential flow valid even when the viscosity μ is very large. The exact solution of this problem is given by

$$\mathbf{u} = \frac{\omega a^2}{r}\,\mathbf{e}_\theta \tag{7.1}$$

and it is a potential flow solution of the Navier-Stokes equations with a circulation

$$\Gamma = -2\pi a^2 \omega \tag{7.2}$$

which satisfies the no-slip condition. The viscosity enters this problem through the couple

$$M = 2\mu\Gamma \tag{7.3}$$

required to turn the cylinder.

The same solution (7.1) for the potential vortex holds for a second-order fluid (see Joseph [1990], page 489) and for a linear viscoelastic fluid with $U = 0$ in the steady case. Deiber and Schowalter [1992] have shown how the potential vortex flow (7.1) might be used as a prototype for predictions of polymer behavior in unsteady and turbulent flow. They point out that it is the rotation of the principal axis of stretch as one follows a fluid particle in its circular orbit that distinguishes this flow from the pure stretching flows familiar to polymer rheologists. Unfortunately, the potential vortex is not likely to exist in a class of deformations more severe than ones for which a second order approximation is valid (see section 9).

Joseph and Fosdick [1973] gave a theory of rod climbing based on a retarded motion expansion of the stress for small ω. At first order they get (7.1), (7.2) and (7.3). If the fluid is neutrally wetting with a flat horizontal contact at the rod, the motion vanishes at second-order and the climb can be computed from the normal stress balance at second-order. The same solution can be obtained by assuming that the flow is a potential vortex solution of a second order fluid.

In another kind of application, uniform flow past a circular cylinder, Taneda [1977] has shown that outside a narrow vorticity layer the potential flow solution (no separation) with zero circulation can be achieved by oscillating the cylinder around its axis. The calculation of Ackeret [1952], using the dissipation method, yields $D = 8\pi\mu U$ as the drag per unit length independent of radius a of the cylinder. The same drag per unit length holds for steady uniform flow of a second-order fluid or a linear viscoelastic fluid past a cylinder. Taneda's experiments (his figure 54) were carried out a Reynolds number R_e = 35, diameter = 0.5 cm, U = .33 cm/sec. The frequency of the torsional oscillation was 2Hz and the angle of oscillation 45°. The oscillation periodically disrupts the boundary layer producing a reverse flow at the top and then the bottom of the cylinder in each cycle. We do not know if this periodic disruption of the boundary layer at $R_e = 35$ is sufficient to reduce the contribution to the drag of the vorticity boundary layer to relatively small values and we have no idea of what might develop if Taneda's experiments were to be carried out in a viscoelastic fluid.

8. Force and moment on a two-dimensional body in the flow of a viscous fluid, a second-order fluid and a linear viscoelastic fluid

The main results concerning force and moment of a two-dimensional body in the potential flow of an ideal fluid can be obtained from the Blasius integral formulas. These formulas have been extended to viscous potential flow by Joseph, Liao and Hu [1992]. Here we are seeking a different extension to viscoelastic potential flow of a second-order fluid which contains the viscous fluid as a special case.

Let

$$X\mathbf{e_x} + Y\mathbf{e_y} \overset{\text{def}}{=} \oint_{\partial B} \hat{\mathbf{n}} \bullet \mathbf{T}dl = -\oint_{\partial B} \mathbf{n} \bullet \mathbf{T}dl \tag{8.1}$$

and

$$M \overset{\text{def}}{=} \oint_{\partial B} \mathbf{x} \wedge (\hat{\mathbf{n}} \bullet \mathbf{T}dl) = -\oint_{\partial B} \mathbf{x} \wedge (\mathbf{n} \bullet \mathbf{T}dl) \tag{8.2}$$

where $\hat{\mathbf{n}} = -\mathbf{n}$ is the outward unit normal to the body, $\mathbf{x} \overset{\text{def}}{=} x\mathbf{e_x} + y\mathbf{e_y}$ is the position vector from the origin $\mathbf{o}$, X and Y are forces on the body, and M is the moment about the origin $\mathbf{o}$. The velocity of the flow is given by $\mathbf{u} \overset{\text{def}}{=} u\mathbf{e_x} + v\mathbf{e_y}$. Using the two-dimensional control volume Ω in Figure 8.1, the balance of momentum and balance of angular momentum can be expressed as

$$\frac{d}{dt}\iint_{\Omega} \rho\mathbf{u}dS = \int_{\partial\Omega} \mathbf{n} \bullet \mathbf{T}dl - \int_{\partial\Omega} \rho\mathbf{u}(\mathbf{u} \bullet \mathbf{n})dl + \iint_{\Omega} \rho\mathbf{g}dS, \tag{8.3}$$

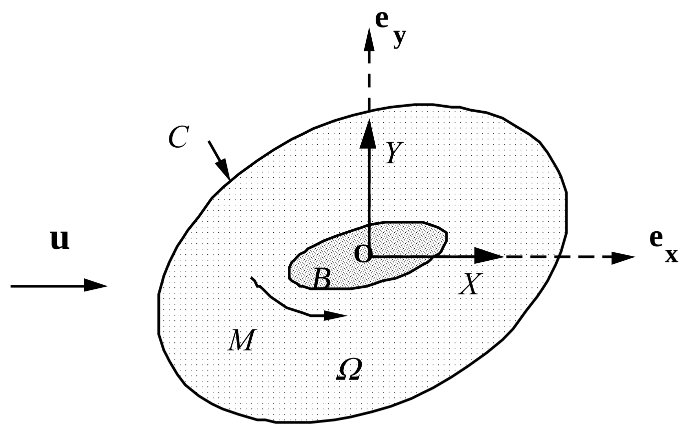

Figure 8.1. In two-dimensional space an arbitrary body B is enclosed by a two-dimensional control volume Ω with outer boundary C and inner boundary ∂B. Here X and Y are the components of the force exerted by the fluid on the body and M is the hydrodynamic couple. $\mathbf{e_x}$ and $\mathbf{e_y}$ are the base vectors in a Cartesian coordinate system with origin **o** inside the body such that at infinity the flow velocity is $\mathbf{u} = U\,\mathbf{e_x}$.

and

$$\frac{d}{dt}\iint_{\Omega}\rho\mathbf{x}\wedge\mathbf{u}dS = \int_{\partial\Omega}\mathbf{x}\wedge(\mathbf{n}\bullet\mathbf{T})dl - \int_{\partial\Omega}\rho\mathbf{x}\wedge[\mathbf{u}(\mathbf{u}\bullet\mathbf{n})]dl + \iint_{\Omega}\rho\mathbf{x}\wedge\mathbf{g}dS \tag{8.4}$$

where $\partial\Omega = C\cup\partial B$. Using (2.1) and (4.4), and applying (8.1) and (8.2), we find that (8.3) and (8.4) can be written as

$$X\mathbf{e_x} + Y\mathbf{e_y} = X_I\mathbf{e_x} + Y_I\mathbf{e_y} + \oint_C \mathbf{n}\bullet\mathbf{S}dl - \oint_C \hat{\beta}\frac{\gamma^2}{2}\mathbf{n}dl \tag{8.5}$$

and

$$M = M_I + \oint_C \mathbf{x}\wedge(\mathbf{n}\bullet\mathbf{S})dl - \oint_C\left(\hat{\beta}\frac{\gamma^2}{2}\right)\mathbf{x}\wedge\mathbf{n}dl \tag{8.6}$$

where

$$X_I\mathbf{e_x} + Y_I\mathbf{e_y} = \oint_C\left(\rho\frac{|\mathbf{u}|^2}{2} - C(t)\right)\mathbf{n}dl - \oint_C \rho\mathbf{u}(\mathbf{u}\bullet\mathbf{n})dl + \oint_{\partial B}\rho\frac{\partial\phi}{\partial t}\hat{\mathbf{n}}dl - \oint_{\partial B}\rho(\mathbf{g}\bullet\mathbf{x})\hat{\mathbf{n}}dl \tag{8.7}$$

and

$$M_I = \oint_C\left(\rho\frac{|\mathbf{u}|^2}{2} - C(t)\right)\mathbf{x}\wedge\mathbf{n}dl - \oint_C \rho\mathbf{x}\wedge\mathbf{u}(\mathbf{u}\bullet\mathbf{n})dl + \oint_{\partial B}\rho\frac{\partial\phi}{\partial t}\mathbf{x}\wedge\hat{\mathbf{n}}dl - \oint_{\partial B}\rho(\mathbf{g}\bullet\mathbf{x})\mathbf{x}\wedge\hat{\mathbf{n}}dl\,. \tag{8.8}$$

We have used the condition $\mathbf{u}\bullet\mathbf{n} = 0$ on ∂B to eliminate integrals $\oint_{\partial B} \rho\mathbf{u}(\mathbf{u}\bullet\mathbf{n})dl$ and $\oint_{\partial B} \rho\mathbf{x}\wedge\mathbf{u}(\mathbf{u}\bullet\mathbf{n})dl$. Notice that the last integrals in (8.7) and (8.8) can also be written as

$$\oint_{\partial B} \rho(\mathbf{g}\bullet\mathbf{x})\hat{\mathbf{n}}dl = \iint_B \nabla(\rho\mathbf{g}\bullet\mathbf{x})dS = M_o\mathbf{g} \tag{8.9}$$

and

$$\oint_{\partial B} \rho(\mathbf{g}\bullet\mathbf{x})(\mathbf{x}\wedge\hat{\mathbf{n}})dl = \iint_B \mathbf{x}\wedge\nabla(\rho\mathbf{g}\bullet\mathbf{x})dS = \mathbf{x}_{\mathrm{cm}}\wedge M_o\mathbf{g} \tag{8.10}$$

where $M_o \overset{\text{def}}{=} \iint_B \rho dS$ is the mass of fluid per unit length displaced by the body and $\mathbf{x}_{\mathrm{cm}} \overset{\text{def}}{=} \frac{1}{M_o}\iint_B \rho\mathbf{x}dS$. Substituting $\mathbf{S}$ from (4.1) and using the relations

$$\mathbf{n}dl = \left(n_{\mathbf{x}}\mathbf{e}_{\mathbf{x}} + n_{\mathbf{y}}\mathbf{e}_{\mathbf{y}}\right)dl = dy\mathbf{e}_{\mathbf{x}} - dx\mathbf{e}_{\mathbf{y}} \text{ on } C,$$

$$\hat{\mathbf{n}}dl = \left(\hat{n}_{\mathbf{x}}\mathbf{e}_{\mathbf{x}} + \hat{n}_{\mathbf{y}}\mathbf{e}_{\mathbf{y}}\right)dl = dy\mathbf{e}_{\mathbf{x}} - dx\mathbf{e}_{\mathbf{y}} \text{ on } \partial B,$$

and the fact that the velocity potential ϕ satisfies Laplace's equation, we find, using the definitions of $\hat{\beta}$ and γ^2, that (8.5) and (8.6) can be written as

$$\begin{aligned} X - iY = X_I - iY_I - 2i\mu\oint_C\left(\frac{dW}{dz}\right)dz \\ -2i\alpha_1\left\{\oint_C\left(\frac{\partial}{\partial t}\left[\frac{dW}{dz}\right]\right)dz + \oint_C\left(\overline{W}\frac{d^2W}{dz^2}\right)dz + \oint_C\left|\frac{dW}{dz}\right|^2 d\bar{z}\right\} \end{aligned} \tag{8.11}$$

and

$$\begin{aligned} M = M_I + \mathrm{Re}\left\{2\mu\oint_C\left(z\frac{dW}{dz}\right)dz\right\} \\ +\mathrm{Re}\left\{2\alpha_1\left[\oint_C z\frac{\partial}{\partial t}\left[\frac{dW}{dz}\right]dz + \oint_C z\overline{W}\frac{d^2W}{dz^2}dz + \oint_C z\left|\frac{dW}{dz}\right|^2 d\bar{z}\right]\right\}. \end{aligned} \tag{8.12}$$

Also, (8.7) and (8.8) become

$$X_I - iY_I = i\frac{\rho}{2}\oint_C W^2dz - i\oint_{\partial B}\rho\left(\mathbf{g}\bullet\mathbf{x} - \frac{\partial\phi}{\partial t}\right)d\bar{z} \tag{8.13}$$

and

$$M_I = \mathrm{Re}\left\{ \frac{-\rho}{2} \oint_C z W^2 \, dz + \oint_{\partial B} \rho \left(\mathbf{g} \bullet \mathbf{x} - \frac{\partial \phi}{\partial t} \right) z d\bar{z} \right\} \tag{8.14}$$

where $W = u - iv$ is the complex velocity, an analytic function of the complex variable $z = x + iy$ and the overbar denotes a complex conjugate. (8.13) and (8.14) are the classical Blasius integral formulas for the flow of an ideal fluid. (8.11) and (8.12) are the generalized formulas for the flow of a second-order fluid.

Since we can always choose a coordinate system such that the flow has $\mathbf{u} = U\mathbf{e_x}$ at infinity, the far-field form of the potential $F(z)$ for flow past a finite body of arbitrary shape is given by

$$F(z) = z\,U + \frac{m+i\Gamma}{2\pi} \ln z + \sum_{k=1}^{\infty} \frac{a_k + ib_k}{z^k} \tag{8.15}$$

where the Γ is the circulation, which is positive if clockwise, m is the volume flux across the boundary of the cylinder, which vanishes for a solid body, and a_k, b_k are real time-dependent constants which are determined by the shape of the body. The complex form of the velocity at far-field is then given by

$$W = \frac{dF}{dz} = U + \frac{m+i\Gamma}{2\pi z} - \sum_{k=1}^{\infty} k \frac{a_k + ib_k}{z^{k+1}} \, .$$

Inserting (8.9), (8.10), (8.13) and (8.14) into (8.11) and (8.12) and letting the outer boundary C approach infinity, we obtain, in view of the asymptotic behavior of W,

$$X\mathbf{e_x} + Y\mathbf{e_y} = X_I\mathbf{e_x} + Y_I\mathbf{e_y} = -\rho m U \mathbf{e_x} + \rho \Gamma U \mathbf{e_y} + \oint_{\partial B} \rho \frac{\partial \phi}{\partial t} \hat{\mathbf{n}} \, dl - M_o \mathbf{g} \tag{8.16}$$

and

$$M = M_I + 2\mu\Gamma + 2\alpha_1 \frac{\partial \Gamma}{\partial t} \tag{8.17}$$

where

$$M_I = -2\rho\pi U b_1 + \frac{\rho m \Gamma}{2\pi} + \oint_{\partial B} \rho \frac{\partial \phi}{\partial t} \mathbf{x} \wedge \hat{\mathbf{n}} \, dl - \mathbf{x}_{\mathrm{cm}} \wedge M_o \mathbf{g} \, .$$

The viscoelastic properties of the fluid do not enter into the expression (8.16) for the forces. The parameter α_2 of the second-order fluid does not enter into the expression (8.17) for the moment and $2\alpha_1 \frac{\partial \Gamma}{\partial t}$ vanishes in steady flow. The forces and moment on an arbitrary simply connected body in two-dimensional steady potential flow of a second-order fluid are the same as in potential flow of a viscous fluid with viscosity μ. Moreover, (8.17) shows that there is moment $M = 2\mu\Gamma + 2\alpha_1 \frac{\partial \Gamma}{\partial t}$ even without a stream.

After carrying out calculations similar to the ones above using the two-dimensional form of the extra-stress (6.1) and the Bernoulli equation (6.6), we find that the force on a two-dimensional body in the flow of a linear viscoelastic fluid is

$$X - iY = X_I - iY_I - 2i \int_{-\infty}^{t} \left[G(t-\tau) \oint_C \left(\frac{dW}{dz} \right) dz \right] d\tau$$

and the moment is given by

$$M = M_I + \mathrm{Re} \left\{ 2 \int_{-\infty}^{t} \left[G(t-\tau) \oint_C \left(z \frac{dW}{dz} \right) dz \right] d\tau \right\}.$$

The far-field potential (8.15) holds here and shows that $X - iY = X_I - iY_I$ and

$$M = M_I + 2 \int_{-\infty}^{t} [G(t-\tau)\Gamma(\tau)] d\tau. \qquad (8.18)$$

Again, (8.18) reduces to (8.17) when Γ is slowly varying, in view of (6.3) and (6.4).

9. Special potential flow solutions of models like Maxwell's

Most models of a viscoelastic fluid will not admit a Bernoulli equation in general. But there are certain potential flows which satisfy the required conditions even for models which do not generally have a Bernoulli equation. For example, uniform flow is a potential flow solution for every model. So too is any motion for which div **S** = 0, say **S** is independent of **x**, as in extensional flow. A less trivial example, the potential vortex, is more representative. Among all of the interpolated Maxwell models, only the upper convected model (UCM) and lower convected model (LCM) can support a potential vortex. The existence of a potential flow solution is a precise mathematical problem equivalent to an examination of the conditions for the existence of solutions to an over-determined problem. We can formulate this problem as follows. The six stress equations in the six components of the extra-stress **S** can generally be solved when the flow is prescribed; that is, for each and every potential flow. The compatibility condition for potential flow

$$\nabla \wedge (\nabla \bullet \mathbf{S}) = 0 \tag{9.1}$$

gives rise to three extra equations for the six components of the stress so that we have three equations too many. In two dimensions we find four equations for three unknowns. When this over-determined system of equations allows a solution, we may solve (2.9) for ψ and the pressure is then given by

$$p = -\rho \frac{\partial \phi}{\partial t} - \rho \frac{|\nabla \phi|^2}{2} + \psi + C(t). \tag{9.2}$$

Potential vortex and sink flow are used to illustrate the concept. And the constitutive equations considered in this section are of the form

$$\lambda\left(\frac{\partial \mathbf{S}}{\partial t} + (\mathbf{u} \bullet \nabla)\mathbf{S} - \frac{1+a}{2}(\mathbf{LS} + \mathbf{SL}^T) + \frac{1-a}{2}(\mathbf{SL} + \mathbf{L}^T\mathbf{S}) \right) + \mathbf{SF} = 2\eta \mathbf{D} \tag{9.3}$$

where $-1 \le a \le 1$ and $\mathbf{F} = \mathbf{I}$ (the unit tensor) for the interpolated Maxwell model, $\mathbf{F} = \mathbf{I} + (\alpha\lambda / \eta)\mathbf{S}$ for the Giesekus model and $\mathbf{F} = [1 + (\varepsilon\lambda / \eta)\mathrm{tr}\mathbf{S}]\mathbf{I}$ for the Phan-Thien and Tanner model, where α and ε are constants. It is convenient to study vortex and sink flow in a plane polar coordinate system. The stress dyad then takes the form

$$\mathbf{S} = \sigma \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \tau \hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}} + \tau \hat{\boldsymbol{\theta}} \otimes \hat{\mathbf{r}} + \gamma \hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}} = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix}.$$

For plane potential flows, (9.1) and (9.3) may be expressed in component form as

$$\begin{cases} -\frac{1}{r}\sigma_{,\theta r} - \frac{1}{r^2}\sigma_{,\theta} + \tau_{,rr} - \frac{1}{r^2}\tau_{,\theta\theta} + \frac{3}{r}\tau_{,r} + \frac{1}{r^2}\gamma_{,\theta} + \frac{1}{r}\gamma_{,\theta r} = 0, \\ \dot{\sigma} + \frac{f_\sigma}{\lambda} + \sigma_{,r}\phi_{,r} + \frac{1}{r^2}\sigma_{,\theta}\phi_{,\theta} - 2a\sigma\phi_{,rr} + \frac{2(a-1)}{r^2}\tau\phi_{,\theta} - \frac{2a}{r}\tau\phi_{,r\theta} = 2G\phi_{,rr}, \\ \dot{\tau} + \frac{f_\tau}{\lambda} + \frac{(a+1)}{r^2}\sigma\phi_{,\theta} - \frac{a}{r}\sigma\phi_{,r\theta} + \tau_{,r}\phi_{,r} + \frac{1}{r^2}\tau_{,\theta}\phi_{,\theta} - \frac{a}{r}\gamma\phi_{,r\theta} + \frac{(a-1)}{r^2}\gamma\phi_{,\theta} = \frac{2G}{r}(\phi_{,r\theta} - \frac{1}{r}\phi_{,\theta}), \\ \dot{\gamma} + \frac{f_\gamma}{\lambda} + \frac{2(a+1)}{r^2}\tau\phi_{,\theta} - \frac{2a}{r}\tau\phi_{,r\theta} + \frac{1}{r^2}\gamma_{,\theta}\phi_{,\theta} + \gamma_{,r}\phi_{,r} - \frac{2a}{r^2}\gamma\phi_{,\theta\theta} - \frac{2a}{r}\gamma\phi_{,r} = \frac{2G}{r}(\frac{1}{r}\phi_{,\theta\theta} + \phi_{,r}) \end{cases} \tag{9.4}$$

where $G = \eta/\lambda$, and $\overset{\nabla}{g} \overset{\text{def}}{=} \partial g / \partial t$. To distinguish between different models f_σ, f_τ, and f_γ are assigned according to Table 9.1.

Model	f_σ	f_τ	f_γ
Interpolated Maxwell	σ	τ	γ
Giesekus	$\sigma + \frac{\alpha}{G}(\sigma^2 + \tau^2)$	$\tau + \frac{\alpha}{G}(\sigma + \gamma)\tau$	$\gamma + \frac{\alpha}{G}(\gamma^2 + \tau^2)$
Phan-Thien and Tanner	$\sigma + \frac{\varepsilon}{G}(\sigma + \gamma)\sigma$	$\tau + \frac{\varepsilon}{G}(\sigma + \gamma)\tau$	$\gamma + \frac{\varepsilon}{G}(\sigma + \gamma)\gamma$

Table 9.1. f_σ, f_τ and f_γ for different models

Consider the potential vortex, $\phi(\theta) = b\theta$, where $b = \omega r_0^2$, ω is a constant angular velocity, and $\omega r_0^2 / r$ is the velocity (in circles). For steady, axisymmetric flow, (9.4) reduces to

$$\begin{cases} \tau_{,rr} + \frac{3}{r}\tau_{,r} = 0, \\ \frac{f_\sigma}{\lambda} + \frac{2(a-1)b}{r^2}\tau = 0, \\ \frac{f_\tau}{\lambda} + \frac{(a+1)b}{r^2}\sigma + \frac{(a-1)b}{r^2}\gamma = -\frac{2Gb}{r^2}, \\ \frac{f_\gamma}{\lambda} + \frac{2(a+1)b}{r^2}\tau = 0. \end{cases} \tag{9.5}$$

A solution of (9.5) for the interpolated model is given by

$$\begin{cases} \tau = C_1 r^{-2} + C_0, \\ \sigma = \frac{-2(a-1)b\lambda}{r^2}\tau, \\ \tau = -\frac{2Gb\lambda}{r^2 - 4(a^2-1)b^2\lambda^2 r^{-2}}, \\ \gamma = \frac{-2(a+1)b\lambda}{r^2}\tau \end{cases} \tag{9.6}$$

where C_1 and C_0 are constants. Equating the first and third equations of (9.6), we get

$$\left(2Gb\lambda + C_1\right) - 4C_1(a^2-1)b^2\lambda r^{-4} + C_0 r^2 - 4C_0(a^2-1)b^2\lambda r^{-2} = 0\,. \tag{9.7}$$

Since (9.7) is true for all $r > r_0$, the coefficients of different powers of r must vanish; this implies $C_0 = 0$, $C_1 = -2Gb\lambda$ and $a^2 - 1 = 0$. Thus, solutions exist only when $a = 1$ or -1. When $a = 1$ (UCM), we have

$$\mathbf{S} = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix} = 2G\frac{\lambda b}{r^2}\begin{bmatrix} 0 & -1 \\ -1 & 4\lambda b / r^2 \end{bmatrix},$$

$$\psi = \frac{2G\lambda^2 b^2}{r^4} + C,$$

and

$$p(r) = -\frac{\rho b^2}{2r^2} + \frac{2G\lambda^2 b^2}{r^4} + C.$$

When $a = -1$ (LCM), we have

$$\mathbf{S} = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix} = 2G\frac{\lambda b}{r^2}\begin{bmatrix} -4\lambda b / r^2 & -1 \\ -1 & 0 \end{bmatrix},$$

$$\psi = -\frac{6G\lambda^2 b^2}{r^4} + C,$$

and

$$p(r) = -\frac{\rho b^2}{2r^2} - \frac{6G\lambda^2 b^2}{r^4} + C.$$

We next evaluate (9.5) for the Giesekus model. When $\tau = 0$,

- if $\alpha = \frac{a-1}{2}$ then $\mathbf{S} = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2G/(a-1) \end{bmatrix}$ and $\alpha \neq 0 \Rightarrow a \neq 1$,
- if $\alpha = \frac{a+1}{2}$ then $\mathbf{S} = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix} = \begin{bmatrix} -2G/(a+1) & 0 \\ 0 & 0 \end{bmatrix}$ and $\alpha \neq 0 \Rightarrow a \neq -1$,
- if $\alpha = a$ then $\mathbf{S} = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix} = -\frac{G}{a}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\alpha \neq 0 \Rightarrow a \neq 0$.

Since $\mathbf{S}$ is constant in this case, we have a constant ψ and the pressure is given by the usual Bernoulli equation.

If, on the other hand, $\tau = C_1 r^{-2} + C_0 \neq 0$, which is the solution of the first equation of (9.5), then we may solve the second and fourth equations of (9.5), and obtain

$$\sigma(r) = -\frac{G}{2\alpha} \pm \frac{\sqrt{g(r)}}{2\alpha r^2} \quad \text{and} \quad \gamma(r) = -\frac{G}{2\alpha} \pm \frac{\sqrt{h(r)}}{2\alpha r^2}$$

where

$$g(r) = -4\alpha^2 (C_1 + C_0 r^2)^2 + 8\alpha b G\lambda (C_1 + C_0 r^2)(1-a) + G^2 r^4$$

and

$$h(r) = -4\alpha^2 (C_1 + C_0 r^2)^2 - 8\alpha b G\lambda (C_1 + C_0 r^2)(1+a) + G^2 r^4 .$$

These solutions must satisfy the third equation, which, after some calculation, may be written as

$$(2\alpha - a)\frac{G\lambda b}{\alpha} + C_0 \frac{\pm\sqrt{g(r)}}{2G} + C_0 \frac{\pm\sqrt{h(r)}}{2G}$$
$$+\left(\frac{(a+1)\lambda b G}{\alpha} + C_1\right)\frac{\pm\sqrt{g(r)}}{2Gr^2} + \left(\frac{(a-1)\lambda b G}{\alpha} + C_1\right)\frac{\pm\sqrt{h(r)}}{2Gr^2} = 0$$

where the $\pm$ signs in front of $\sqrt{g(r)}$ and $\sqrt{h(r)}$ are independent. This equation holds only when all the coefficients vanish, implying that $a + 1 = a - 1$ which is impossible. Therefore, the only solutions are those with $\tau = 0$. However, this is a strange potential vortex without torque and constant normal stresses. It does not appear to be physically acceptable.

If the Phan-Thien and Tanner model is adopted, we find that

$$\mathbf{S} = G\begin{bmatrix} -1 + (a-1)/(2\varepsilon) & 0 \\ 0 & 1 - (a+1)/(2\varepsilon) \end{bmatrix}$$

for $\tau = 0$. As mentioned before, this stress is unacceptable. When $\tau = C_1 r^{-2} + C_0 \neq 0$ we find two solutions for σ and γ:

$$\begin{bmatrix} \sigma \\ \gamma \end{bmatrix} = \begin{bmatrix} G(1-a)(1-q(r))/(4a\varepsilon) \\ -G(1+a)(1-q(r))/(4a\varepsilon) \end{bmatrix} \text{ and } \begin{bmatrix} \sigma \\ \gamma \end{bmatrix} = \begin{bmatrix} G(1-a)(1+q(r))/(4a\varepsilon) \\ -G(1+a)(1+q(r))/(4a\varepsilon) \end{bmatrix} \quad (9.8)$$

where $q(r) = \frac{1}{r^2}\sqrt{r^4 - C_0 16 ab\varepsilon\lambda r^2 / G - 16 C_1 ab\varepsilon\lambda / G}$. These solutions must also satisfy the rate equation for τ which may be expressed as

$$\left[(a+1)\lambda b + c\left(C_1 + C_o r^2\right)\right]\sigma + \left[(a-1)\lambda b + c\left(C_1 + C_o r^2\right)\right]\gamma + 2\lambda G b + \left(C_1 + C_o r^2\right) = 0. \quad (9.9)$$

Substituting σ and γ from (9.8) into (9.9), we obtain

$$(1-a^2)G\lambda b + 4a\varepsilon\lambda G b + a\varepsilon C_1 + a\varepsilon C_o r^2 \pm q(r)\left[(1-a^2)G\lambda b - a\varepsilon C_1 - a\varepsilon C_o r^2\right] = 0.$$

Hence, $C_0 = 0$, $(1-a^2)G\lambda b + 4a\varepsilon\lambda G b + a\varepsilon C_1 = 0$ and $(1-a^2)G\lambda b - a\varepsilon C_1 = 0$. The last two equations imply $a^2 - 2\varepsilon a - 1 = 0$. So, we have $a = \varepsilon \pm \sqrt{\varepsilon^2 + 1}$, $C_1 = -2G\lambda b$ and $q(r) = r^{-2}\sqrt{r^4 + 32a\varepsilon b^2\lambda^2}$. This $q(r)$ is well-defined because the adjustable constant ε is non-negative. Since $-1 \le a \le 1$, only one of the two solutions for a is acceptable. The extra-stress is

$$\mathbf{S} = \begin{bmatrix} -G(1 \pm \frac{1}{r^2}\sqrt{r^4 + 32a\varepsilon b^2\lambda^2})/[2(1+a)] & -2Gb\lambda / r^2 \\ -2Gb\lambda / r^2 & G(1 \pm \frac{1}{r^2}\sqrt{r^4 + 32a\varepsilon b^2\lambda^2})/[2(1-a)] \end{bmatrix}.$$

Tnis solution rules out the case when $a = 1$ or -1. We also have

$$\psi = \frac{-G}{2(1+a)}(1 \pm \frac{1}{r^2}\sqrt{r^4 + 32a\varepsilon b^2\lambda^2}) - \frac{G}{2(1-a^2)}\left[\log(r) \pm \int \frac{\sqrt{r^4 + 32a\varepsilon b^2\lambda^2}}{r^3} dr\right]$$

and

$$p(r) = C - \frac{\rho b^2}{2r^2} - \frac{G}{2(1+a)}(1 \pm \frac{1}{r^2}\sqrt{r^4 + 32a\varepsilon b^2\lambda^2}) - \frac{G}{2(1-a^2)}\left[\log(r) \pm \int \frac{\sqrt{r^4 + 32a\varepsilon b^2\lambda^2}}{r^3} dr\right].$$

Potential vortex solutions of Maxwell models are possible only for the upper and lower convected models. The Giesekus and Phan-Thien and Tanner models replace the linear term $\mathbf{S}/\lambda$ with a nonlinear term, chosen so as to fix up the fluid response, to avoid unpleasant singularities and other maladies. The potential vortex solutions of these nonlinear models are not unique. One of the two solutions is unphysical and the other requires non-generic relations among the material parameters. The FENE-P model does not even produce a solution (see Appendix C).

We next examine the possibility of superposing a potential vortex and sink, confining our study to the interpolated Maxwell model. The reader may verify that the potential $\phi = m\log(r)$ for a sink of constant strength m satisfies (9.4) for steady, axisymmetric flow when

$$\tau = 0,$$

$$\sigma(r) = r^{-2a}\exp\left(-\frac{r^2}{2m\lambda}\right)\cdot\left\{\int_r^\infty\left[2Gr^{(2a-1)}\exp\left(\frac{r^2}{2m\lambda}\right)\right]dr + C_1\right\},$$

and

$$\gamma(r) = -r^{2a}\exp\left(-\frac{r^2}{2m\lambda}\right)\cdot\left\{\int_r^\infty\left[2Gr^{(-2a-1)}\exp\left(\frac{r^2}{2m\lambda}\right)\right]dr + C_2\right\}$$

for all values of a $[-1, 1]$. Moreover,

$$\begin{aligned}\psi = -2G\log(r) - \int_r^\infty \frac{1}{\lambda m\exp\left[r^2/(2\lambda m)\right]}\Big\{ & C_1 r^{-1-2a}\left(\lambda m - 2a\lambda m - r^2\right) \\ & - C_2\lambda m r^{-1+2a} + 2\lambda m G r^{-1+2a}\int_r^\infty s^{-1-2a}\exp\left[s^2/(2\lambda m)\right]ds \\ & - 2Gr^{-1-2a}(-\lambda m + 2a\lambda m + r^2)\int_r^\infty s^{-1+2a}\exp\left[s^2/(2\lambda m)\right]ds\Big\}dr\end{aligned}$$

and

$$\begin{aligned}p(r) = -\frac{\rho b^2}{2r^2} + C - 2G\log(r) - \int_r^\infty \frac{1}{\lambda m\exp\left[r^2/(2\lambda m)\right]}\Big\{ & C_1 r^{-1-2a}\left(\lambda m - 2a\lambda m - r^2\right) \\ & - C_2\lambda m r^{-1+2a} + 2\lambda m G r^{-1+2a}\int_r^\infty s^{-1-2a}\exp\left[s^2/(2\lambda m)\right]ds \\ & - 2Gr^{-1-2a}(-\lambda m + 2a\lambda m + r^2)\int_r^\infty s^{-1+2a}\exp\left[s^2/(2\lambda m)\right]ds\Big\}dr.\end{aligned}$$

Since (9.1) and the constitutive equations are nonlinear, the superposition of two potential flow solutions is not automatically a solution. Consider the superposition of the sink flow and the potential vortex under the assumption that the components of stress only depend on r. We find that (9.4) reduces to

$$
\begin{cases}
\tau_{,rr} + \dfrac{3}{r}\tau_{,r} = 0, \\
\dfrac{\sigma}{\lambda} + \sigma_{,r}\dfrac{m}{r} + 2a\sigma\dfrac{m}{r^2} + \dfrac{2(a-1)b}{r^2}\tau = -\dfrac{2Gm}{r^2}, \\
\dfrac{\tau}{\lambda} + \dfrac{(a+1)b}{r^2}\sigma + \dfrac{m}{r}\tau_{,r} + \dfrac{(a-1)b}{r^2}\gamma = -\dfrac{2Gb}{r^2}, \\
\dfrac{\gamma}{\lambda} + \dfrac{2(a+1)b}{r^2}\tau + \gamma_{,r}\dfrac{m}{r} - \dfrac{2am}{r^2}\gamma = \dfrac{2Gm}{r^2}.
\end{cases}
\tag{9.10}
$$

When $\tau = 0$, (9.10) gives

$$
\begin{cases}
a(a+1)\sigma = G\left(1 - \dfrac{1}{2m\lambda}r^2 + a\right), \\
a(a-1)\gamma = G\left(1 - \dfrac{1}{2m\lambda}r^2 - a\right).
\end{cases}
\tag{9.11}
$$

We have to verify that (9.11) will satisfy the second and fourth equations of (9.10). Multiplying the second equation of (9.10) by $r^2 a(a+1)/m$ and using (9.11), we get

$$
\begin{aligned}
ra(a+1)\sigma_{,r} + (\frac{1}{m\lambda}r^2 + 2a)a(a+1)\sigma &= rG\left(-\frac{1}{m\lambda}r\right) + (\frac{1}{m\lambda}r^2 + 2a)G\left(1 - \frac{1}{2m\lambda}r^2 + a\right) \\
&= 2G\left(a(a+1) - \frac{1}{4m^2\lambda^2}r^4\right) \neq -2a(a+1)G.
\end{aligned}
$$

Hence, there is no solution when $\tau = 0$.

When $\tau = C_1 r^{-2} + C_0 \neq 0$, we may solve for σ and γ from the second and fourth equations of (9.10) and obtain

$$
\begin{aligned}
\sigma = & \frac{C_2}{r^{2a}\exp[r^2/(2\lambda m)]} \\
& + \frac{2}{mr^{2a}\exp[r^2/(2\lambda m)]}\int r^{2a-1}\exp[r^2/(2\lambda m)]\{b(1-a)(C_1 r^{-2} + C_0) - Gm\}dr
\end{aligned}
\tag{9.12}
$$

and

$$
\begin{aligned}
\gamma = & \frac{r^{2a}C_3}{\exp[r^2/(2\lambda m)]} \\
& - \frac{2r^{2a}}{m\exp[r^2/(2\lambda m)]}\int r^{-2a-1}\exp[r^2/(2\lambda m)]\{b(1+a)(C_1 r^{-2} + C_0) - Gm\}dr.
\end{aligned}
\tag{9.13}
$$

After substituting (9.12) and (9.13) into the third equation of (9.10), we obtain

$$
\begin{aligned}
&\frac{C_0 m}{\lambda} r^2 \exp[r^2/(2\lambda m)] + (\frac{C_1}{\lambda} + 2Gb - C_1 \frac{2m}{r^2}) m \exp[r^2/(2\lambda m)] \\
&\quad + C_2 m(a+1) b r^{-2a} + C_3 r^{2a} m(a-1) b \\
&\quad + C_1 2 r^{-2a} (1-a^2) b^2 \int r^{2a-3} \exp[r^2/(2\lambda m)] dr \\
&\quad + C_1 2 r^{2a} (1-a^2) b^2 \int r^{-2a-3} \exp[r^2/(2\lambda m)] dr \\
&\quad + C_0 2 r^{-2a} (1-a^2) b^2 \int r^{2a-1} \exp[r^2/(2\lambda m)] dr \\
&\quad + C_0 2 r^{2a} (1-a^2) b^2 \int r^{-2a-1} \exp[r^2/(2\lambda m)] dr \\
&\quad - 2 r^{2a} (1-a) b G m \int r^{-2a-1} \exp[r^2/(2\lambda m)] dr \\
&\quad - 2 r^{-2a} (a+1) b G m \int r^{2a-1} \exp[r^2/(2\lambda m)] dr = 0.
\end{aligned}
\tag{9.14}
$$

We find that (9.14) can be satisfied only when $a = 1$ (UCM) or $a = -1$ (LCM). When a=1, (9.14) reduces to

$$
\frac{C_0}{\lambda m} r^2 + \frac{2bC_2}{mr^2 \exp\left(r^2/(2\lambda m)\right)} + \left(2G\lambda b + C_1\right)\left(\frac{1}{\lambda m} - \frac{2}{r^2}\right) = 0.
$$

This implies that $C_0 = 0$, $C_2 = 0$, and $C_1 = -2G\lambda b = -2\eta b$. Hence

$$
\tau = \frac{-2G\lambda b}{r^2},
$$

$$
\sigma = -\frac{2G\lambda m}{r^2},
$$

$$
\gamma = \frac{C_3 r^2}{\exp\left(r^2/(2m\lambda)\right)} - \frac{2G\lambda b^2}{mr^2} - \frac{G(b^2+m^2)}{m^2}\left(1 - \frac{r^2 \mathrm{Ei}\left[r^2/(2m\lambda)\right]}{2\lambda m \exp\left(r^2/(2m\lambda)\right)}\right),
$$

$$
\begin{aligned}
\psi = &-\frac{G\lambda(m^2+b^2)}{mr^2} + \log[r]\frac{G(b^2+m^2)}{m^2} \\
&- C_3\left(m\lambda - \frac{m\lambda}{\exp\left(r^2/(2m\lambda)\right)}\right) + \frac{G(b^2+m^2)}{2\lambda m^3}\int_r^\infty \frac{r \mathrm{Ei}\left[r^2/(2m\lambda)\right]}{\exp\left(r^2/(2m\lambda)\right)} dr,
\end{aligned}
$$

and

$$p = C - \frac{(\rho m + 2G\lambda)(m^2+b^2)}{2mr^2} + \log[r]\frac{G(b^2+m^2)}{m^2}$$
$$- C_3\left(m\lambda - \frac{m\lambda}{\exp\left(r^2/(2m\lambda)\right)}\right) + \frac{G(b^2+m^2)}{2\lambda m^3}\int_r^\infty \frac{r\mathrm{Ei}\left[r^2/(2m\lambda)\right]}{\exp\left(r^2/(2m\lambda)\right)}dr$$

where C_3 is a constant and $\mathrm{Ei}[z]$ is an exponential integral function defined by

$$\mathrm{Ei}[z] \overset{def}{=} -\int_{-z}^{\infty}\frac{e^{-t}}{t}dt.$$

When $a=-1$, (9.14) reduces to

$$\frac{C_0}{\lambda m}r^2 - \frac{2bC_3}{mr^2\exp\left(r^2/(2\lambda m)\right)} + \left(2G\lambda b + C_1\right)\left(\frac{1}{\lambda m} - \frac{2}{r^2}\right) = 0$$

which implies that $C_0 = 0$, $C_3 = 0$ and $C_1 = -2G\lambda b = -2\eta b$. Hence

$$\tau = \frac{-2G\lambda b}{r^2}, \tag{9.15}$$

$$\sigma = \frac{C_2 r^2}{\exp\left(r^2/(2m\lambda)\right)} - \frac{2G\lambda b^2}{mr^2} + \frac{G(b^2+m^2)}{m^2}\left(1 - \frac{r^2\mathrm{Ei}\left[r^2/(2m\lambda)\right]}{2\lambda m\exp\left(r^2/(2m\lambda)\right)}\right), \tag{9.16}$$

$$\gamma = \frac{2G\lambda m}{r^2}, \tag{9.17}$$

$$\psi = \frac{G\lambda(m^2-b^2)}{mr^2} + \frac{G(b^2+m^2)}{m^2}(1+\log[r]) + C_2\left(\frac{r^2 - m\lambda}{\exp\left(r^2/(2m\lambda)\right)} + m\lambda\right)$$
$$-\frac{G(b^2+m^2)}{2\lambda m^3}\left(\frac{r^2\mathrm{Ei}\left[r^2/(2m\lambda)\right]}{\exp\left(r^2/(2m\lambda)\right)} - \int_r^\infty \frac{r\mathrm{Ei}\left[r^2/(2m\lambda)\right]}{\exp\left(r^2/(2m\lambda)\right)}dr\right), \tag{9.18}$$

and

$$p = \frac{2G\lambda(m^2-b^2) - \rho m(m^2+b^2)}{2mr^2} + \frac{G(b^2+m^2)}{m^2}(1+\log[r]) + C_2\left(\frac{r^2 - m\lambda}{\exp\left(r^2/(2m\lambda)\right)} + m\lambda\right)$$
$$-\frac{G(b^2+m^2)}{2\lambda m^3}\left(\frac{r^2\mathrm{Ei}\left[r^2/(2m\lambda)\right]}{\exp\left(r^2/(2m\lambda)\right)} - \int_r^\infty \frac{r\mathrm{Ei}\left[r^2/(2m\lambda)\right]}{\exp\left(r^2/(2m\lambda)\right)}dr\right) + C \tag{9.19}$$

where C_2 is a constant. Equations (9.15) through (9.16) define potential flow fields which are generated by a superposed sink and potential vortex.

We turn next to three dimensions and look for a solution for the components of the extra-stress in the interpolated Maxwell model for sink flow, $\phi = m/r$, using (9.1) and (9.3) (such solutions are incompletely discussed by Joseph [1990]). The general formulas for (9.1) and (9.3) with $\mathbf{F} = \mathbf{I}$ in spherical coordinates (r, θ, φ) are given in Appendix C. Substituting $\phi = m/r$ into those formulas and assuming that the components of $\mathbf{S}$ depend only on r, we obtain nine equations for the six components of the extra-stress

$$\mathbf{S} = \sigma\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \gamma\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}} + \beta\hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\varphi}} + \tau\left(\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}} \otimes \hat{\mathbf{r}}\right) + \delta\left(\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\theta}}\right) + \kappa\left(\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}} \otimes \hat{\mathbf{r}}\right).$$

These equations take the following form:

$$\frac{\cot\theta}{r}\tau_{,r} - \frac{2}{r^2}\delta + \frac{3\cot\theta}{r^2}\kappa = 0\,,$$

$$\frac{1}{r}\tau_{,r} + \tau_{,rr} + \frac{2\cot\theta}{r}\delta_{,r} + \frac{3}{r}\kappa_{,r} = 0\,,$$

$$\frac{\cot\theta}{r}\gamma_{,r} - \frac{\cot\theta}{r}\beta_{,r} + \frac{1}{r^2\sin^2\theta}\tau + \frac{4}{r}\tau_{,r} + \tau_{,rr} = 0,$$

$$\frac{1}{\lambda}\sigma - \frac{m}{r^2}\sigma_{,r} - \frac{4am}{r^3}\sigma = \frac{4Gm}{r^3}\,,$$

$$\frac{1}{\lambda}\gamma - \frac{m}{r^2}\gamma_{,r} + \frac{2am}{r^3}\gamma = -\frac{2Gm}{r^3}\,,$$

$$\frac{1}{\lambda}\beta - \frac{m}{r^2}\beta_{,r} + \frac{2am}{r^3}\beta = -\frac{2Gm}{r^3}\,,$$

$$\frac{1}{\lambda}\tau - \frac{m}{r^2}\tau_{,r} - \frac{am}{r^3}\tau = 0\,,$$

$$\frac{1}{\lambda}\delta - \frac{m}{r^2}\delta_{,r} + \frac{2am}{r^3}\delta = 0,$$

and

$$\frac{1}{\lambda}\kappa - \frac{m}{r^2}\kappa_{,r} - \frac{am}{r^3}\kappa = 0$$

where first three equations come from (9.1) and the last six equations come from the constitutive equation. Solving these nine equations, we find that

$$\mathbf{S} = \begin{bmatrix} \sigma & \tau & \kappa \\ \tau & \gamma & \delta \\ \kappa & \delta & \beta \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

$$= \operatorname{diag}\begin{bmatrix} r^{-4a}\exp\left[r^3/(3\lambda m)\right]\cdot\left\{4G\int_r^\infty r^{(-1+4a)}\exp\left[-r^3/(3\lambda m)\right]dr + C_1\right\} \\ -r^{2a}\exp\left[r^3/(3\lambda m)\right]\cdot\left\{2G\int_r^\infty r^{(-1-2a)}\exp\left[-r^3/(3\lambda m)\right]dr + C_2\right\} \\ -r^{2a}\exp\left[r^3/(3\lambda m)\right]\cdot\left\{2G\int_r^\infty r^{(-1-2a)}\exp\left[-r^3/(3\lambda m)\right]dr + C_2\right\} \end{bmatrix} \tag{9.20}$$

where C_1 and C_2 are constants, and obtain

$$\begin{aligned} \psi = {} & C_1 r^{-4a}\exp\left[r^3/(3\lambda m)\right] - 2C_1\int_r^\infty r^{-4a-1}\exp\left[r^3/(3\lambda m)\right]dr \\ & -2C_2\int_r^\infty r^{2a-1}\exp\left[r^3/(3\lambda m)\right]dr \\ & +4Gr^{-4a}\exp\left[r^3/(3\lambda m)\right]\int_r^\infty r^{(-1+4a)}\exp\left[-r^3/(3\lambda m)\right]dr \\ & -8G\int_r^\infty\left\{r^{-4a-1}\exp\left[r^3/(3\lambda m)\right]\int_r^\infty s^{(-1+4a)}\exp\left[-s^3/(3\lambda m)\right]ds\right\}dr \\ & -4G\int_r^\infty\left\{r^{2a-1}\exp\left[r^3/(3\lambda m)\right]\int_r^\infty s^{(-1-2a)}\exp\left[-s^3/(3\lambda m)\right]ds\right\}dr \end{aligned} \tag{9.21}$$

and

$$\begin{aligned} p = {} & C - \frac{\rho m^2}{2r^4} + C_1 r^{-4a}\exp\left[r^3/(3\lambda m)\right] - 2C_1\int_r^\infty r^{-4a-1}\exp\left[r^3/(3\lambda m)\right]dr \\ & -2C_2\int_r^\infty r^{2a-1}\exp\left[r^3/(3\lambda m)\right]dr \\ & +4Gr^{-4a}\exp\left[r^3/(3\lambda m)\right]\int_r^\infty r^{(-1+4a)}\exp\left[-r^3/(3\lambda m)\right]dr \end{aligned}$$

$$-8G\int_r^\infty \left\{ r^{-4a-1} \exp\left[r^3/(3\lambda m)\right] \int_r^\infty s^{(-1+4a)} \exp\left[-s^3/(3\lambda m)\right] ds \right\} dr$$
$$-4G\int_r^\infty \left\{ r^{2a-1} \exp\left[r^3/(3\lambda m)\right] \int_r^\infty s^{(-1-2a)} \exp\left[-s^3/(3\lambda m)\right] ds \right\} dr\,. \qquad (9.22)$$

The formulas (9.20) through (9.22) define the fields generated by a sink (or source) flow of an interpolated Maxwell model in three dimensions.

10. Discussion

The theory of potential flows of an inviscid fluid can be readily extended to a theory of potential flow of viscoelastic fluids which admit a pressure (Bernoulli) function. We have developed some of this theory for Newtonian fluids, linearly viscoelastic fluids and second-order fluids. The unsteady drag on a body in potential flow is independent of the viscosity and of the viscoelastic parameters for the models studied. However, there are additional viscous and unsteady viscoelastic moments associated with circulation in planar motions. These additional moments could play a role in the dynamics of flow in doubly connected regions of three-dimensional space, e.g. in the dynamics of vortex rings. It is evident that the various vorticity and circulation theorems which are at the foundation of the theory of inviscid potential flow hold also when the viscosity and model viscoelastic parameters are not zero. In addition, the theory of viscous and viscoelastic potential flow admits approximations to real flows through the use of dissipation and vorticity layer methods. For example, the dissipation theory predicts that the drag on a rising spherical gas bubble in a viscoelastic fluid is the same as the (Levich) drag on this bubble in a viscous fluid with the same viscosity and density when the rise velocity is steady but not when it is unsteady. The pressure on solid bodies and bubbles in viscous liquids is well approximated by potential flow when separation is suppressed even when, as for the solid body, the drag is determined by the dissipation in the viscous vorticity layer at the boundary. It is therefore not unreasonable to hope that the shapes of gas bubbles rising in viscoelastic fluids at moderate and perhaps moderately large speeds can be predicted from forces associated with viscoelastic potential flows.

Concepts from the theory of viscous and viscoelastic potential flow have something to say about the phenomenon of vortex inhibition. Gordon and Balakrishnan [1972] report that"...remarkably small quantities of certain high molecular weight polymers inhibit the tendency of water to form a vortex, as it drains from a large tank..." and they discuss the phenomenon from a molecular point of view, noting that the same high molecular weight polymers which are effective drag reducers also work to inhibit the 'bathtub" vortex. The "bathtub" vortex for an inviscid fluid is frequently modeled by superposing a potential vortex and a sink subject to the condition that the pressure at the unknown position of the free surface is atmospheric. In more sophisticated models account is taken for the fact that the vortex core does not reduce its diameter indefinitely, but tends to a constant value obtained by superposing a potential vortex and a uniform axial motion subject to the same pressure condition. This asymptotic regime is in the long straight part of the vortex tube near the drain hole shown in the sketch of Figure 1 of Gordon and Balakrishnan [1972] and in the first panel of the photograph of the same experiment shown as Figure 2.5-11 in Bird, Armstrong, and Hassager [1987]. We can imagine an exact harmonic function which satisfies all the asymptotic limits which we have listed and is such that the pressure in the Bernoulli equation is atmospheric at the free surface $z = h(r)$. Exactly the same solution satisfies the equations for viscous potential flow with the added caveat that the vanishing of the shear stress at the free surface cannot be satisfied by viscous potential flow. However, the "Levich type" vorticity layer which would develop at the free surface to accommodate this missing condition can be expected to be weak in the sense that its relative strength in an energy balance as well as its thickness will decrease as the Reynolds number increases.

Obviously the aforementioned modeling fails dismally for most models and for some of the currently most popular models of a viscoelastic fluid and if we think that the dilute solutions used in the experiments of Gordon and Balakrishnan [1972] are viscoelastic, then we should expect vortex inhibition even without the molecular arguments. Indeed, molecular ideas seem to involve the idea of strong extensional flow, but the steady vortex which drains from the hole is perhaps modeled by the superposition of a potential vortex and a uniform axial flow which has no extensional component whatever.

The polymeric solutions used in the vortex inhibition experiments are in the same range of extreme dilution, say 10 ppm, as in experiments on drag reduction (see Berman [1978] for a review) or the anomalous transport of heat and mass in the flow across wires (see Joseph [1990] for a review). It is apparent that in spite of the fact that the aqueous polymeric liquids used in these experiments have surpassingly small weight fractions, they are responding like viscoelastic liquids. In fact the usual ideas like those of Rouse and his followers do not work since the drag reduction is never linear in the concentration, no matter how small (see Berman [1978], p.56).

The theory of rod climbing is based on the potential vortex at the lowest order in an expansion in which the second order fluid is the first nontrivial approximation to the stress for slow motions. This theory shows that for small $r < \sqrt{4\hat{\beta}/\rho}$, where $\hat{\beta} = \lambda\eta$ is the climbing constant for Maxwell models, the effect of normal stresses is cause the free surface to rise rather than sink. For aqueous drag reducers we may guess that $\eta \approx 10^{-2}$, $\lambda \approx 2\times10^{-3}$ (Joseph [1990]) so that in the region $r < 10^{-1}$ mm the vortex inhibition is suppressed by normal stresses.

Our analysis has led us to definite conclusions about potential flows of viscous and viscoelastic fluids. Some special fluids, like inviscid, viscous, linear viscoelastic and second order fluids, admit potential flow generally and give rise to Bernoulli functions. Other fluids will not admit potential flows unless the compatibility condition (9.1) is satisfied. This leads to an over-determined system of equations for the components of the stress. Special potential flow solutions, like uniform flow and simple extension, satisfy these extra conditions automatically and other special solutions can satisfy the equations for some models and not for others. It appears that only very simple potential flows are admissible for general models. This lack of general admissibility greatly complicates the study of boundary layers for viscoelastic liquids.

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Appendix A: **The divergence of the extra stress tensor**

$\nabla \bullet \mathbf{S}$ in (4.2) is computed as follows:

First, $\nabla \bullet \mathbf{A}$, $\nabla \bullet \mathbf{B}$, and $\nabla \bullet \mathbf{A}^2$ are calculated:

$$(\nabla\bullet\mathbf{A})_i = \frac{\partial}{\partial x_k}A_{ik} = u_{i,kk}+u_{k,ik} = (\nabla^2\mathbf{u})_i + \frac{\partial}{\partial x_i}(\nabla\bullet\mathbf{u}) = (\nabla^2\mathbf{u})_i \qquad \text{(A.1)}$$

Since

$$\frac{\partial(\nabla\bullet\mathbf{A})}{\partial t} + (\mathbf{u}\bullet\nabla)(\nabla\bullet\mathbf{A}) = \frac{d(\nabla\bullet\mathbf{A})}{dt},$$

$$\begin{aligned}
L_{nk}\frac{\partial A_{ik}}{\partial x_n} + L_{nk}\frac{\partial A_{in}}{\partial x_k} + L_{ni}\frac{\partial A_{nk}}{\partial x_k} &= L_{kn}\frac{\partial A_{in}}{\partial x_k} + L_{nk}\frac{\partial A_{in}}{\partial x_k} + L_{ni}\frac{\partial A_{nk}}{\partial x_k}\\
&= (L_{kn}+L_{nk})\frac{\partial A_{in}}{\partial x_k} + L_{ni}\frac{\partial A_{nk}}{\partial x_k} = A_{nk}\frac{\partial A_{ni}}{\partial x_k} + L_{ni}\frac{\partial A_{nk}}{\partial x_k}\\
&= \frac{\partial A_{nk}A_{ni}}{\partial x_k} - \frac{\partial A_{nk}}{\partial x_k}A_{in} + (\mathbf{L}^T)_{in}\frac{\partial A_{nk}}{\partial x_k}\\
&= (\nabla\bullet\mathbf{A}^2)_i + \left[(\mathbf{L}^T-\mathbf{A})\bullet(\nabla\bullet\mathbf{A})\right]_i,
\end{aligned}$$

$$\begin{aligned}
A_{in}\frac{\partial L_{nk}}{\partial x_k} &= A_{in}\frac{\partial A_{nk}}{\partial x_k} - A_{in}\frac{\partial L_{kn}}{\partial x_k} = A_{in}(\nabla\bullet\mathbf{A})_n - A_{in}\frac{\partial L_{kk}}{\partial x_n}\\
&= [\mathbf{A}\bullet(\nabla\bullet\mathbf{A})]_i - A_{in}\frac{\partial(\nabla\bullet\mathbf{u})}{\partial x_n} = [\mathbf{A}\bullet(\nabla\bullet\mathbf{A})]_i,
\end{aligned}$$

and

$$A_{nk}\frac{\partial L_{ni}}{\partial x_k} = A_{nk}\frac{\partial L_{nk}}{\partial x_i} = A_{nk}\frac{\partial A_{nk}}{\partial x_i} - A_{nk}\frac{\partial L_{kn}}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{tr\mathbf{A}^2}{2}\right) - A_{nk}\frac{\partial L_{ni}}{\partial x_k}$$

$$\square\ A_{nk}\frac{\partial L_{ni}}{\partial x_k} = \frac{\partial}{\partial x_i}\left(\frac{tr\mathbf{A}^2}{4}\right) = \left[\nabla\left(\frac{tr\mathbf{A}^2}{4}\right)\right]_i,$$

we have

$$\begin{aligned}
(\nabla\bullet\mathbf{B})_i &= \frac{\partial(\nabla\bullet\mathbf{A})_i}{\partial t} + \{\nabla\bullet[(\mathbf{u}\bullet\nabla)\mathbf{A}]\}_i + [\nabla\bullet(\mathbf{AL}+\mathbf{L}^T\mathbf{A})]_i\\
&= \frac{\partial(\nabla\bullet\mathbf{A})_i}{\partial t} + (\mathbf{u}\bullet\nabla)(\nabla\bullet\mathbf{A})_i + L_{nk}\frac{\partial A_{ik}}{\partial x_n} + \frac{\partial}{\partial x_k}(A_{in}L_{nk}+L_{ni}A_{nk})\\
&= \frac{\partial(\nabla\bullet\mathbf{A})_i}{\partial t} + (\mathbf{u}\bullet\nabla)(\nabla\bullet\mathbf{A})_i + L_{nk}\frac{\partial A_{ik}}{\partial x_n}\\
&\qquad + L_{nk}\frac{\partial A_{in}}{\partial x_k} + L_{ni}\frac{\partial A_{nk}}{\partial x_k} + A_{in}\frac{\partial L_{nk}}{\partial x_k} + A_{nk}\frac{\partial L_{ni}}{\partial x_k}\\
&= \left\{\frac{d}{dt}(\nabla\bullet\mathbf{A}) + \mathbf{L}^T\bullet(\nabla\bullet\mathbf{A}) + \nabla\left(\frac{tr\mathbf{A}^2}{4}\right) + \nabla\bullet\mathbf{A}^2\right\}_i. \qquad \text{(A.2)}
\end{aligned}$$

$$\left(\nabla \bullet \mathbf{A}^2\right)_i = \frac{\partial A_{kn}A_{ni}}{\partial x_k} = A_{kn}\frac{\partial A_{ni}}{\partial x_k} + A_{ni}\frac{\partial A_{kn}}{\partial x_k} = A_{nk}\frac{\partial L_{nk}}{\partial x_i} + A_{kn}\frac{\partial L_{ik}}{\partial x_n} + \left[\mathbf{A}\bullet(\nabla \bullet \mathbf{A})\right]_i$$

$$= \frac{\partial}{\partial x_i}\left(\frac{tr\mathbf{A}^2}{2}\right) - A_{nk}\frac{\partial L_{kn}}{\partial x_i} + A_{kn}\frac{\partial L_{ik}}{\partial x_n} + \left[\mathbf{A}\bullet(\nabla \bullet \mathbf{A})\right]_i$$

$$= \left[\nabla\left(\frac{tr\mathbf{A}^2}{2}\right) + \mathbf{A}\bullet(\nabla \bullet \mathbf{A})\right]_i + A_{kn}\frac{\partial}{\partial x_n}\left(L_{ik} - L_{ki}\right)$$

$$= \left[\nabla\left(\frac{tr\mathbf{A}^2}{2}\right) + \mathbf{A}\bullet(\nabla \bullet \mathbf{A})\right]_i + \frac{\partial \Omega_{ik}}{\partial x_n}A_{kn}$$

$$= \left[\nabla\left(\frac{tr\mathbf{A}^2}{2}\right) + \mathbf{A}\bullet(\nabla \bullet \mathbf{A}) + (\nabla\mathbf{\Omega})\bullet \mathbf{A}\right]_i \qquad \text{(A.3)}$$

where $\Omega_{ij} \equiv \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = L_{ij} - L_{ji} = -\varepsilon_{ijk}\omega_k$, and $\nabla\mathbf{\Omega} = \frac{\partial}{\partial x_k}\Omega_{ij}\hat{\mathbf{i}}\otimes\hat{\mathbf{j}}\otimes\hat{\mathbf{k}}$. That is, $\mathbf{\Omega}$ + $\mathbf{L}-\mathbf{L}^T$, twice the antisymmetric part of $\mathbf{L}$. Hence $\mathbf{L} = \frac{\mathbf{A}+\mathbf{\Omega}}{2}$ and $\mathbf{L}^T = \frac{\mathbf{A}-\mathbf{\Omega}}{2}$.

Combining (A.1), (A.2) and (A.3), we have

$$\nabla \bullet \mathbf{S} = \mu\nabla \bullet \mathbf{A} + \alpha_1\nabla \bullet \mathbf{B} + \alpha_2\nabla \bullet \mathbf{A}^2$$

$$= \mu\nabla \bullet \mathbf{A} + \alpha_1\frac{d}{dt}(\nabla \bullet \mathbf{A}) + \alpha_1\mathbf{L}^T \bullet (\nabla \bullet \mathbf{A}) + \alpha_1\nabla\left(\frac{tr\mathbf{A}^2}{4}\right)$$
$$+(\alpha_1 + \alpha_2)\left[\nabla\left(\frac{tr\mathbf{A}^2}{2}\right) + \mathbf{A}\bullet(\nabla \bullet \mathbf{A}) + \nabla\mathbf{\Omega}\bullet \mathbf{A}\right]$$

$$= \mu\nabla^2\mathbf{u} + \alpha_1\left[\frac{d\nabla^2\mathbf{u}}{dt} + \mathbf{L}^T \bullet \left(\nabla^2\mathbf{u}\right)\right]$$
$$+ (\alpha_1 + \alpha_2)\left[\mathbf{A}\bullet\left(\nabla^2\mathbf{u}\right) + \nabla\mathbf{\Omega}\bullet\mathbf{A}\right] + \frac{\hat{\beta}}{2}\nabla\gamma^2$$

where $\hat{\beta} = 3\alpha_1 + 2\alpha_2$, $\gamma^2 = \frac{tr(\mathbf{A}^2)}{2}$, and $\nabla \bullet \mathbf{A} = \nabla^2\mathbf{u}$.

Appendix B: **The pressure equation for a rising spherical bubble in a second-order fluid**

The computation of $\partial\phi / \partial t$ is somewhat delicate because

$$\phi = -\frac{1}{2}U\frac{a^3}{r^2}\cos\theta \tag{B.1}$$

is computed relative to an origin moving with the sphere, so that even when U is constant the motion is not steady. Following the analysis given in section 15·33 by Milne-Thomson [1960] we find that

$$\frac{\partial\phi}{\partial t} = U^2\frac{a^3}{2r^3}(\sin^2\theta - 2\cos^2\theta) - \dot{U}\frac{a^3}{2r^2}\cos\theta . \tag{B.2}$$

To compute the pressure formula (4.4) we need to compute γ^2 where **A** is calculated on potential flow relative to spherical coordinates. Relative to this basis, we have

$$[\mathbf{A}] = \begin{bmatrix} 2\frac{\partial u_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta} & 0 \\ r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta} & 2\left(\frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r}\right) & 0 \\ 0 & 0 & 2\left(\frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right) \end{bmatrix}$$

$$= -3U\frac{a^3}{r^4}\begin{bmatrix} 2\cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -\cos\theta \end{bmatrix}. \tag{B.3}$$

Hence,

$$\gamma^2 = \frac{1}{2}tr(\mathbf{A}^2) = 9U^2\frac{a^6}{r^8}\left(2\cos^2\theta + 1\right). \tag{B.4}$$

The pressure now can be computed from (4.4) using (B.1), (B.2) and (B.4). We find that

$$p - p_\infty = \frac{\rho}{2}\dot{U}\frac{a^3}{r^2}\cos\theta + \frac{\rho}{2}U^2\frac{a^3}{r^3}\left(3\cos^2\theta - 1\right) - \frac{\rho}{8}U^2\frac{a^6}{r^6}\left(1 + 3\cos^2\theta\right)$$
$$+ \frac{9\hat{\beta}}{2}U^2\frac{a^6}{r^8}\left(2\cos^2\theta + 1\right) + \rho\mathbf{g}\bullet\mathbf{x} . \tag{B.5}$$

The pressure is not required for the dissipation calculation. Notice that, (B.2) is also used to compute the time derivative in the tensor **B** which occurs in the extra stress tensor **S**.

Appendix C: **The nonexistence of a potential vortex in a FENE-P fluid**

The constitutive equation of the FENE-P model is (see Bird, Armstrong, and Hassanger [1987], p. 410)

$$\frac{1}{\lambda}Z\mathbf{S}+\left(\frac{\partial\mathbf{S}}{\partial t}+(\mathbf{u}\bullet\nabla)\mathbf{S}-\mathbf{L}\mathbf{S}-\mathbf{S}\mathbf{L}^{T}\right)$$
$$-\left(\mathbf{S}+(1-\varepsilon b)G\mathbf{I}\right)\left\{\frac{\partial\ln Z}{\partial t}+(\mathbf{u}\bullet\nabla)\ln Z\right\}=2(1-\varepsilon b)G\mathbf{D} \tag{C.1}$$

where $G=\eta/\lambda$, $\varepsilon=2/(b^{2}+2b)$, b is a constant, and

$$Z=1+\left(\frac{3}{b}\right)\left(1+\frac{\lambda\mathrm{tr}\mathbf{S}}{3\eta}\right).$$

The compatibility condition for potential flow is

$$\nabla\wedge(\nabla\bullet\mathbf{S})=0. \tag{C.2}$$

In polar coordinates, (C.1) and (C.2), with stress $\mathbf{S}=\sigma\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}+\tau(\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}}+\hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}})+\gamma\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}$ and velocity potential ϕ, can be written as

$$-\frac{1}{r}\sigma_{,\theta r}-\frac{1}{r^{2}}\sigma_{,\theta}+\tau_{,rr}-\frac{1}{r^{2}}\tau_{,\theta\theta}+\frac{3}{r}\tau_{,r}+\frac{1}{r^{2}}\gamma_{,\theta}+\frac{1}{r}\gamma_{,\theta r}=0, \tag{C.3}$$

$$\frac{Z}{\lambda}\sigma+\left(\dot{\sigma}+\sigma_{,r}\phi_{,r}+\frac{1}{r^{2}}\sigma_{,\theta}\phi_{,\theta}-2\sigma\phi_{,rr}-\frac{2}{r}\tau\phi_{,r\theta}\right)$$
$$-\left[\frac{1}{bG}\sigma+\frac{1-\varepsilon b}{b}\right]\frac{1}{Z}\left(\dot{\sigma}+\dot{\gamma}+\phi_{,r}(\sigma_{,r}+\gamma_{,r})+\frac{\phi_{,\theta}}{r^{2}}(\sigma_{,\theta}+\gamma_{,\theta})\right)=2(1-\varepsilon b)G\phi_{,rr},, \tag{C.4}$$

$$\frac{Z}{\lambda}\tau+\left(\dot{\tau}+\frac{2}{r^{2}}\sigma\phi_{,\theta}-\frac{1}{r}\sigma\phi_{,\theta r}+\tau_{,r}\phi_{,r}+\frac{1}{r^{2}}\tau_{,\theta}\phi_{,\theta}-\frac{1}{r}\gamma\phi_{,r\theta}\right)$$
$$-\tau\frac{1}{bG}\frac{1}{Z}\left(\dot{\sigma}+\dot{\gamma}+\phi_{,r}(\sigma_{,r}+\gamma_{,r})+\frac{\phi_{,\theta}}{r^{2}}(\sigma_{,\theta}+\gamma_{,\theta})\right)=(1-\varepsilon b)\frac{2G}{r}\left(\phi_{,r\theta}-\frac{1}{r}\phi_{,\theta}\right), \tag{C.5}$$

$$\frac{Z}{\lambda}\gamma+\left(\dot{\gamma}+\frac{4}{r^{2}}\tau\phi_{,\theta}-\frac{2}{r}\tau\phi_{,\theta r}+\frac{1}{r^{2}}\gamma_{,\theta}\phi_{,\theta}+\gamma_{,r}\phi_{,r}-\frac{2}{r^{2}}\gamma\phi_{,\theta\theta}-\frac{2}{r}\gamma\phi_{,r}\right)$$

$$-\left[\frac{1}{bG}\gamma+\frac{1-\varepsilon b}{b}\right]\frac{1}{Z}\left(\dot{\sigma}+\dot{\gamma}+\phi_{,r}(\sigma_{,r}+\gamma_{,r})+\frac{\phi_{,\theta}}{r^2}(\sigma_{,\theta}+\gamma_{,\theta})\right)$$
$$=(1-\varepsilon b)\frac{2G}{r}\left(\frac{1}{r}\phi_{,\theta\theta}+\phi_{,r}\right) \tag{C.6}$$

where

$$Z=1+\left(\frac{3}{b}\right)\left(1+\frac{\lambda \mathrm{tr}\mathbf{S}}{3\eta}\right)=1+\frac{3}{b}+\frac{1}{bG}(\sigma+\gamma). \tag{C.7}$$

For potential vortex $\phi(\theta)=\beta\theta$, where $\beta=\omega r_0^2$ and ω is a constant angular velocity, and under the assumption that **S** is axisymmetric and steady, (C.3) through (C.6) reduces to:

$$\tau_{,rr}+\frac{3}{r}\tau_{,r}=0, \tag{C.8}$$

$$\frac{Z}{\lambda}\sigma=0, \tag{C.9}$$

$$\frac{Z}{\lambda}\tau+\frac{2\beta}{r^2}\sigma=-(1-\varepsilon b)\frac{2G\beta}{r^2}, \tag{C.10}$$

and

$$\frac{Z}{\lambda}\gamma+\frac{4\beta}{r^2}\tau=0. \tag{C.11}$$

The solution of (C.8) is

$$\tau=C_1 r^{-2}+C_0 \tag{C.12}$$

where C_0 and C_1 are to-be-determined constants. On the other hand, since Z □ 0, (C.9) implies $\sigma=0$, and thus, (C.7), (C.10) and (C.11) reduce to

$$Z=1+\frac{3}{b}+\frac{1}{bG}\gamma, \tag{C.13}$$

$$\frac{Z}{\lambda}\tau=-(1-\varepsilon b)\frac{2G\beta}{r^2}, \tag{C.14}$$

and

$$\frac{Z}{\lambda}\gamma + \frac{4\beta}{r^2}\tau = 0. \tag{C.15}$$

Substituting (C.13) into (C.14) and (C.15), and then eliminating γ, we obtain another equation for τ as follows:

$$\left(\frac{bG^2\eta\beta(1-\varepsilon b)^2}{r^2} + \frac{G^2(b+3)(1-\varepsilon b)}{2}\tau\right) + \tau^3 = 0. \tag{C.16}$$

The constants C_0 and C_1 should be determined from the compatibility between (C.14) and (C.16). Hence, after inserting (C.14) into (C.16), we get

$$C_1^3 r^{-6} + 3C_0 C_1^2 r^{-4} + \left(bG^2\eta\beta(1-\varepsilon b)^2 + C_1\frac{(b+3)G^2(1-\varepsilon b)}{2} + 3C_1C_0^2\right)r^{-2} + \left(C_0^3 + C_0\frac{(b+3)G^2(1-\varepsilon b)}{2}\right) = 0. \tag{C.17}$$

Since (C.17) is true for all $r>r_0$, the coefficients of different powers of r must vanish; this implies $bG^2\eta\beta(1-\varepsilon b)^2 = 0$ which is impossible. Therefore, the FENE-P model cannot support a potential vortex.

Appendix D: **Formulas for the interpolated Maxwell model and the compatibility condition for potential flow in spherical coordinates**

In this appendix, we will list all the necessary formulas to express (9.1) and (9.3) in spherical coordinates with basis $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\}$ as shown in Figure D.1. The general formulas for the interpolated Maxwell model will be given first without assuming the potential flow, then we give reduced formulas for potential flow and finally we give formulas for the compatibility condition $\boxed{\nabla\wedge(\nabla\bullet\mathbf{S}) = 0}$.

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}},\ \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{r}},\ \frac{\partial\hat{\mathbf{r}}}{\partial\varphi} = \hat{\boldsymbol{\varphi}}\sin\theta,\ \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\varphi} = \hat{\boldsymbol{\varphi}}\cos\theta,\ \frac{\partial\hat{\boldsymbol{\varphi}}}{\partial\varphi} = -\hat{\mathbf{r}}\sin\theta - \hat{\boldsymbol{\theta}}\cos\theta,$$

$$\frac{\partial f}{\partial\xi} \equiv f_{,\xi} \text{ and } \frac{\partial f}{\partial t} \equiv \dot{f},$$

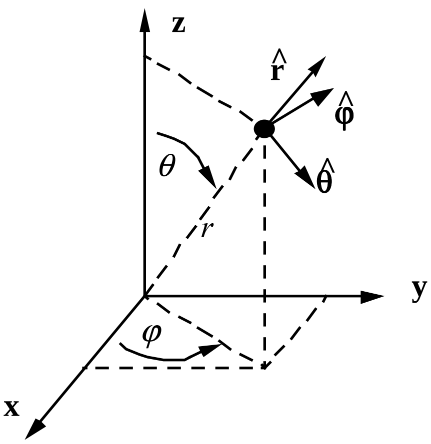

Figure D.1. Spherical coordinates

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi},$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2},$$

$$[\mathbf{d}] = \begin{bmatrix} d_r \\ d_\theta \\ d_\varphi \end{bmatrix} \leftrightarrow \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{bmatrix} \text{ (matrix form of the vector } \mathbf{d} = d_r\hat{\mathbf{r}} + d_\theta\hat{\boldsymbol{\theta}} + d_\varphi\hat{\boldsymbol{\varphi}}\text{)},$$

$$\nabla \wedge \mathbf{d} = \frac{1}{r^2\sin\theta}\begin{bmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ d_r & rd_\theta & r\sin\theta d_\varphi \end{bmatrix} = \begin{bmatrix} \frac{\cot\theta}{r}d_\varphi + \frac{1}{r}\frac{\partial d_\varphi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial d_\theta}{\partial \varphi} \\ \frac{1}{r\sin\theta}\frac{\partial d_r}{\partial \varphi} - \frac{\partial d_\varphi}{\partial r} - \frac{d_\varphi}{r} \\ \frac{d_\theta}{r} + \frac{\partial d_\theta}{\partial r} - \frac{1}{r}\frac{\partial d_r}{\partial \theta} \end{bmatrix},$$

$$\begin{aligned}\mathbf{k} = {} & k_{rr}\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} + k_{r\theta}\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}} + k_{r\varphi}\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\varphi}} + k_{\theta r}\hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}} \\ & + k_{\theta\theta}\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}} + k_{\theta\varphi}\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\varphi}} + k_{\varphi r}\hat{\boldsymbol{\varphi}}\otimes\hat{\mathbf{r}} + k_{\varphi\theta}\hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\theta}} + k_{\varphi\varphi}\hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\varphi}},\end{aligned}$$

$$[\mathbf{k}] = \begin{bmatrix} k_{rr} & k_{r\theta} & k_{r\varphi} \\ k_{\theta r} & k_{\theta\theta} & k_{\theta\varphi} \\ k_{\varphi r} & k_{\varphi\theta} & k_{\varphi\varphi} \end{bmatrix} \leftrightarrow \begin{bmatrix} \hat{\mathbf{r}}\otimes\hat{\mathbf{r}} & \hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}} & \hat{\mathbf{r}}\otimes\hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}} & \hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{\varphi}}\otimes\hat{\mathbf{r}} & \hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\varphi}} \end{bmatrix},$$

$$\nabla \bullet \mathbf{k} = \begin{bmatrix} \frac{\partial k_{rr}}{\partial r} + \frac{\partial k_{\theta r}}{r\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial k_{\varphi r}}{\partial \varphi} + 2\frac{k_{rr}}{r} - \frac{k_{\theta\theta}}{r} - \frac{k_{\varphi\varphi}}{r} + \frac{k_{\theta r}\cot\theta}{r} \\ \frac{\partial k_{r\theta}}{\partial r} + \frac{\partial k_{\theta\theta}}{r\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial k_{\varphi\theta}}{\partial \varphi} + 2\frac{k_{r\theta}}{r} + \frac{k_{\theta r}}{r} + \frac{k_{\theta\theta}\cot\theta}{r} - \frac{k_{\varphi\varphi}\cot\theta}{r} \\ \frac{\partial k_{r\varphi}}{\partial r} + \frac{\partial k_{\theta\varphi}}{r\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial k_{\varphi\varphi}}{\partial \varphi} + 2\frac{k_{r\varphi}}{r} + \frac{k_{\varphi r}}{r} + \frac{k_{\theta\varphi}\cot\theta}{r} + \frac{k_{\varphi\theta}\cot\theta}{r} \end{bmatrix},$$

$$\mathbf{u} = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

$$\mathbf{S} = \sigma\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \gamma\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}} + \beta\hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\varphi}} + \tau\left(\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}} \otimes \hat{\mathbf{r}}\right)$$
$$+ \delta\left(\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\theta}}\right) + \kappa\left(\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}} \otimes \hat{\mathbf{r}}\right),$$

$$\mathbf{S} = \begin{bmatrix} s_{rr} & s_{r\theta} & s_{r\varphi} \\ s_{\theta r} & s_{\theta\theta} & s_{\theta\varphi} \\ s_{\varphi r} & s_{\varphi\theta} & s_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} \sigma & \tau & \kappa \\ \tau & \gamma & \delta \\ \kappa & \delta & \beta \end{bmatrix},$$

$$\mathbf{L} = u_{,r}\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \left(\frac{u_{,\theta}}{r} - \frac{v}{r}\right)\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}} + \left(\frac{u_{,\varphi}}{r\sin\theta} - \frac{w}{r}\right)\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\varphi}} + v_{,r}\hat{\boldsymbol{\theta}} \otimes \hat{\mathbf{r}} + \left(\frac{v_{,\theta}}{r} + \frac{u}{r}\right)\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}$$
$$+ \left(\frac{v_{,\varphi}}{r\sin\theta} - \frac{w\cot\theta}{r}\right)\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\varphi}} + w_{,r}\hat{\boldsymbol{\varphi}} \otimes \hat{\mathbf{r}} + \frac{w_{,\theta}}{r}\hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\theta}} + \left(\frac{u}{r} + \frac{v\cot\theta}{r} + \frac{w_{,\varphi}}{r\sin\theta}\right)\hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\varphi}},$$

$$\mathbf{L} = \begin{bmatrix} u_{,r} & \frac{u_{,\theta}}{r} - \frac{v}{r} & \frac{u_{,\varphi}}{r\sin\theta} - \frac{w}{r} \\ v_{,r} & \frac{v_{,\theta}}{r} + \frac{u}{r} & \frac{v_{,\varphi}}{r\sin\theta} - \frac{w\cot\theta}{r} \\ w_{,r} & \frac{w_{,\theta}}{r} & \frac{u}{r} + \frac{v\cot\theta}{r} + \frac{w_{,\varphi}}{r\sin\theta} \end{bmatrix},$$

Rate equations for interpolated Maxwell models:

We should express the rate equation

$$\lambda\left(\frac{\partial \mathbf{S}}{\partial t} + (\mathbf{u} \bullet \nabla)\mathbf{S} - \frac{1+a}{2}(\mathbf{LS} + \mathbf{SL}^T) + \frac{1-a}{2}(\mathbf{SL} + \mathbf{L}^T\mathbf{S})\right) + \mathbf{S} = 2\eta\mathbf{D}$$

in spherical polar coordinates in component forms with $G = \eta / \lambda$.

$\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}$ component:

$$\dot{\sigma}+(\frac{1}{\lambda}-2au_{,r})\sigma+u\sigma_{,r}+\frac{v}{r}\sigma_{,\theta}+\frac{w}{r\sin\theta}\sigma_{,\varphi}-\tau\left((1+a)\frac{u_{,\theta}}{r}+(1-a)(\frac{v}{r}-v_{,r})\right)$$
$$-\kappa\left((1+a)\frac{u_{,\varphi}}{r\sin\theta}+(1-a)(\frac{w}{r}-w_{,r})\right)=2u_{,r}G.$$

$\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}$ component:

$$\dot{\gamma}+\left\{\frac{1}{\lambda}-\frac{2a}{r}\left(v_{,\theta}+u\right)\right\}\gamma+u\gamma_{,r}+\frac{v}{r}\gamma_{,\theta}+\frac{w}{r\sin\theta}\gamma_{,\varphi}+\tau\left((1-a)\frac{u_{,\theta}}{r}+(1+a)(\frac{v}{r}-v_{,r})\right)$$
$$-\frac{\delta}{r}\left((1+a)\frac{v_{,\varphi}}{\sin\theta}+(1-a)(w\cot\theta-w_{,\theta})\right)=\frac{2G}{r}\left(v_{,\theta}+u\right).$$

$\hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\varphi}}$ component:

$$\dot{\beta}+\left\{\frac{1}{\lambda}-\frac{2a}{r}\left(u+v\cot\theta+\frac{w_{,\varphi}}{\sin\theta}\right)\right\}\beta+u\beta_{,r}+\frac{v}{r}\beta_{,\theta}+\frac{w}{r\sin\theta}\beta_{,\varphi}$$
$$+\kappa\left((1-a)\frac{u_{,\varphi}}{r\sin\theta}+(1+a)(\frac{w}{r}-aw_{,r})\right)+\frac{\delta}{r}\left((1-a)\frac{v_{,\varphi}}{\sin\theta}+(1+a)(w\cot\theta-w_{,\theta})\right)$$
$$=\frac{2G}{r}\left(u+v\cot\theta+\frac{w_{,\varphi}}{\sin\theta}\right).$$

$\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}}$ components:

$$\dot{\tau}+\left\{\frac{1}{\lambda}-a\left(\frac{v_{,\theta}}{r}+\frac{u}{r}+u_{,r}\right)\right\}\tau+u\tau_{,r}+\frac{v}{r}\tau_{,\theta}+\frac{w}{r\sin\theta}\tau_{,\varphi}+\frac{\sigma}{2}\left((1-a)\frac{u_{,\theta}}{r}+(1+a)(\frac{v}{r}-v_{,r})\right)$$
$$-\frac{\gamma}{2}\left((1+a)\frac{u_{,\theta}}{r}+(1-a)(\frac{v}{r}-v_{,r})\right)-\frac{\delta}{2}\left((1+a)\frac{u_{,\varphi}}{r\sin\theta}+(1-a)(\frac{w}{r}-w_{,r})\right)$$
$$-\frac{\kappa}{2r}\left((1+a)\frac{v_{,\varphi}}{\sin\theta}+(1-a)(w\cot\theta-w_{,\theta})\right)=G\left(v_{,r}+\frac{u_{,\theta}}{r}-\frac{v}{r}\right).$$

$\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\theta}}$ components:

$$\begin{aligned}
&\dot{\delta}+\left\{\frac{1}{\lambda}-\frac{a}{r}\left(2u+v_{,\theta}+v\cot\theta+\frac{w_{,\varphi}}{\sin\theta}\right)\right\}\delta+u\delta_{,r}+\frac{v}{r}\delta_{,\theta}+\frac{w}{r\sin\theta}\delta_{,\varphi}\\
&+\frac{\gamma}{2r}\left((1-a)\frac{v_{,\varphi}}{\sin\theta}+(1+a)(w\cot\theta-w_{,\theta})\right)-\frac{\beta}{2r}\left((1+a)\frac{v_{,\varphi}}{\sin\theta}+(1-a)(w\cot\theta-w_{,\theta})\right)\\
&+\frac{\tau}{2}\left((1-a)\frac{u_{,\varphi}}{r\sin\theta}+(1+a)(\frac{w}{r}-w_{,r})\right)+\frac{\kappa}{2}\left((1-a)\frac{u_{,\theta}}{r}+(1+a)(\frac{v}{r}-v_{,r})\right)\\
&=\frac{G}{r}\left(\frac{v_{,\varphi}}{\sin\theta}-w\cot\theta+w_{,\theta}\right).
\end{aligned}$$

$\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{\varphi}}\otimes\hat{\mathbf{r}}$ components:

$$\begin{aligned}
&\dot{\kappa}+\left\{\frac{1}{\lambda}-a\left(u_{,r}+\frac{u}{r}+\frac{v\cot\theta}{r}+\frac{w_{,\varphi}}{r\sin\theta}\right)\right\}\kappa+u\kappa_{,r}+\frac{v}{r}\kappa_{,\theta}+\frac{w}{r\sin\theta}\kappa_{,\varphi}\\
&+\frac{\sigma}{2}\left((1-a)\frac{u_{,\varphi}}{r\sin\theta}+(1+a)(\frac{w}{r}-w_{,r})\right)-\frac{\beta}{2}\left((1+a)\frac{u_{,\varphi}}{r\sin\theta}+(1-a)(\frac{w}{r}-w_{,r})\right)\\
&+\frac{\tau}{2r}\left((1-a)\frac{v_{,\varphi}}{\sin\theta}+(1+a)(w\cot\theta-w_{,\theta})\right)-\frac{\delta}{2}\left((1+a)\frac{u_{,\theta}}{r}+(1-a)(\frac{v}{r}-v_{,r})\right)\\
&=G\left(\frac{u_{,\varphi}}{r\sin\theta}-\frac{w}{r}+w_{,r}\right).
\end{aligned}$$

Rate equations for interpolated Maxwell model in potential flow:

In this case $\mathbf{L}=\mathbf{L}^T$ and $\dfrac{\partial\mathbf{S}}{\partial t}+\dfrac{1}{\lambda}\mathbf{S}+(\mathbf{u}\bullet\nabla)\mathbf{S}-a(\mathbf{LS}+\mathbf{SL})=2G\mathbf{D}$.

$\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}$ component:

$$\begin{aligned}
&\dot{\sigma}+(\frac{1}{\lambda}-2a\phi_{,rr})\sigma+\phi_{,r}\sigma_{,r}+\frac{\phi_{,\theta}}{r^2}\sigma_{,\theta}+\frac{\phi_{,\varphi}}{r^2\sin^2\theta}\sigma_{,\varphi}\\
&-\frac{2\tau}{r}\left(a\phi_{,r\theta}+(1-a)\frac{\phi_{,\theta}}{r}\right)-\frac{2\kappa}{r\sin\theta}\left(a\phi_{,\varphi r}+(1-a)\frac{\phi_{,\varphi}}{r}\right)=2G\phi_{,rr}.
\end{aligned}$$

$\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}$ component:

$$\dot{\gamma}+\left(\frac{1}{\lambda}-\frac{2a}{r}\left(\frac{\phi_{,\theta\theta}}{r}+\phi_{,r}\right)\right)\gamma+\phi_{,r}\gamma_{,r}+\frac{\phi_{,\theta}}{r^2}\gamma_{,\theta}+\frac{\phi_{,\varphi}}{r^2\sin^2\theta}\gamma_{,\varphi}-\frac{2\tau}{r}\left(a\phi_{,r\theta}-(1+a)\frac{\phi_{,\theta}}{r}\right)$$
$$-\frac{2\delta}{r^2\sin\theta}\left(a\phi_{,\varphi\theta}+(1-a)\cot\theta\phi_{,\varphi}\right)=\frac{2G}{r}\left(\frac{\phi_{,\theta\theta}}{r}+\phi_{,r}\right).$$

$\hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\varphi}}$ component:

$$\dot{\beta}+\left(\frac{1}{\lambda}-\frac{2a}{r}\left(\phi_{,r}+\frac{\phi_{,\varphi\varphi}}{r\sin^2\theta}+\frac{\cot\theta}{r}\phi_{,\theta}\right)\right)\beta+\phi_{,r}\beta_{,r}+\frac{\phi_{,\theta}}{r^2}\beta_{,\theta}+\frac{\phi_{,\varphi}}{r^2\sin^2\theta}\beta_{,\varphi}$$
$$-\frac{2\kappa}{r\sin\theta}\left(a\phi_{,\varphi r}-(1+a)\frac{\phi_{,\varphi}}{r}\right)-\frac{2\delta}{r^2\sin\theta}\left(a\phi_{,\varphi\theta}-(1+a)\cot\theta\phi_{,\varphi}\right)$$
$$=\frac{2G}{r}\left(\phi_{,r}+\frac{\phi_{,\varphi\varphi}}{r\sin^2\theta}+\frac{\cot\theta}{r}\phi_{,\theta}\right).$$

$\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}}$ components:

$$\dot{\tau}+\left(\frac{1}{\lambda}-a\left(\phi_{,rr}+\frac{\phi_{,\theta\theta}}{r^2}+\frac{\phi_{,r}}{r}\right)\right)\tau+\phi_{,r}\tau_{,r}+\frac{\phi_{,\theta}}{r^2}\tau_{,\theta}+\frac{\phi_{,\varphi}}{r^2\sin^2\theta}\tau_{,\varphi}+\frac{\sigma}{r}\left(\frac{\phi_{,\theta}}{r}+a\left(\frac{\phi_{,\theta}}{r}-\phi_{,r\theta}\right)\right)$$
$$-\frac{\gamma}{r}\left(a\phi_{,r\theta}+(1-a)\frac{\phi_{,\theta}}{r}\right)-\frac{\kappa}{r^2\sin\theta}\left(a\phi_{,\varphi\theta}+(1-a)\cot\theta\phi_{,\varphi}\right)$$
$$-\frac{\delta}{r\sin\theta}\left(a\phi_{,\varphi r}+(1-a)\frac{\phi_{,\varphi}}{r}\right)=\frac{2G}{r}\left(\phi_{,r\theta}-\frac{\phi_{,\theta}}{r}\right).$$

$\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\theta}}$ components:

$$\dot{\delta}+\left(\frac{1}{\lambda}+a\phi_{,rr}\right)\delta+\phi_{,r}\delta_{,r}+\frac{\phi_{,\theta}}{r^2}\delta_{,\theta}+\frac{\phi_{,\theta}}{r^2}\kappa+\frac{\phi_{,\varphi}}{r^2\sin^2\theta}\delta_{,\varphi}$$
$$-\frac{\gamma}{r^2\sin\theta}\left(a\phi_{,\varphi\theta}-(1+a)\cot\theta\phi_{,\varphi}\right)-\frac{\beta}{r^2\sin\theta}\left(a\phi_{,\varphi\theta}+(1-a)\cot\theta\phi_{,\varphi}\right)$$
$$-\frac{\tau}{r\sin\theta}\left(a\phi_{,\varphi r}-(1+a)\frac{\phi_{,\varphi}}{r}\right)-\frac{a\kappa}{r}\left(\phi_{,r\theta}-\frac{\phi_{,\theta}}{r}\right)=\frac{2G}{r^2\sin\theta}\left(\phi_{,\varphi\theta}-\cot\theta\phi_{,\varphi}\right).$$

$\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{\varphi}} \otimes \hat{\mathbf{r}}$ components:

$$\dot{\kappa} + \left(\frac{1}{\lambda} + \frac{a}{r}(\phi_{,r} + \frac{\phi_{,\theta\theta}}{r}) \right) \kappa + \phi_{,r}\kappa_{,r} + \frac{\phi_{,\theta}}{r^2}\kappa_{,\theta} + \frac{\phi_{,\varphi}}{r^2 \sin^2\theta}\kappa_{,\varphi} - \frac{\sigma}{r\sin\theta}\left(a\phi_{,\varphi r} - (1+a)\frac{\phi_{,\varphi}}{r} \right)$$
$$- \frac{\beta}{r\sin\theta}\left(a\phi_{,\varphi r} + (1-a)\frac{\phi_{,\varphi}}{r} \right) - \frac{\tau}{r^2\sin\theta}\left(a\phi_{,\varphi\theta} - (1+a)\cot\theta\phi_{,\varphi} \right)$$
$$- \frac{\delta}{r}\left(a\phi_{,r\theta} + (1-a)\frac{\phi_{,\theta}}{r} \right) = \frac{2G}{r\sin\theta}\left(\phi_{,\varphi r} - \frac{\phi_{,\varphi}}{r} \right).$$

Formulas for the compatibility condition $\boxed{\nabla \wedge (\nabla \bullet \mathbf{S}) = 0}$ **written for potential flow:**

$\hat{\mathbf{r}}$ component:

$$-\frac{\cot\theta}{r^2\sin\theta}\gamma_{,\varphi} - \frac{1}{r^2\sin\theta}\gamma_{,\theta\varphi} + \frac{\cot\theta}{r^2\sin\theta}\beta_{,\varphi} + \frac{1}{r^2\sin\theta}\beta_{,\varphi\theta} + \frac{\cot\theta}{r}\tau_{,r} - \frac{3}{r^2\sin\theta}\tau_{,\varphi} + \frac{1}{r}\tau_{,r\theta}$$
$$- \frac{1}{r\sin\theta}\tau_{,r\varphi} - \frac{2}{r^2}\delta + \frac{\cot\theta}{r^2}\delta_{,\theta} + \frac{1}{r^2}\delta_{,\theta\theta} - \frac{1}{r^2\sin^2\theta}\delta_{,\varphi\varphi} + \frac{3\cot\theta}{r^2}\kappa + \frac{3}{r^2}\kappa_{,\theta} = 0.$$

$\hat{\boldsymbol{\theta}}$ component:

$$\frac{1}{r\sin\theta}\sigma_{,r\varphi} + \frac{2}{r^2\sin\theta}\sigma_{,\varphi} - \frac{1}{r^2\sin\theta}\gamma_{,\varphi} - \frac{1}{r^2\sin\theta}\beta_{,\varphi} - \frac{1}{r\sin\theta}\beta_{,\varphi r} - \frac{1}{r}\tau_{,r}$$
$$+ \frac{\cot\theta}{r^2\sin\theta}\tau_{,\varphi} - \tau_{,rr} + \frac{1}{r^2\sin\theta}\tau_{,\theta\varphi} - \frac{2\cot\theta}{r}\delta_{,r} - \frac{1}{r}\delta_{,\theta r} - \frac{3}{r}\kappa_{,r} + \frac{1}{r^2\sin^2\theta}\kappa_{,\varphi\varphi} = 0.$$

$\hat{\boldsymbol{\varphi}}$ component:

$$-2\frac{1}{r^2}\sigma_{,\theta} - \frac{1}{r}\sigma_{,r\theta} + \frac{\cot\theta}{r}\gamma_{,r} + \frac{1}{r^2}\gamma_{,\theta} + \frac{1}{r}\gamma_{,\theta r} - \frac{\cot\theta}{r}\beta_{,r} + \frac{1}{r^2}\beta_{,\theta} + \frac{1}{r^2\sin^2\theta}\tau + \frac{4}{r}\tau_{,r}$$
$$- \frac{\cot\theta}{r^2}\tau_{,\theta} + \tau_{,rr} - \frac{1}{r^2}\tau_{,\theta\theta} + \frac{1}{r\sin\theta}\delta_{,\varphi r} + \frac{\cot\theta}{r^2\sin\theta}\kappa_{,\varphi} - \frac{1}{r^2\sin\theta}\kappa_{,\varphi\theta} = 0.$$

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