Long wave and lubrication theories for core-annular flow

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Different nonlinear amplitude equations for long waves in core-annular flow are compared. Each equation has its own limits of validity that can be critically assessed by comparing the linearization of approximate and exact theories. […printed abstract started this way.]

We compare different nonlinear amplitude equations for long waves in core-annular flow. Each equation has its own limits of validity which can be critically assessed by comparing the linearization of approximate and exact theories. Long wave theory gets the dispersion relation for the longest waves correctly but cannot accommodate cases like capillary instability, in which the most dangerous wave is not surpassingly long. Small gap lubrication based theories accommodate shorter waves of the size of the core when various extra conditions are satisfied, but various stabilizing mechanisms associated with inertia may not be well represented. One theory in which lubrication theory is used in the water film but not in the core captures the shear stabilization of inertia when the gap is small enough. The criterion for small enough is not uniform in the viscosity ratio and surpassingly small films are required for validity when the oil viscosity is large. The results of lubrication theory are not robust with respect to changes to larger gaps outside the regime of asymptotic validity; for example, the stabilizing effects of the inertia of the core and annulus may reverse for larger, but still small thicknesses.

1. Introduction

Stability problems for core-annular flow in lubricated pipelines have been studied by Joseph, Renardy and Renardy[1], Preziosi, Chen and Joseph[2], Hu and Joseph[3], Chen, Bai and Joseph[4], Chen and Joseph[5], Bai, Chen and Joseph[6] and Hu, Lundgren and Joseph[7]. The linear theory is fairly complete and in satisfactory agreement with experiments. A weakly nonlinear approach based on the Ginzburg-Landau equation was studied by Chen and Joseph[5]. Ginzburg-Landau amplitude equations describe weakly nonlinear modulations of monochromatic waves of wave length $2\pi/\alpha_c$, where α_c is the critical wave number of linear theory at the nose of a neutral curve. Typically, core-annular flows are unstable. The neutral curves have no nose, no critical points near which perfect core-annular flows are stable for all wave numbers. In less typical cases, there are two critical points, one on the upper branch of the neutral curve and one on the lower branch. When the Reynolds number is between the upper and lower critical points, perfect core-annular flow is stable (as in Figure 8 of Preziosi, Chen and Joseph[2]). In the case of very long waves, it would be impossible to obtain an amplitude equation of the Ginzburg-Landau type. The critical wave number at the nose tends to zero so that the wave supposed to be modulated is already hugely long. (We are indebted to A. Frenkel for this remark. He noted that to have a Ginzburg-Landau equation the sideband width $\Delta\alpha$ ought to be small relative to the wave number α on which it centers). In this situation, there are other types of approximation which give rise to nonlinear amplitude equations describing a slowly varying waveform rather than the slowly varying envelope of a modulated wave as in the Ginzburg-Landau equation. These types of approach was pioneered by Benny[8]. For core-annular flow in a pipe, similar approach based on the lubrication approximation in the annulus has been developed by Frenkel, Babchin, Levich, Shlang and Sivashinsky[9], Frenkel[10], and Papageorgiou, Maldarelli and Rumschitzki[11]. Inertia of the fluid in the annulus is completely neglected in the derivation of the nonlinear amplitude equation which arises at lowest order. This means that such theories automatically rule out motions in which secondary flows are involved. It is thought that such theories are particularly relevant to very thin films in situations in which the most amplified wave of linear theory is very long.

In general, the inertia terms in the Navier-Stokes equation $\rho(\mathbf{u}\cdot\ )\mathbf{u}$ induce wave number multiplication which produces shorter and shorter waves. A monochromatic linear wave proportional to $\exp(i\tilde{\alpha}\, x)$, where $\tilde{\alpha}$ is the wave number of maximum linear growth, undergoes multiplications leading to $\exp(\pm 2i\tilde{\alpha}\, x)$ and mean terms, the nonlinear interaction of these lead to $\exp(\pm 4i\tilde{\alpha}\, x)$, and so on. The nonlinear terms therefore rapidly produce short waves from long ones. This type of generation of short waves from long ones due to inertia is automatically removed by assumption in lubrication-based theories which allow inertia to be treated only as a perturbation, if at all. The amplitude equations arise at lowest order and these equations do allow for wave number multiplication of the thickness function $\eta(x,t)$, which is another matter. If the dissipative terms do not dissipate the short waves, they will begin to dominate the dynamics, and their effects will not be captured by the amplitude equation. For this reason, we think that if the problem is to follow the evolution of the amplitude of a wave under conditions in which the wave number of maximum growth is bounded strictly away from zero, the application of a long wave equation could lead to irrelevant results. Solutions of a hierarchy of equations pinned on the amplitude equation which converge and preserve slow scales may fail to satisfy the Navier-Stokes equations. This problem appears not to have been studied with the tools of analysis, numerical analysis or by comparison with experiments.

We have been trying to determine the conditions under which the predictions of the lubrication-based amplitude equations for lubricated pipelining may be realized in an experiment. These amplitude equations are appropriate when there is a thin lubricating film of water on the pipe. Very thin water films have not yet been seen in experiments in which the flow rates of oil and water are prescribed. In these cases the film thickness is a functional of the solution. If the water flow rate is reduced or the flow rate of oil is increased, the oil will stick to the wall and/or the water will emulsify into the oil, leading to a failure of lubrication. It may be hard to achieve the conditions required to test the predictions of the nonlinear amplitude equations based on lubrication approximation. On the theoretical side, we were led to a study of the conditions of validity of the amplitude equations based on lubrication theory. For water lubricated pipelining, the oil in the core typically is 100 times more viscous than the lubricating water. For the lubrication based theory to be applicable even under the moderate operating condition of order one core Reynolds number, the water film has to be extremely thin. If it is not thin, the inertia of the water film will not be negligible. Indeed examination of the special case of long waves in section 4 shows that the lubrication based theory applies only when the dimensionless film thickness is small compared to $m^{2/3}$, where $m = \frac{\mu_{water}}{\mu_{oil}}$, typically 10^{-2} or smaller for oil in water.

2. Amplitude equation of Hooper and Grimshaw

Hooper and Grimshaw[12] derived a nonlinear amplitude equation of the Kuramoto-Sivashinsky type for Yih's problem: Couette and Poiseuille flow between parallel plates with an undisturbed interface at y=0 with $\mathbf{u}=\mathbf{e}_x u(y)$, $u_l(y)=a_l y^2+b_l y+1$, $l=1$ for $y<0$, $l=2$ for $y>0$, $a_l = G/2\mu_l$, where G is the pressure gradient and μ_l are fluid viscosities. Yih[13] found the dispersion relation for the linear problem when the disturbance wave length is long compared to the depths of both fluids,

$$c = c_o + i\,\alpha \boldsymbol{R} \left\{ \hat{J}(m, \zeta, b, a_1) - \alpha^2 S(m, b) \right\} + O(\alpha^2) \tag{1}$$

where c is the complex wave speed, α is the disturbance wave number, c_o is real, $m=\mu_2/\mu_1$, $\zeta=\rho_2/\rho_1$, $a=d_2/d_1$ are the viscosity, density and depth ratio, $S=T/\rho_1 d_1 U_0^2$ is a surface tension parameter, U_0 is the value of the interfacial velocity, $\hat{J}$ is a rational function of parameters given in Yih[13] and it is positive for certain values of m and ζ, and $\boldsymbol{R}=U_0 d_1/\nu_1$ is the Reynolds number. When S is O(1) its contribution is smaller than the error term in (2).

Hooper and Grimshaw[12] did a weakly nonlinear long wave analysis assuming

$$\eta = \varepsilon A(\zeta,\tau)\,, \quad \xi = \varepsilon (x - c_0 t)\,, \quad \tau = \varepsilon^2 t \tag{2}$$

where ε is a small parameter, otherwise unspecified, η is the deviation of the fluids interface from its flat position. The perturbed stream function is expanded as

$$\psi\,(x,y,t;\,\varepsilon) = \varepsilon\,\psi_0(\zeta,y,\tau) + \varepsilon^2\,\psi_1\,(\zeta,\,y,\,\tau) + \ldots \tag{3}$$

At $O(\varepsilon^\circ)$ they recover the basic flow and at $O(\varepsilon)$ they get Yih's stability result with eigenfunctions expressed in terms of the unknown amplitude $A(\xi,\tau)$ determined by solvability conditions at second order which give rise to the amplitude equation

$$A_\tau + \boldsymbol{R}\,\hat{J}\;A_{\xi\xi} + \alpha^2\,\boldsymbol{R}\,S\,A_{\xi\xi\xi\xi} = -\,l\,(m,\,\zeta)\,AA_\xi \tag{4}$$

where the subscripts are the partial derivatives with respect to the corresponding variables and l $(m,\,\zeta)$ is a rational function found by analysis.

The linear part of the amplitude equation (4) is exact in the sense that, linearization of (4) results in the exact dispersion relation (1). Thus the linear stabilization or destabilization mechanism for the longest waves are preserved in the amplitude equation (4). It is important to note that, although the linear instability caused by viscosity stratification discovered by Yih[13] persists at arbitrarily small Reynolds numbers, it is necessary to maintain all the inertia terms in the governing equations when performing the stability analysis, as is evident in the dispersion relation (1). The second term on the right hand side of equation (1), excluding the surface tension contribution, is the sum of the contributions from the inertias of both fluids, and it is the term determining the linear instability or stability of the problem to the leading order.

An amplitude equation similar to (4) can be derived for core-annular flow of two fluids in a circular pipe by following the procedure of Hooper and Grimshaw[12] when the wave is long relative to both the core radius and annulus thickness. This amplitude equation has the same linear part as that of the linearization of the full problem in the same limit, which includes both the contributions from the core and the annulus, but only up to and not including terms of $O(a^2)$. This equation should not be used to describe the nonlinear evolution of systems, like those driven by capillary instability, in which the wave number $\alpha=\alpha_c$ of maximum growth is bounded strictly away from zero. In such cases the expression for α_c would involve terms of order α^2 neglected in the analysis. Other types of amplitude equation which do not produce the same dispersion relation as the exact linear theory can also be derived for core-annular flow, as we shall see in section 3, but different conditions must be imposed for the validity of such theory.

3. The amplitude equations of Frenkel et al and Papageorgiou et al

Frenkel et al[9] consider a core-annular flow of two fluids with matched viscosities and densities. Their analysis requires that the wave length be long relative to the gap, but not to the core. Unlike the "longer" wave analysis of the Hooper-Grimshaw type, this analysis does accommodate capillary instability. Their analysis predicts that in a certain range of parameters, capillary instabilities of perfect core-annular flow saturate nonlinearly, producing chaotic waves rather than film rupturing. Since viscosity differences are neglected, the linear mechanism of shear stabilization through interfacial friction which is routinely observed in experiments is absent from the analysis. Besides requiring that the layer thickness be small relative to the core radius, their analysis requires small Reynolds numbers in the annulus and other conditions. Under these conditions, they derive a Kuramoto-Sivashinsky equation of the form (4) or (10) with DI=0. This equation gives rise to bounded but chaotic solutions which are said to saturate the linear instability.

Frenkel[10], responding to results of the analysis of stabilizing effects of interfacial friction found in Preziosi, Chen and Joseph[2], found a method for dealing with the dynamical effects of the core in which lubrication approximations are inappropriate. He used lubrication theory in the gap and a different approximation, which is basically not restricted to long waves, in the core. He obtained nonlinear amplitude equations with additional linear terms, equivalent to the term DI in (10), which give rise to dispersion and dissipation. Smooth rather than chaotic waves then saturate unstable core-annular flow (see Papageorgiou et al[11]). He identifies four parameters

$$\varepsilon_1 = \frac{J\,\varepsilon^2}{\boldsymbol{R}_1}\ , \quad \varepsilon_2 = \frac{\boldsymbol{R}_1}{\varepsilon\, J}\ , \quad \varepsilon_3 = \frac{1}{m-1}\ , \quad \varepsilon_4 = \boldsymbol{R}_1 \qquad (5)$$

which must be separately small (and positive) for his theory to be valid. ε in (5) is defined as $\frac{R_2-R_1}{R_1}$, where R_2, R_1 are the radius of the pipe and the oil core respectively. $\boldsymbol{R}_1$ and J are core Reynolds number and dimensionless surface tension parameter defined in Preziosi et al[2] . Since $\varepsilon_1\,\varepsilon_2=\varepsilon$, the film thickness must be small.

Frenkel[10] also uses his equation to discuss the case of lubricated pipelining in which m<1 and $\varepsilon_3<0$, but he does not give the justification of his theory for this case. He notes that his equations give rise to shear stabilization due to viscosity stratification (interfacial friction) without nonlinear effects when

$$\boldsymbol{R}_1\,\alpha \ll 1 \quad \text{and} \quad \frac{m\,J\,\varepsilon}{\boldsymbol{R}_1^{\,2}} \ll 1\ . \qquad (6)$$

This can be interpreted as shear stabilization for small gaps, $\varepsilon\ll 1$ and long waves $\alpha\ll 1$. We shall see that this shear stabilization is correct in the case of very thin films to which the lubrication theory applies.

Frenkel's idea was further developed and systematized in the work of Papageorgiou et al[11]. They obtain a nonlinear equation for the evolution of the interface from the analysis of a solution of the problem in powers of the film thickness, $\varepsilon\ll 1$. They found that the inertia of the water in the thin annulus may be neglected at leading order and they took into account the dynamics of the core in an approximation introduced by Frenkel[10]. They found that in addition to the requirement that $\varepsilon\ll 1$ it was also necessary to guarantee that

$$\frac{\varepsilon J}{\boldsymbol{R}_1} = O(1)\ . \qquad (7)$$

This condition can be satisfied in two ways:

$$\boldsymbol{R}_1 = \varepsilon\,, \quad J = O(1) \tag{8}$$

which gives rise to a low Reynolds number approximation, and

$$\boldsymbol{R}_1 = O(1)\,, \quad J = \frac{1}{\varepsilon} \tag{9}$$

which is a large surface tension approximation. The condition $(9)_2$ may be written as $\boldsymbol{R}_1 = \frac{\rho \nu_1^2}{T\,\varepsilon}$, where T is the surface tension. For heavy crudes, T=20 dyne/cm, $\mu_1=10^3$ Poise, $\varepsilon=10^{-1}$, say, then $R_1=O(10^6 \text{cm})$. Obviously, (9) is a very restrictive condition.

If $\varepsilon \ll 1$ and (7) is satisfied, then they find an amplitude equation in the form

$$\eta_\tau - \frac{2}{m}\ \eta\eta_z + \frac{J\,\varepsilon}{3\boldsymbol{R}_1 m}\ (\eta+\eta_{zz})_{zz} + DI = 0 \tag{10}$$

where t, x are physical variables such that

$$\tau = \varepsilon^2 t\,, \quad z = \varepsilon(x - W(1)\,t), \tag{11}$$

W(1) being the interfacial velocity of the basic flow. The term DI is a linear term which depends globally on the wave number k. Equation (10) was given by Frenkel[10] but the term DI was expressed in an abstract rather than explicit manner.

In the small Reynolds number case (8), Papageorgiou et al[11] find that the core dynamics is governed by Stokes flow and can be solved by the method of Fourier transforms. The transform of the stream function satisfies an ordinary differential equation which can be solved by Bessel functions. The stream function for the core can be expressed by the inversion integral which is a global expression in wave number space and not restricted to long waves. The core quantities on the interface can be expressed in terms of this inversion integral and they appear in the term DI of equation (12) below:

$$DI = \frac{i}{\pi m}\left(1-\frac{1}{m}\right)\int_{-\infty}^{\infty} N_B(k) \int_{-\infty}^{\infty} \eta(y,\tau) \exp\ \{ik(z-y)\}\ dydk\ , \tag{12}$$

with

$$N_B(k) = \tag{13}$$

where $I_0(k)$ and $I_1(k)$ are Bessel functions. This linear term DI given by (12) and (13) is purely dispersive so that is not possible to stabilize the capillary instability. In fact, shear stabilization, as we have and shall again see, is associated with the inertia of the basic flow, here neglected both in the core and the annulus. We don't have much shear stabilization at very low Reynolds numbers, so the analysis and the physics are not incompatible.

In the second case (9), where the surface tension is supposed to be large, a Stokes flow approximation in the core is not appropriate. The core inertia of the basic flow must be taken into account and this can lead to shear stabilization. In the analysis of Frenkel[10] and Papageorgiou et al[11] the inertia of the core is fully represented. The solution of the core stream function can be obtained by the method of Fourier transforms, as indicated already, and the stream function can be expressed by the inverse transform. This solution is again global in wave numbers, being expressed as an integral over all wave numbers which leads again to (10) but now with

$$DI = \frac{-i}{2\pi m}\left(1-\frac{1}{m}\right)\int_{-\infty}^{\infty} N_K(k) \int_{-\infty}^{\infty} \eta(y,\tau) \exp\ \{k(z-y)\}\ dydk\ , \tag{14}$$

where

$$N_K(k) = \frac{I_1(k) \exp[-\lambda]\ M(\Lambda, 2, 2\lambda)}{N_1(k)\ I_0(k) - N_2(k)\ I_1(k)} \tag{15}$$

with

$$\lambda = \frac{1}{2}\sqrt{k\boldsymbol{R}_1}\;\exp\left[-i\,\frac{\pi}{4}\right],$$

$$\Lambda = 1 + \frac{k^2}{8\lambda} - \frac{\lambda}{2},$$

$$N_1(k) = \int_0^1 (I_1(k)K_1(kt) - I_1(kt)K_1(k))\; t^2 \exp[-\lambda t^2]\, M(\Lambda, 2, 2\lambda t^2)\, dt,$$

$$N_2(k) = \int_0^1 I_0(k)K_1(kt) + I_1(kt)K_0(k))\, t^2 \exp[-\lambda t^2]\, M(\Lambda, 2, 2\lambda t^2)\, dt$$

and M is the confluent hypergeometric function (the Kummer function). The functions I's and K's are the modified Bessel functions of various orders.

The analyses of Frenkel[10] and Papageorgiou et al[11] and the calculation in section 4 below show that DI given by (14) and (15) gives rise to dissipation as well as to dispersion. We get shear stabilization from (14) because it represents the part of the shear stabilization arising from the inertia of the core. The amplitude equation (10) is quite general in the sense that it is capable of describing waves with wave lengths comparable to the core radius, although the wave lengths are required to be long compared to the thickness of the annulus. On the other hand, contributions from the annulus inertia is totally neglected in equation (10). Under what circumstances that this omission of annulus inertia is compatible with the underlying physics and thus equation (10) can be safely applied is a question not adequately addressed. For the special case of long waves, waves with wave lengths long relative to both the core radius and annulus thickness, we will give a critical assessment of the validity of the amplitude equation (10) in section 5.

4. Long wave expansions for the amplitude equation (10) when $\boldsymbol{R}_1 = O(1)$

We like to compare the stability criteria for long waves which arise from linearizing the nonlinear amplitude equation (10) around $\eta=0$ with stability criteria for long waves for the linearized full problem when $\boldsymbol{R}_1=O(1)$.

We first substitute normal modes

$$\eta\,(z,\tau) = \hat{\eta}\ \exp\,[i\alpha(z-ct)] \tag{16}$$

into the linearized version of (10)

$$\eta_\tau + \frac{J\,\varepsilon}{3\boldsymbol{R}_1 m}(\eta+\eta_{zz})_{zz} - \frac{i\,(m-1)}{2\pi m^2}\int_{-\infty}^{\infty} N_K(k)\int_{-\infty}^{\infty}\eta(y,\tau)\,\exp\,\{ik(z-y)\}\,dydk = 0\,. \tag{17}$$

After using the theory of Fourier transforms in the form

$$N_K(\alpha)e^{i\alpha z} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N_K(k)\,e^{ikz}\,e^{i(k-\alpha)y}\,dkdy\,.$$

we find that

$$i\alpha c = (\alpha^4-\alpha^2)\,\frac{J\,\varepsilon}{3\boldsymbol{R}_1 m} - i\,\frac{m-1}{m^2}\,N_K(\alpha)\,. \tag{18}$$

Asymptotic development of the Kummer function for small α leads us to

$$N_K(\alpha) = -\,4\,\alpha - i\,\alpha^2\,\frac{\boldsymbol{R}_1}{12}\,. \tag{19}$$

Combining (18) and (19) and writing $c=c_0+\alpha c_1$ we find that

$$c = 4\,\frac{m-1}{m^2} + i\,\alpha\left[\frac{J\,\varepsilon}{3\boldsymbol{R}_1 m} + \frac{m-1}{m^2}\,\frac{\boldsymbol{R}_1}{12}\right] = c_0 + \alpha\,c_1\,. \tag{20}$$

Finally we note that the stability of the solution η=0 of (10) to long waves depends on the sign of the growth rate α Im c corresponding to (16) and to leading order in α we have

$$\alpha \,\mathrm{Im}\, c = \alpha^2 \left[\frac{J\,\varepsilon}{3\boldsymbol{R}_1 m} + \frac{m-1}{m^2}\frac{\boldsymbol{R}_1}{12} \right] . \tag{21}$$

We may recall that the derivation leading to (10) shows that the eigenvalue c, (20), or lubrication theory is related to the eigenvalue C of the exact linearized theory by

$$C - W(1) \sim \varepsilon^2 c + O(\varepsilon^3) . \tag{22}$$

In the next section we will show that the result (20) of lubrication theory is exact in the limit $\varepsilon \varnothing 0$, to which it is said to apply provided that J=1/ε and $\boldsymbol{R}_1$=O(1). Equation (21) shows that perfect core-annular flow is always unstable when $\boldsymbol{R}_1$ is small or the less viscous fluid is inside, m>1. We get shear stabilization when the more viscous fluid is inside and $\boldsymbol{R}_1$ is increased past a critical value defined by the first zero of (21). This shear stabilization is solely due to the inertia of the core which is maintained in the analysis leading to equation (10) when $\boldsymbol{R}_1$=O(1).

5. Exact stability results for long waves

Now we shall study the stability of perfect core-annular flow to long waves α=0 without using the approximations (one of which is $\varepsilon \varnothing 0$) of lubrication theory. We will show that the two theories give rise to the same result in the limit $\alpha \varnothing 0$ and $\varepsilon \varnothing 0$. However, the results of lubrication are not robust; they are changed qualitatively when ε is finite. In particular for larger values of $\varepsilon > \tilde{\varepsilon}(m)$, the core becomes destabilizing and even $\tilde{\varepsilon}(m) \varnothing 0$ as $m \varnothing 0$. Moreover, the inertia of the lubricating fluid in the annulus which is negligible when $\varepsilon \varnothing 0$ becomes important when ε is relatively large.

The analysis of long wave solutions of the linearized theory of stability of perfect core-annular flow in a series of powers of the wave number α was carried out by Preziosi et al[2] for horizontal flow and Hickox[14], Chen et al[4] for vertical flow. These results were obtained following the solution procedure of Yih[13]. After expanding the eigenvalue C and the radial component of perturbation velocity u(r) in powers of series of wave number α,

$$C = C^{(0)} + \alpha\, C^{(1)} + O(\alpha^2),$$

$$u(r, \alpha) = u^{(0)}(r) + \alpha u^{(1)}(r) + O(\alpha^2)\,, \tag{23}$$

and solving the $O(\alpha^0)$ problem, they found that $C^{(0)}$ is real,

$$C^{(0)} = \frac{a^2\,(a^2-1)}{a^4+m-1}\,,$$

$$u_1^{(0)}(r) = \left[(a^2-1)^2-m\right] r + m\, r^3\,,$$

$$u_2^{(0)}(r) = \frac{(a^2-r^2)^2}{r}\,, \tag{24}$$

where the subscripts 1 and 2 refer to the core and annulus. Thus the stability at the leading order is determined by the sign of the imaginary part of $C^{(1)}$ which can be obtained from the solution of the $O(\alpha)$ problem. One disadvantage of Yih's procedure is that, in the final dispersion relation obtained, the contribution from the core inertia and that from the annulus inertia are not explicitly distinguishable. We noticed that, however, $C^{(1)}$ can be obtained by invoking the Fredholm alternative without actually solving the $O(\alpha)$ problem. The contributions from the inertia of the core and the annulus to $C^{(1)}$ are explicit and separate in the formula obtained by this method and this allows us to not only trace the origin of the instability, but also evaluate the relative importance of each contribution. We obtained, by using the Fredholm alternative, for the case of matched density,

$$C^{(1)} = i\left\{\boldsymbol{R}_1 \frac{1-m}{16\, m\, (a^2+m-1)\, (a^4+m-1)^2}\, (I_c + B_c + I_a + B_a) \;+ \right. \tag{25}$$

$$\left. \frac{J}{\boldsymbol{R}_1}\, \frac{-(2a^2 + m-2)\,(2a^2 + m-1) - (a^2-1)^2+(a^4 + m-1)\,(4\, ln\, a + 1 + m)}{16\, m\, (a^4 + m-1)} \right\},$$

where

$$I_c = 8\,m \int_0^1 r^2\, q_1(r)\, [W_1(r) - C^{(0)}]\, dr\,,$$

$$B_c = q_1(1)\,\{[W(1) - C^{(0)}]\,(u_1^{(0)\prime}(1) + u_1^{(0)}(1)) - W'(1)\, u_1^{(0)}(1)\}\,,$$

$$I_a = 8 \int_1^a r^2\, q_2(r)\, [W_2(r) - C^{(0)}]\, dr\,,$$

$$B_a = q_1(1)\,\{[W(1) - C^{(0)}]\,(u_2^{(0)\prime}(1) + u_2^{(0)}(1)) - W_2'(1)\, u_2^{(0)}(1)\}\,,$$

$$W_1(r) = 1 - \frac{m\,r^2}{a^2 + m - 1}\,, \tag{26}$$

$$W_2(r) = \frac{a^2 - r^2}{a^2 + m - 1}\,,$$

$$q_1(r) = D_1\, r^3 + F_1\, r\,,$$

$$q_2(r) = D_2\, r^3 + E_2\, r\,(2\,\ln r - 1) + F_2\, r + \frac{G_2}{r}\,,$$

$$D_1 = m\,(a^2 - 1)^2\,,$$

$$F_1 = (2a^2 + m - 2)\,(2a^2 + m - 1) + (1 - m)\,(a^2 - 1)^2 - (a^4 + m - 1)\,(4\,\ln a + 1 + m),$$

$$D_2 = -\,(2a^2 + m - 2)\,,$$

$$E_2 = 2(a^4 + m - 1)\,,$$

$$F_2 = 2\,\{a^2\,(2a^2 + m - 2) - 2\,(a^4 + m - 1)\,\ln a\}\,,$$

$$G_2 = (1 - m)\,(a^2 - 1)^2 + a^4 + m - 1\,,$$

$$q_1(1) = (2a^2 + m - 2)\,(2a^2 + m - 1) + (a^2 - 1)^2 - (a^4 + m - 1)\,(4\,\ln a + 1 + m) < 0\,,\ \forall m,\ a > 1.$$

Core-annular flow is stable when $\mathrm{Im}(C^{(1)})<0$, and unstable when $\mathrm{Im}(C^{(1)})>0$. It is obvious from (25) that when the viscosities are matched, m=1, only capillary instability without shear stabilization or destabilization is possible. In (25), I_c , I_a are the contributions from the bulk fluids of core and of annulus, B_c, B_a are the boundary contributions from the core and the annulus. Thus, the total inertia contributions from the core and the annulus are I_c+B_c and I_a+B_a respectively. Equation (25) enables us to separate the contribution of the annulus inertia from that of the core inertia and a direct comparison of these contributions will indicate under what circumstances that the annulus contribution is negligible and thus justify the use of lubrication theory for the annulus.

For the case of lubricated pipelining, m<1 and (25) shows that if $I_c+B_c<0$, then the core inertia is stabilizing, otherwise it is destabilizing. Similar criteria apply to the annulus inertia I_a+B_a. In Figure 1, we have plotted the contributions of inertia due to the oil and the water, which lead to shear stabilization or shear destabilization in formula (25) for $C^{(1)}$, against $a=1+\varepsilon$, using m as a parameter. The inertia I_c+B_c of the oil core is stabilizing when $\varepsilon=a-1$ is small and is destabilizing when ε is greater than a critical value $\tilde{\varepsilon}(m)$ which depends strongly on m and tends to zero with m. When m<0.8, the inertia I_a+B_a of the water annulus is stabilizing for $\varepsilon<\tilde{\tilde{\varepsilon}}(m)$, otherwise it is destabilizing. $\tilde{\tilde{\varepsilon}}(m)$ ∅0 as m∅0.8. For most of the examples given in Figure 1, $\tilde{\tilde{\varepsilon}}(m)$ is larger than the maximum value of ε plotted and thus the destabilization portions of I_a+B_a are not shown in these figures. It is evident from these comparisons that when m<0.8, both the core and the annulus contributions are stabilizing when ε is small, and the core contribution is dominant for the very smallest values of ε. But when ε exceeds certain critical value $\hat{\varepsilon}(m)$, the annulus contribution I_a+B_a becomes larger than that from the the core I_c+B_c and $\hat{\varepsilon}(m)$ tends to zero with m. This indicates that the range of ε within which the annulus contribution is negligible depends strongly on the values of m.

Figure 2 shows how the combined action of inertia $I_a+B_a+I_c+B_c$ in the core and annulus produce shear stabilization and destabilization. Though the core contribution and the annulus contribution are separately sensitive to the value of m, the combined contribution depends only weakly on m. The flow is stabilized by interfacial friction when $\varepsilon<0.5$, say, independent of m and is destabilized when ε is larger. When m is small, the shear stabilization at relatively large values of ε is achieved mainly through the contributions from the annulus inertia. For example, when m=0.1, the core inertia is destabilizing when $\varepsilon>0.15$, as shown in Figure 1(c), but shear stabilization persists up to $\varepsilon=0.5$ due to the strong stabilizing effect of the annulus inertia at relatively large values of ε (from Figure 1(c), it is evident that the annulus contribution to stabilization dominates when $\varepsilon>0.1$), as shown in Figure 2(a).

A quantitative estimate for the range of ε within which the core contribution dominates and thus the annulus contribution can be safely neglected, can be obtained by examining the asymptotic expansions for the contributions of inertia in the core and the annulus when the annulus is thin. When $\varepsilon \ll 1$, we have

$$I_c + B_c = -\frac{4}{3} m^2 \varepsilon^2 + O(\varepsilon^3),$$

$$I_a + B_a = \frac{128}{15}(-4 + 5\,m)\ \varepsilon^5 + O(\varepsilon^6). \tag{27}$$

For lubricated case m<1 and a thin annulus $\varepsilon \ll 1$, the contribution I_c+B_c of the core is always negative and stabilizing, while the contribution I_a+B_a of the annulus is negative and stabilizing only when m<0.8. Thus, when m<0.8, both the core and the annulus are stabilizing , and the contribution from the annulus is negligible if

$$-\frac{128}{15}(-4 + 5m)\ \varepsilon^5 \ll \frac{4}{3}\ m^2 \varepsilon^2 .$$

This leads to

$$\varepsilon \ll \left(\frac{5}{32}\,\frac{m^2}{4-5m}\right)^{1/3} . \tag{28}$$

When m is small, (6.8) reduces to

$$\varepsilon \ll 0.34\, m^{2/3} . \tag{29}$$

When m=0.05, (29) indicates that the inertia in the annulus can be safely neglected when $\varepsilon \ll 0.046$, which is consistent with Figure 1(b).

We may derive a criterion like (29) by quantitative arguments when m<1. The film Reynolds number is defined by

$$\boldsymbol{R}_f = \frac{W(1)\,(R_2 - R_1)}{\nu_2} . \tag{30}$$

$\boldsymbol{R}_f$ is related to the core Reynolds number $\boldsymbol{R}_1$ by the relation

$$\boldsymbol{R}_f = \boldsymbol{R}_1 \frac{\varepsilon}{m} \frac{\varepsilon\,(2+\varepsilon)}{\varepsilon\,(2+\varepsilon)+m} . \tag{31}$$

The expansion scheme leading to the omission of inertia in the annulus and to the amplitude equation (10), is actually an expansion in the parameter $\varepsilon \boldsymbol{R}_f$. For the inertia of the annulus to be negligible, one requires that

$$\varepsilon \boldsymbol{R}_f \ll 1.$$

When ε is also small, (31) reduces to

$$\boldsymbol{R}_f = \boldsymbol{R}_1 \frac{\varepsilon^2}{m^2} .$$

Hence

$$\varepsilon \ll m^{2/3} , \tag{32}$$

when $\boldsymbol{R}_1 = O(1)$. (32) is consistent with the criterion (29).

When 0.8<m<1, inertia in the core is stabilizing but inertia in the annulus is destabilizing when ε is small. In this situation, the contribution from the annulus is negligible if

$$\varepsilon \ll \left(\frac{5}{32}\,\frac{m^2}{5m-4}\right)^{1/3}. \tag{33}$$

When the core fluid is less viscous, m>1, and the multiplier in the first term on the right hand side of (25) is negative. The asymptotic formula (27) indicates that when $\varepsilon \ll 1$ the core inertia is destabilizing and the annulus inertia is stabilizing. However, when (33) is satisfied, the stabilizing effect of the annulus can be neglected. Thus the numerical studies carried out for the case m>1 by Papageorgiou et al[11] is accurate because (33) is satisfied.

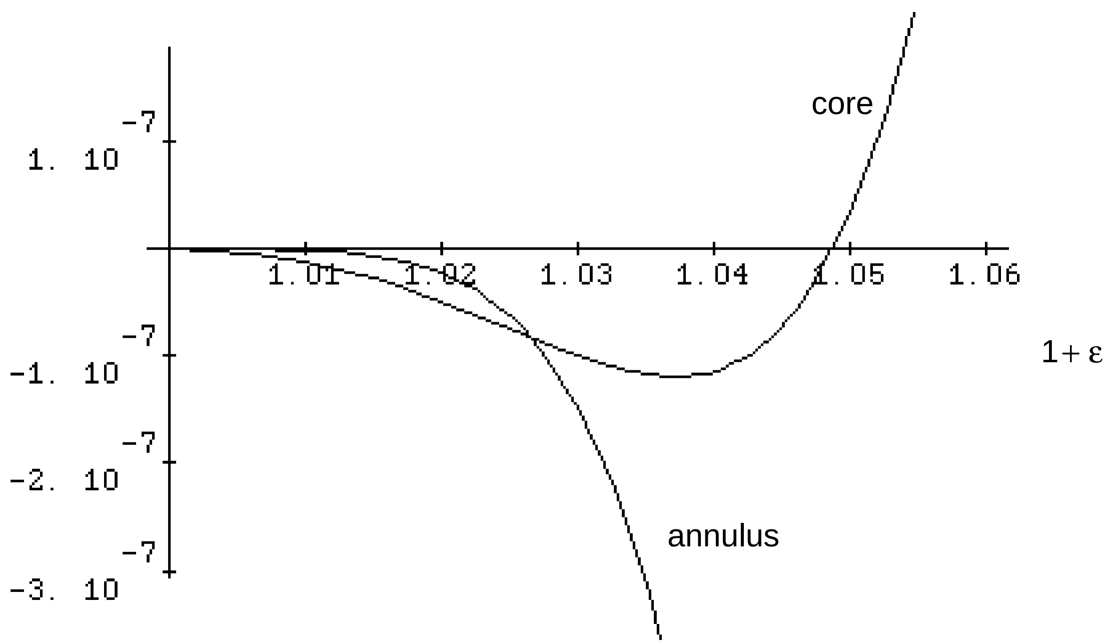

Figure 1(a) m = 0.01

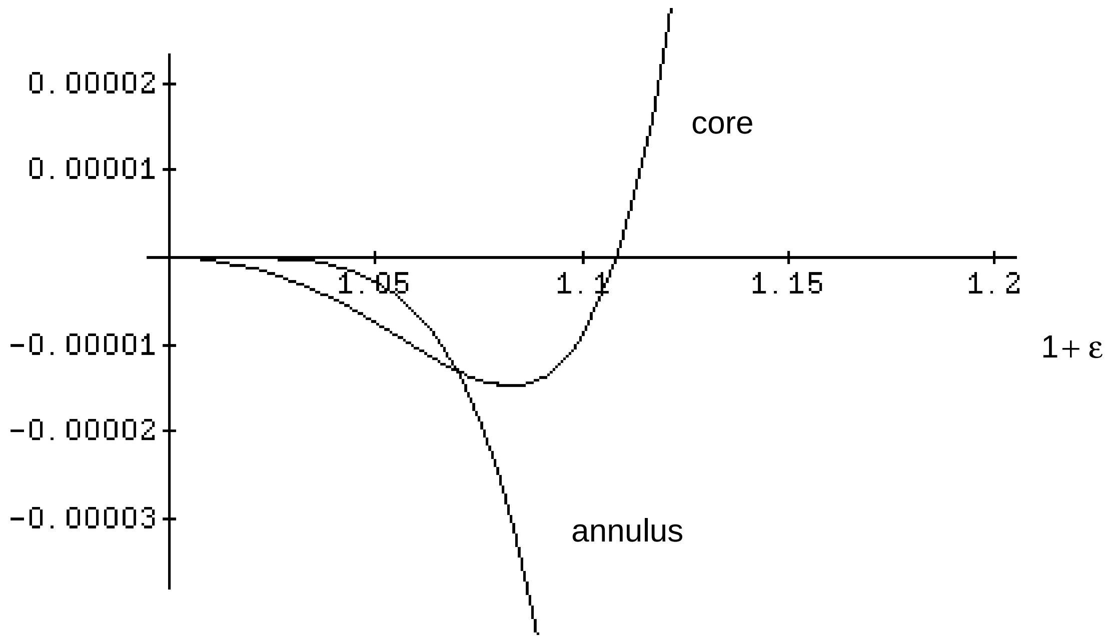

Figure 1(b) m = 0.05

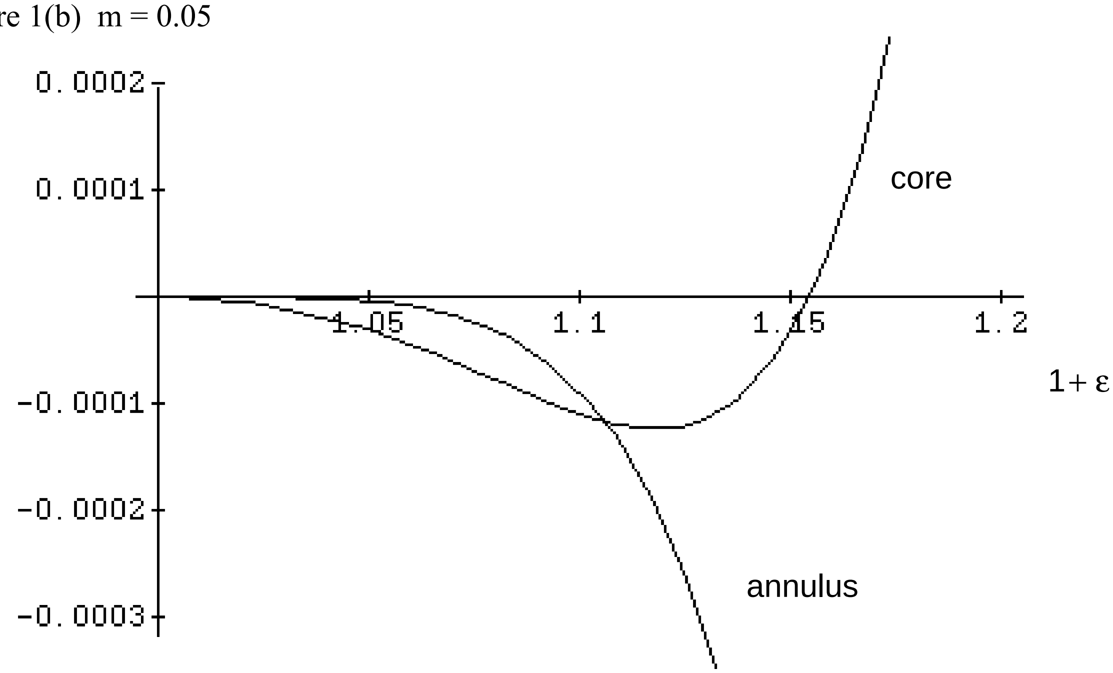

Figure 1(c) m = 0.1

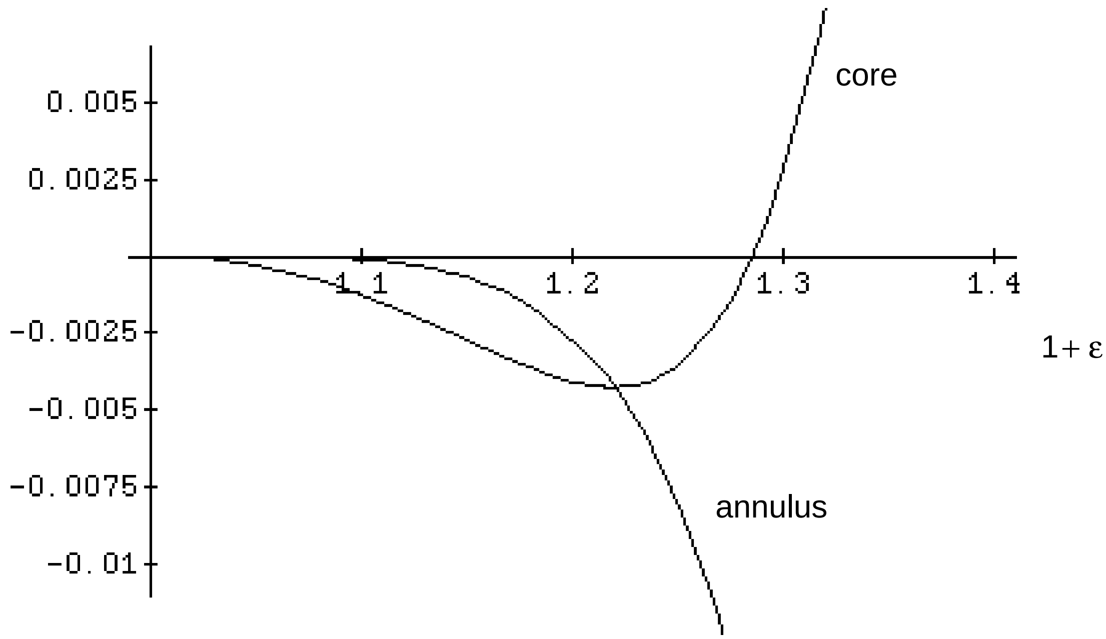

Figure 1(d) m = 0.3

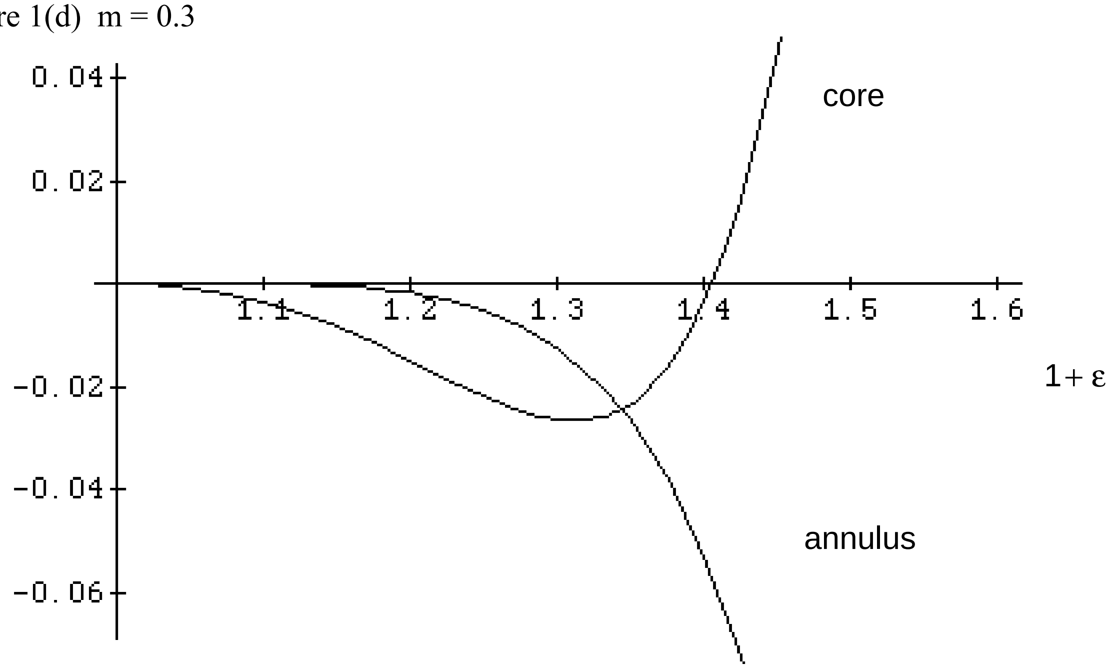

Figure 1(e) m = 0.5

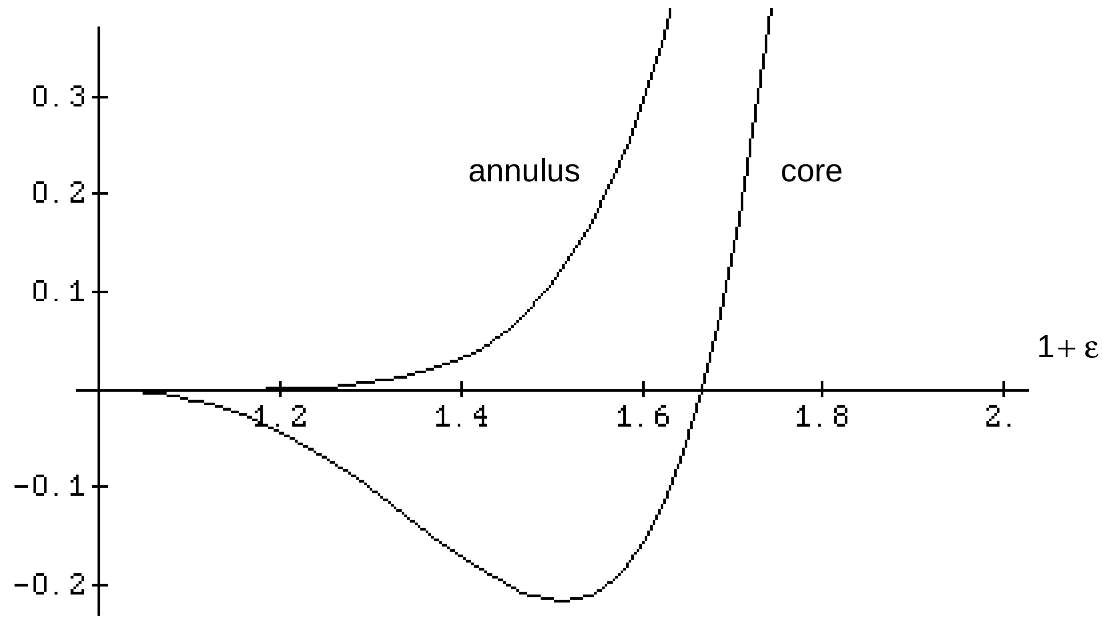

Figure 1(f) m = 0.8

Figure 1 Comparison of the contributions of inertia of the water annulus I_a+B_a and oil core I_c+B_c to the long wave instability of core-annular flow as a function of a: (a) m=0.01, (b) m=0.05, (c) m=0.1, (d) m=0.3, (e) m=0.5, (f) m=0.8.

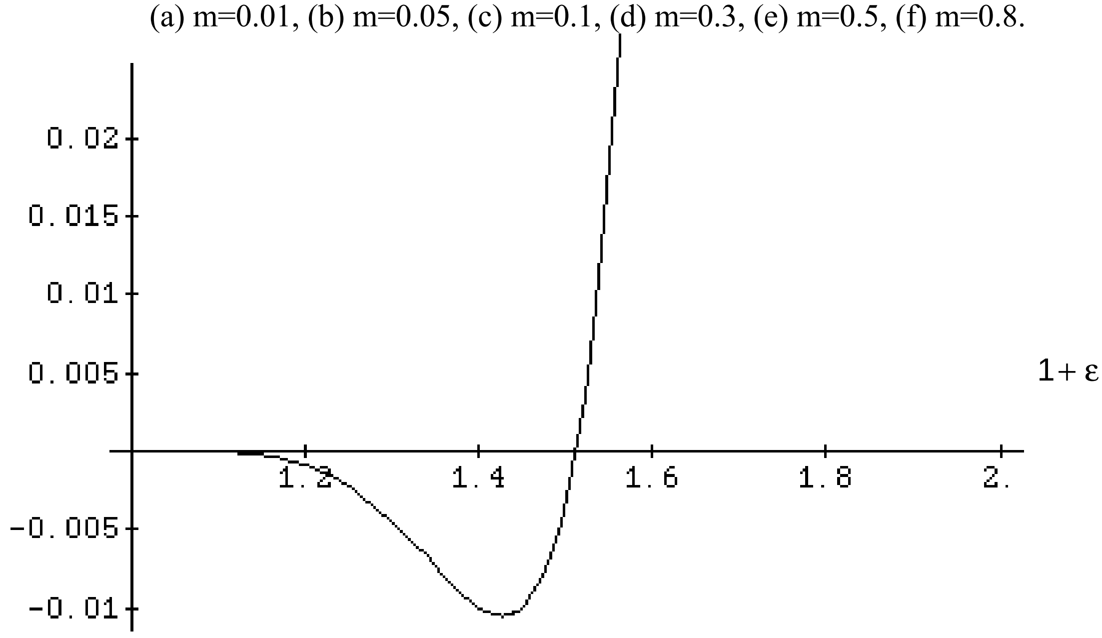

Figure 2(a) m = 0.1

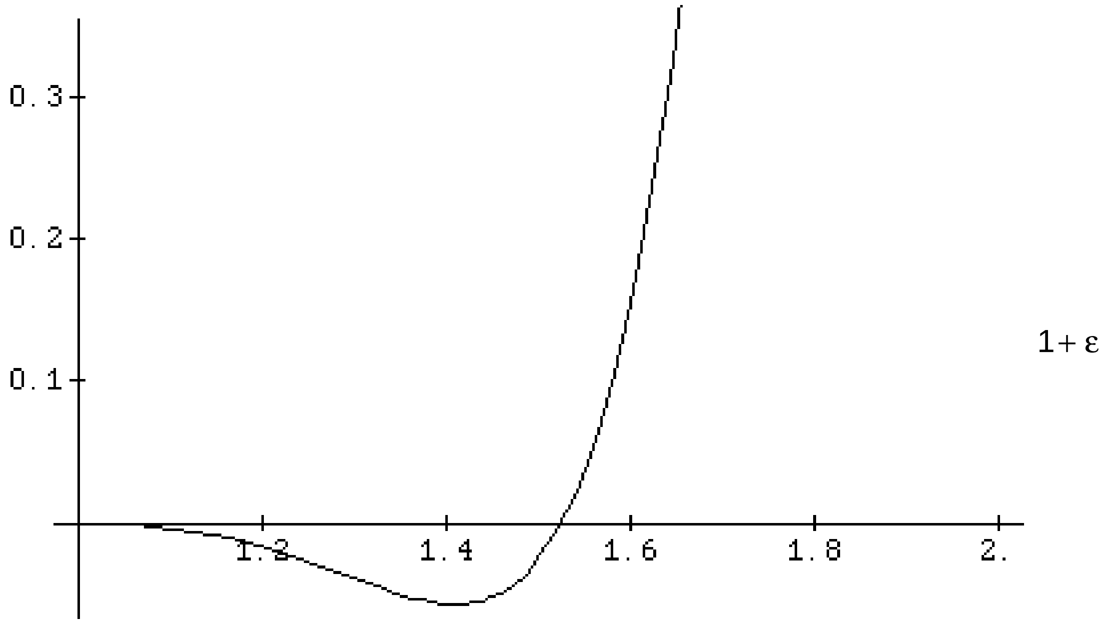

Figure 2(b) m = 0.5

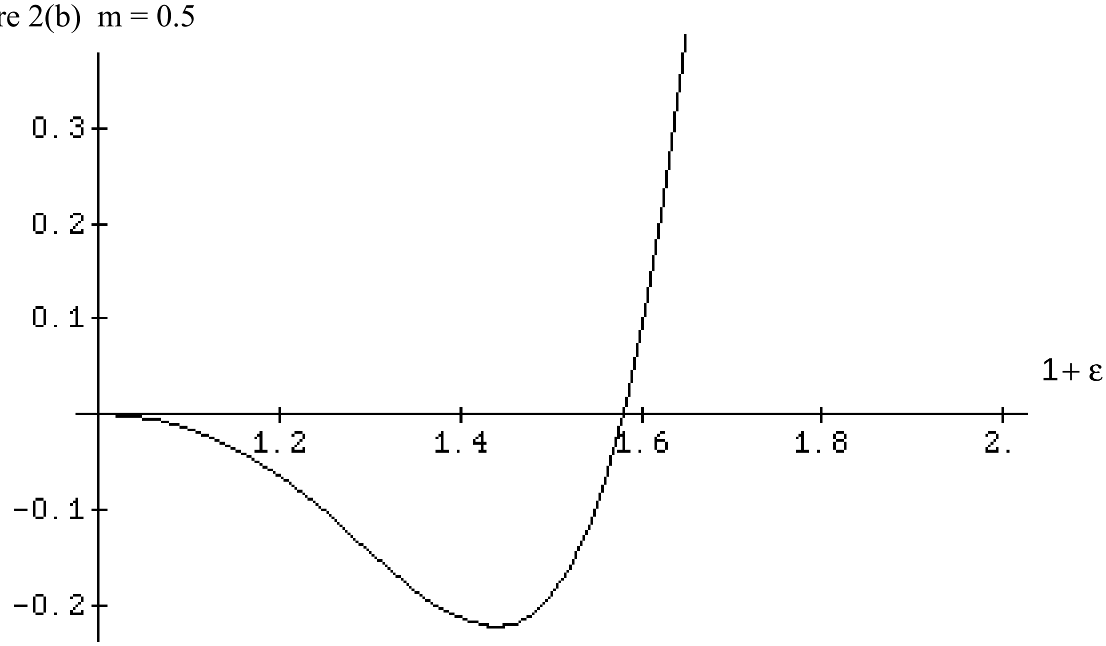

Figure 2(c) m = 0.95

Figure 2 Total contribution of inertia of the core plus the annulus $I_a+B_a+I_c+B_c$ to instability of core-annular flow. Though the separate contributions of the core and the annulus are sensitive to the values of m<1, the total contribution is insensitive. (a) m=0.1, (b) m=0.5, (c) m=0.95.

6. Discussion

For the lubricated case m<1, the linear dispersion relation resulting from the amplitude equation (10) in the limit $\alpha \varnothing 0$ approaches that of the exact problem when the condition (28) is satisfied. This can be shown by using the asymptotic formula (27). When the condition (28) is satisfied, the contribution from inertia in the annulus I_a+B_a can be neglected and we have

$$C - W(1) = \varepsilon^2 4 \frac{m-1}{m^2} + i\,\alpha\,\varepsilon^2 \left[\frac{J\,\varepsilon}{3\boldsymbol{R}_1 m} + \frac{m-1}{m^2}\frac{\boldsymbol{R}_1}{12} \right] + O(\varepsilon^3).$$

Comparing this to the dispersion relation (20), we find that

$$C - W(1) = \varepsilon^2 c + O(\varepsilon^3)\,,$$

which is in agreement with (22).

At least two types weakly nonlinear amplitude equations can be obtained for core-annular flow when the annulus is thin. The first is the one analogue to that of Hooper and Grimshaw[12] for the pipe geometry. The wave length is required to be long compared with both the core radius and the annulus thickness. The linear part of this amplitude equation is exact in the sense that the linear dispersion relation reduces to that of the full problem in the same limit. The linear mechanism of shear stabilization or destabilization is fully preserved in this amplitude equation, but the dynamics are restricted to the longest waves which, as in the case of capillary instability, are not the most strongly amplified. The second approach is that of Frenkel[10] and Papageorgiou et al[11]. Their amplitude equation is more general, because the wave length is required to be long only compared with the annulus thickness and waves with wave length comparable to the core radius are incorporated in the analysis. The trade-off of this is that, besides the strict restriction $\frac{\varepsilon J}{\boldsymbol{R}_1}$ =O(1), the contribution from inertia in the annulus is completely neglected and the linear part of the amplitude equation is only an approximation to exact linear problem. When certain extra conditions are not satisfied, important physics could be missed. In the case of lubricated pipelining, m<1, the theory is restricted to cases where the annulus thickness, $\varepsilon = a - 1$, is very small. Specifically one requires that

$$\varepsilon \ll \left(\frac{5}{32} \frac{m^2}{4 - 5\,m} \right)^{1/3} ,$$

for m<0.8. When m is very small

$$\varepsilon \ll 0.34\, m^{2/3} .$$

This condition may be hard to achieve for lubricated pipelinings, since typical values of m are of the order of 10^{-2} or smaller.

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List of figure captions

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