Kelvin-Helmholtz mechanism for side branching in the displacement of light with heavy fluid under gravity

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Abstract. The problem of stability of smooth fingering motions which may develop from the Rayleigh-Taylor instability when the initial data is analytic is considered. A second-order ordinary linear differential equation with time-dependent coefficients is derived for the evolution of a small wavy perturbation of the interface in a local approximation when the waves are short. It is possible to have a stable, but Hadamard unstable, perturbation if the time-dependent coefficients satisfy certain growth conditions. The strongest Hadamard instabilities occur as Kelvin-Helmholtz instabilities associated with a velocity difference at the sides of falling fingers.

1. Introduction

The problem of linearized instability of the interface between inviscid fluids, with surface tension neglected as it evolves, is studied by local analysis. The basic flow is time-dependent and nonlinear, and it may be regarded as arising initially from a Rayleigh-Taylor instability. Such flows are ill-posed, but they may be solved in a finite time interval before blow-up, provided that the initial data is analytic. We may also regard the basic flow as smoothed, say, by small viscous or surface tension effects which are neglected in the stability analysis. In fact, the details of the basic flow are left vague in the analysis and our conclusions which apply to arbitrary basic flows are independent of the precise details.

The main idea of the paper is that once the heavy liquid starts to fall, a velocity difference will develop across the interface. In this case, the possibility of Kelvin-Helmholtz instabilities arises from discontinuities in the tangential components of velocity which develop in the basic flow as it evolves. It appears natural to think that the largest discontinuities of velocity are at the sides rather than at the tips of unstable falling fingers. In fact, Hadamard instability of the Kelvin-Helmholtz problem is stronger than the Rayleigh-Taylor instability with a growth rate proportional to the wave number k instead of r(k). Kelvin-Helmholtz instabilities at the sides of fingers which develop from Rayleigh-Taylor instability, should emerge from every numerical study of the nonlinear problem. For example, the side branches are clearly evident in Figs. 4.1, 4.2 and 4.3 of the numerical study reported in the Ph.D. thesis of Tryggvason [1985] and in Figs. 18–24 of Gardner, Glimm, McBryan, Menikoff, Sharp and Zhang [1988]. Since the basic flow is unsteady, we need to account for the time-dependence of the velocity and the interface as it evolves. In fact, Moore and Griffith-Jones [1974] have shown that these time-dependent terms can stabilize the flow for any fixed k no matter how large, but these stable flows are still Hadamard unstable with unbounded growth as $k^{\varnothing}$_ at any fixed value of t, no matter how large or small (see Joseph and Saut [1990]).

2. Governing equations and the basic flow

We imagine that at some instant (t=0) heavy fluid in the semi-infinite half space y>0 lies above light liquid in y<0. The fluids are incompressible and nearly inviscid with a small value of interfacial tension. If the viscosity and interfacial tension are put to zero, this system will be Hadamard unstable, with a growth rate proportional to the square root of the wave number. In practice, this problem is regularized by surface tension and viscosity but is still unstable. This type of instability is known as Rayleigh-Taylor instability and it leads to fingers of heavy fluid into light fluid. The fingering flows which arise from Rayleigh-Taylor instability are unstable. The tips of the fingers of heavy into light fluid undergo repeated instabilities, called tip splitting (see Homsy [1987]), which can perhaps be understood as a replication of the original Rayleigh-Taylor instability. The fingering flows are plagued by other instabilities which lead to branching off the sides. The side branches are particularly striking in the ill-posed problem where they give rise to scraggly dendritic structures (Nittman and Daccord

[1985]). The analysis given below suggests that the side branches arise from a Kelvin-Helmholtz instability associated with the velocity discontinuities which develop across the sides of fingers accelerating into light fluid.

We suppose that a Rayleigh-Taylor instability has occurred and that fingers of heavy fluid are accelerating into light fluid. The motion of the two fluids may be smooth for a time, even with vanishing viscosity and interfacial tension, if the initial conditions are analytic. We study the stability of this smooth fingering flow using the linearized stability theory in a local approximation whose structure is tailored to the study of short waves.

We choose a special point P on the interface _ such that at $t=t_0$ (put $t_0=0$) the normal **N** to _ at P is colinear with a fixed direction Y. At a later time the interface will have moved so that Y will not be a perpendicular bisector of _ as it is at t=0. We lay down two plane coordinate systems (X*, Y*) and (X, Y) centered on P, as in Figure 1. The two systems are related by an orthogonal transformation with angle Θ.

$$X^* = X \cos\Theta - Y \sin\Theta ,$$
$$Y^* = X \sin\Theta + Y \cos\Theta . \quad (2.1)$$

The gravity vector $\mathbf{g} = -g\mathbf{e}_{Y^*}$ points against Y* increasing and the equations of motion are written in the (X,Y) system. Fluid 1 is over Fluid 2 and both fluids are assumed to be in potential flow

$$U = f(\_\Phi,\_X) , \quad W = f(\_\Phi,\_Y) , \quad {}^2\Phi = 0 . \quad (2.2)$$

Both fluids satisfy Bernouilli's equation with pressure P and density ρ

$$P + f(\rho,2) (U^2 + W^2) + \rho f(\_\Phi,\_t) + \rho g Y^* = 0 . \quad (2.3)$$

Figure 1. Coordinate systems for the falling fingers: ρ_1_ρ_2 is the heavy fluid. The fixed direction Y is colinear to the normal **N** on _ at t=0, but not in general for later t. Another useful representation of _ is r=R(θ,t) where (r,θ) are polar coordinated which are located at the center of curvature of _ at t=0.

We could add conservative time-dependent body forces to our problem to get different basic flows, as was done by Joseph and Saut [1990, p. 198]. These body forces are prescribed and they change the basic flow, but otherwise will not enter into the stability problem. There are two descriptions of the interface _:

$$Y = H(X, t) \quad \text{and} \quad Y^* = F(X^*, t) \tag{2.4}$$

where H and F are related by (2.1). Using $(2.4)_1$, we get

f(dY,dt) = W = f(_H,_t) + U f(_H,_X) (2.5)

as the kinematic equation of motion of the interface _. The jump of restrictions of field variables defined in Fluids One and Two across _ is defined by

û•ô sup4(a(def,=)) $(\bullet)_1 - (\bullet)_2$ (2.6)

and

ûρô = $\rho_1 - \rho_2$ _ 0 . (2.7)

The continuity of the normal component of velocity

ûWô = ûUô f(_H,_X) (2.8)

follows from (2.5). The pressure p in (2.3) is continuous across _. Hence

f(1,2) ûρ (U^2+W^2)ô + ûρ f(_Φ,_t)ô + ûρôg (X sinΘ + H(X,t) cosΘ) = 0 . (2.9)

The governing equations are (2.2), (2.5) and (2.9).

The basic flow is a smooth solution of the governing equations, designated with a subscript zero, and such that

$$f(\_H,\_X)\ (X,0) = 0\ . \qquad (2.10)$$

The condition (2.10) is fulfilled by virtue of our choice of coordinates, so that our analysis applies to all smooth solutions of the governing solutions.

3. Perturbation equations

Now we perturb the basic flow

$$Y = H_0(X,t) + \varepsilon h(X,t)\ , \qquad (3.1)$$

$$\Phi\ (X,Y,t) = \Phi_0(X,Y,t) + \varepsilon\phi(X,Y,t) \qquad (3.2)$$

$$(W,U) = (W_0,U_0)\ (X,Y,t) + \varepsilon(\omega,u)\ (X,Y,t) \qquad (3.3)$$

where

$$(\omega,u) = b(f(\_\phi,\_Y)\ ,\ f(\_\phi,\_X))\ . \qquad (3.4)$$

The only field equation needed in our analysis is

$$f(\_^2\phi,\_X^2) + f(\_^2\phi,\_Y^2) = 0\ . \qquad (3.5)$$

Care must be taken in forming the interface conditions. For example

$$W_0(X,H_0+\varepsilon h) + \varepsilon\omega(X,H_0+\varepsilon h) = f(\_,\_t)\ (H_0+\varepsilon h)$$
$$+ U_0(X,H_0+\varepsilon h)\ f(\_,\_X)\ (H_0+\varepsilon h) + \varepsilon u(X,H_0+\varepsilon h)\ f(\_(H_0+\varepsilon h),\_X) \qquad (3.6)$$

reduces, after using the equation of the basic flow and linearizing, to

$$\omega + h\ f(\_W_0,\_Y) = f(\_h,\_t) + u\ f(\_H_0,\_X) + U_0\ f(\_h,\_X) + h\ f(\_U_0,\_Y)\ f(\_H_0,\_X) \qquad (3.7)$$

where ω, u, W_0 and U_0 are evaluated on $Y=H_0$. The linearization of (2.9) gives rise to

blcûrcô(ρ b(f(_ϕ,_X) U_0 + f(_ϕ,_Y) W_0)) + blcûrcô(ρ f(_ϕ,_t)) + Mh = 0 (3.8)

where

M = blcûrcô(ρ b(U_0 f(_U_0,_Y) + W_0 f(_W_0,_Y))) + blcûrcô(ρ f(_W_0,_t)) + ûρô gcosΘ(3.9)

The governing perturbation equations are (3.5), (3.7), and (3.8). It is convenient to write (3.7) as follows

f(_ϕ,_Y) + A f(_ϕ,_X) + hB = f(_h,_t) + U_0 f(_h,_X) (3.10)

where

A = – f(_H_0,_X)

and

B = f(_W_0,_Y) + f(_U_0,_Y) A .

We may always choose our coordinates at the instant of observation for which (2.10) holds (A=0). But A_t=_A/_t, A_X=_A/_X need not be zero at this instant. The interface equations (3.8) and (3.10) may be written as

ρ_1b(ϕo($^{(1)}$,$_X$)Uo($^{(1)}$,$_0$) + ϕo($^{(1)}$,$_Y$)Wo($^{(1)}$,$_0$)) – ρ_2b(ϕo($^{(2)}$,$_X$)Uo($^{(2)}$,$_0$) + ϕo($^{(2)}$,$_Y$)Wo($^{(2)}$,$_0$)) +
ρ_1ϕo($^{(1)}$,$_t$) – ρ_2ϕo($^{(2)}$,$_t$) + Mh
= 0 , (3.11)

ϕo($^{(1)}$,$_Y$) + ϕo($^{(1)}$,$_X$)A = h_t + Uo($^{(1)}$,$_0$)h_X – $hB^{(1)}$, (3.12)

ϕo($^{(2)}$,$_Y$) + ϕo($^{(2)}$,$_X$)A = h_t + Uo($^{(2)}$,$_0$)h_X – $hB^{(2)}$. (3.13)

We may write (3.11), (3.12) and (3.13) in polar coordinates (r, θ) by putting $dY=dr$, $dX=-rd\theta$. Moore and Griffith-Jones [1974] considered the problem of an expanding vortex at $r=R(t)$ in a fluid of constant density $\rho_1=\rho_2$ expanding radially $\mathbf{e}_r\bullet\mathbf{U}=o(\dot{},R)R/r$ with a vortical azimuthal velocity $\mathbf{e}_\theta\bullet\mathbf{U}_1=\gamma/r$ when $r>R(t)$ and $\mathbf{e}_\theta\bullet\mathbf{U}_2=0$ when $r<R(t)$ where $\Gamma=2\_\gamma$ is the circulation. In this case $W_0=\mathbf{e}_r\bullet\mathbf{U}$ and $U_0=\mathbf{e}_\theta\bullet\mathbf{U}$ and $M=-\rho\gamma^2/R^3$, $A=0$, $B=-o(\dot{},R)/R$ and (3.11), (3.12) and (3.13) become

$$-f(\gamma,R^2)\,\phi o({}^{(1)}{}_{,\theta}) + o(\dot{},R)b(\phi o({}^{(1)}{}_{,r})-\phi o({}^{(2)}{}_{,r})) + b(\phi o({}^{(1)}{}_{,t})-\phi o({}^{(2)}{}_{,t})) - \gamma^2 h/R^3 = 0\,, \qquad (3.14)$$

$$\phi o({}^{(1)}{}_{,r}) = h_t - f(\gamma,R^2)\,h_\theta + f(o(\dot{},R),R)\,h\,, \qquad (3.15)$$

$$\phi o({}^{(2)}{}_{,r}) = h_t + f(o(\dot{},R),R)\,h\,. \qquad (3.16)$$

4. Local approximation for short waves

In the coordinate system being used, the tangent to $H_0(X,t)$ is flat at the X and $t=0$ for which $A=0$. We are going to look at waves so short that the coefficients of ϕ_X, ϕ_Y and h are essentially constant over the length $2\_/k$ of one wave. Then we solve $^2\phi=0$ so that $\phi^{(1)}(X,Y,t)\varnothing 0$ as $Y\varnothing\_$ and $\phi^{(2)}(X,Y,t)\varnothing 0$ as $Y\varnothing 0$. Hence

$$\phi^{(1)} = \psi_1(t)\,\mathrm{esup3}(ikX)\mathrm{esup3}(-kY)\,,$$
$$\phi^{(2)} = \psi_2(t)\,\mathrm{esup3}(ikX)\mathrm{esup3}(kY)\,,$$
$$h = \hbar(t)\mathrm{esup3}(ikX)\,. \qquad (4.1)$$

These equations (4.1) are inserted into (3.11), (3.12) and (3.13) and are evaluated on $Y=H_0(X,t)$, defining

$$y_1(t) = \psi_1\,\mathrm{esup3}(-kH_0)$$
$$y_2(t) = \psi_2\,\mathrm{esup3}(kH_0) \qquad (4.2)$$

we find that

$$k(-1 + iA)\, y_1 + B^{(1)} h = o(^{\bullet}, h) + ikUo(^{(1)},_0)\, h \,, \tag{4.3}$$

$$k(+1 + iA)\, y_2 + B^{(2)} h = o(^{\bullet}, h) + ikUo(^{(2)},_0)\, h \,, \tag{4.4}$$

$$i\rho_1 Uo(^{(1)},_0) k y_1 + \rho_1 k A Uo(^{(1)},_0) y_1 + \rho_1 y_{1t} - i\rho_2 Uo(^{(2)},_0) k y_2 - \rho_2 k A Uo(^{(2)},_0) y_2 - \rho_2 y_{2t} + Mh = 0 \;. \tag{4.5}$$

We have chosen our local coordinates so that A=0 initially. It follows that for the smooth solutions we have in mind, A is small locally in time and we put it to zero. Of course A_t need not be small when A is small.

After eliminating y_1 and y_2

$$o(^{\bullet\bullet}, h) + \Theta_1 o(^{\bullet}, h) + \Theta_2 h = 0 \tag{4.6}$$

where

$$\Theta_1 = f(1,\bullet\rho®)\ bbc\{(2ik\bullet\rho U_0® \ - \bullet\rho B® + i\hat{u}\rho\hat{o}\ A_t) \tag{4.7}$$

$$\Theta_2 = f(1,\bullet\rho®)\ \Big\{ -kM - iA_t\hat{u}\rho B\hat{o} - (kA_t)\hat{u}\rho U_0\hat{o} - ik\bullet\rho BU_0® - k^2\bullet\rho Uo(^2,_0)® - \bullet\rho B_t® + ik\bullet\rho U_{0t}® \Big\} \tag{4.8}$$

and

$$\bullet\bullet®\ sup4(a(def,=))\ (\bullet)_1 + (\bullet)_2 \;.$$

and, upon putting A=0 after differentiation

$$B \;=\; \_W_0 / \_Y \,,$$
$$B_t \;=\; \_W_{0t} / \_Y + A_t \_U_0 / \_Y \,,$$
$$W_0 \;=\; H_{0t} \,,$$
$$W_{0t} \;=\; H_{0tt} - A_t U_0 \,,$$
$$M \;=\; blc\hat{u}rc\hat{o}(\rho\ b(U_0\ f(\_U_0,\_Y)) + W_0\ f(\_W_0,\_Y)) + \hat{u}\rho W_{0t}\hat{o} + \hat{u}\rho\hat{o}\ g\cos\Theta \;. \tag{4.9}$$

We do not expect our local formulation to hold globally. The case A_0 is not too difficult to work out, but it adds more to the length of the equations than to our understanding of them.

5. Special cases

Consider the case of instability of a flat interface, Θ, H_0, H_{0t}, A_t vanish, for uniform flow in the X direction where W_0 is identically zero and

f(_Uo($^{(1)}$,$_0$),_Y) = f(_Uo($^{(2)}$,$_0$),_Y) = 0 ,

Then B=B_t=W_{0t}=0 and M=ûρôg

Θ_1 = f(2ik,•ρ®) •ρU_0® , (5.1)

Θ_2 = f(–k^2,•ρ®) •ρUo(^{2},$_0$)® + f(ik•ρU_{0t}®,•ρ®) – f(kûρôg,•ρ®) (5.2)

Equation (4.6) with coefficients given by (5.1) and (5.2) was presented as equation (4.22) and was discussed by Joseph and Saut [1990]. When U_0 is independent of t we can seek solutions proportional to $e^{\sigma t}$, leading to a dispersion relation of the form

$$\sigma^2 + \Theta_1\sigma + \Theta_2 = 0 \qquad (5.3)$$

Hence

σ = – f(1,2) Θ_1 ± f(1,2) r(Θo(^{2},$_1$)–4Θ_2) (5.4)

This reduces to

σ = ± r(f(k(ρ_1–ρ_2)g,ρ_1+ρ_2))

when U_0_0, as in Rayleigh-Taylor instability, and to

$$\sigma = -\frac{ik}{2}\left[U_0^{(1)} + U_0^{(2)}\right] \pm \frac{k}{2}\left|U_0^{(2)} - U_0^{(1)}\right| ,$$

$$\mathrm{Re}\,\sigma = \frac{k}{2}\left|U_0^{(2)} - U_0^{(1)}\right| .$$

when $\rho_1=\rho_2$ and U_0 is independent of t, as in Kelvin-Helmholtz instability. The Kelvin-Helmholtz instability is “more” ill-posed than the Rayleigh-Taylor instability because it has a growth rate proportional to k, rather than the square root of k.

The problem of Moore and Griffith-Jones [1974] is not a special case of (4.6). The wave number k has been assumed to be independent of t but the equivalent wave number s/R(t) for their case does depend on t. We may obtain a second-order equation of the form (4.6), their equation (2.6), directly from the last three equations of section 3.

6. WKB solutions

When the coefficients Θ_1 and Θ_2 in (4.6) depend on time, we cannot have solutions of the $ce^{\sigma t}$ with constant c and σ. In this case, however, we can find asymptotic solutions, valid for large k, by the WKB method. Applying this method, following ideas introduced by Moore and Griffith-Jones [1974], we introduce the transformation

$$h(t) = \eta(t) \exp\left\{-\frac{1}{2} i\int_0^t \Theta_1(s)\, ds\right\} . \qquad (6.1)$$

Then $\eta(t)$ satisfies the equation

$$\ddot{\eta}(t) - Q(t)\eta(t) = 0 \qquad (6.2)$$

where

$$Q(t) = \frac{\Theta_{1t}}{2} + \frac{\Theta_1^2}{4} - \Theta_2 . \qquad (6.3)$$

Substituting (4.7) and (4.8) into (6.3), we obtain

$$Q(t) = ak^2 + bk + c \tag{6.4}$$

with

$$a = \frac{\rho_1\rho_2}{[\![\rho]\!]^2} |U_0|^2 ,$$

$$b = \frac{[\![\rho]\!]M+2\rho_1\rho_2 A_t|U_0|}{[\![\rho]\!]^2} + i \frac{\rho_1\rho_2}{[\![\rho]\!]^2} |B|\,|U_0| ,$$

$$c = \frac{1}{[\![\rho]\!]^2} \left(\frac{1}{2} [\![\rho]\!][\![\rho B_t]\!] + \frac{1}{4} [\![\rho B]\!]^2 - \frac{1}{4} A_t^2|\rho|^2\right) + \frac{i}{[\![\rho]\!]^2} \left(A_t|\rho B|[\![\rho]\!] - \frac{1}{2} A_t|\rho|[\![\rho B]\!] - \frac{1}{2} A_{tt}|\rho^2|\right) .$$

We now assume that a>0. Then equation (6.2) can be written as

$$\varepsilon^2\ddot{\eta}(t) - \tilde{Q}\eta(t) = 0 \tag{6.5}$$

where $\varepsilon=1/k$, $\tilde{Q}=Q/k^2$ and Q is bounded up to a certain t. The analysis then holds up to that t and we may also consider the case when t is infinite. Equation (6.5) is obviously the canonical singular perturbation form for which WKB solutions were designed. We shall work with the original equation (6.2) with Q(t) given by (6.4). The asymptotic solutions obtained by the standard WKB procedure are of the form

$$\eta(t) \sim \exp\left(kS_0 + S_1 + O\left(\frac{1}{k}\right)\right) \tag{6.6}$$

where

$$S_0 = \pm i\int_0^t \sqrt{a(s)}\, ds \tag{6.7}$$

$$S_1 = \pm i\int_0^t \frac{b(s)}{2\sqrt{a(s)}}\, ds - \frac{1}{4} \ln a(t) . \tag{6.8}$$

The approximation (6.6) will be valid, if S_0 and S_1 and the $O\left(\frac{1}{k}\right)$ term are bounded. The condition on the boundedness of the $O\left(\frac{1}{k}\right)$ term gives the following relation

k » i($_0$,t, blc{(±bbc[(f(1,8) a_{tt}asup3(–3/2) –f(5,32) ao(^{2},$_t$)asup3(–5/2) – f(1,8) b^2asup3(–3/2) + f(1,2) asup3(–1/2) c)))
+ brc}(f(1,4) ba_ta^{-2} – f(1,4) b_ta^{-1})dS . (6.9)

The boundedness assumptions we have laid down for a, b, and c imply that (6.7), (6.8) and (6.9) are bounded for any finite t. For the t∅_ case, the integrands in these three integrals must go to zero fast enough to insure integrability. In an n term WKB approximation, the extra boundedness condition is imposed at 0b(f(1,k^{n-1})). Condition (6.9) is the analog to the condition (4.2) of Moore and Griffith-Jones [1974].

Substituting a, b, and c defined in (6.4) into equations (6.7) and (6.8), using (6.1) and (6.6), we obtain an approximate solution for h(t) in the form

h(t) = f(d_1,$a^{1/4}$) exp bbc{(i($_0$,t, E(s) ds – f(1,2)) i($_0$,t, Θ_1(s) ds + 0b(f(1,k))))
+ f(d_2,$a^{1/4}$) exp bbc{(–i($_0$,t, E(s) ds – f(1,2)) i($_0$,t, Θ_1(s) ds + 0b(f(1,k)))) (6.10)

where d_1, d_2 are constant, and

E(t) = kα_1 ûU_0ô + α_2 f(M(t),ûU_0ô) + $\alpha_3 A_t$ + i f(α_3,2) ûBô .

with

α_1 = r(f($\rho_1\rho_2$,ρ_1+ρ_2))

α_2 = f(1,2r($\rho_1\rho_2$)r(ρ_1+ρ_2))

α_3 = f(r($\rho_1\rho_2$),$(\rho_1+\rho_2)^{3/2}$)

For large k, the dominant part of h(t) in (6.10) is given asymptotically by

h(t,k) ~ f(1,$a^{1/4}$) exp bbc{(k i($_0$,t, b(α_1ûU_0ô –i f(•ρU_0®,•ρ®)) ds)) (6.11)

For any finite t, *h*(t,k)∅_ with k. This corresponds to a Hadamard instability of a Kelvin-Helmholtz problem. Moreover, if the integral in (6.11) remains bounded as t∅_, then for each fixed finite k, no matter how large, *h*(_,k) is bounded. We may say then that *h*(t,k) is exponentially stable in the sense of integrability. The flow will then be stable in this sense but Hadamard unstable.

7. Conclusions

1. Smooth solutions of the Rayleigh-Taylor problem are Hadamard unstable. The most dangerous disturbances are Kelvin-Helmholtz instabilities for which disturbances grow like $|U_1 – U_2| \exp(\sigma t)$ with σ proportional to k for large k. This should be compared with all the other destabilizing factors with growth rates which are proportional to r(k).

2. The prefactor $|U_1 – U_2|$ where U is the tangential component is likely to be greatest at the sides of the fingers, leading to side branching.

3. It is thought that smooth solutions will last only up to a point where a singularity of curvature develops. However, these solutions are Hadamard unstable before they blow up.

4. Instability and Hadamard instability involve different limits. Instability means that certain disturbances become unbounded as t∅_. Hadamard instability means that a disturbance becomes unbounded as k∅_ for any t, however small or large. It is even possible to find stable unsteady flow for which disturbances are not unbounded as t∅_, which are Hadamard unstable.

Acknowledgements

The work of D.D. Joseph and T.Y.J. Liao was supported by the U.S. Army (Mathematics), the Department of Energy, and the National Science Foundation.

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