GENERALIZATION OF THE FOSCOLO-GIBILARO ANALYSIS OF DYNAMIC WAVES

by Daniel D. Joseph

Abstract

A new expression for the particle phase pressure in a fluidized bed generalizing the one used by Foscolo and Gibilaro is derived. In the new theory uniform fluidization is always unstable.

1. Introduction

This note reports an attempt to obtain the particle phase pressure in the Foscolo-Gibilaro [2] one-dimensional, particle in a fluidized bed model, from a constitutive hypothesis which appears to be implied by their work. Our hypothesis leads to their expression plus another term which is proportional to the space derivative of the particle velocity. We think of this term as representing a change in the microstructure, the positions of the particles relative to one another. This term is also missing from the one-dimensional equations which were recently derived by G.K. Batchelor [1]. When the new term is added we find that the state of uniform fluidization is always unstable.

2. Dynamical equations

Foscolo and Gibilaro [2] start with the coupled one-dimensional equation for the particles and fluid phase. The particle phase equations are

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi u_p}{\partial z} = 0, \tag{1}$$

$$\phi \rho_p \left[\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial z} \right] = -\phi \rho_p g + \mathbf{F} - \frac{\partial p_p}{\partial z}. \tag{2}$$

where u_p is the particle velocity, ϕ is the particle volume fraction, $\phi=1-\varepsilon$, where ε is the fluids fraction, ρ_p is the particle density, F is the interaction force, the force that the fluid exerts on the particle, and p_p is the particle phase pressure. The fluid equations are of the same form except that the subscript p is replaced by f, ϕ is replaced by ε and F by minus F.

Foscolo and Gibilaro modeled the interaction force F and the particle phase pressure in a manner that decouples the equations for the fluid and solid phases. This gives rise to a system of equations for the particles only, called the particle bed model.

It is convenient to introduce a dynamic pressure π_p into (2) by writing

$$p_p = P + \pi_p \tag{3}$$

$$\phi \rho_p g + \frac{\partial P}{\partial z} = 0 \tag{4}$$

Then (2) reduces to

$$\phi \rho_p \left[\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial z} \right] = \mathbf{F} - \frac{\partial \pi_p}{\partial z}. \tag{5}$$

<u>3. Drag on a particle in a steady fluidized suspension</u>

To get their equations they first derived an interesting expression $F_d(1)$ for the drag force exerted by the fluid on a single particle in a uniform fluidized suspension. This expression relies strongly on the well-known correlation of Richardson and Zaki for fluidized and sedimenting beds of monosized spherical particles

$$u_f = u_c^p \tag{6}$$

where

$$u_f = u_p\phi + u_f\varepsilon \tag{7}$$

is the composite velocity, the volume flux divided by total area and u_c is independent of z, $\frac{\partial u_f}{\partial z} = 0$. Of course $V=u_f$ when $\varepsilon=1$, the steady terminal velocity of a freely falling single sphere in a sea of fluid. The exponent n depends on the Reynolds number $Re = \frac{dV}{\mu}$ where d is the diameter

$$n = \begin{cases} 4.65 \text{ for } Re < 0.2\,, \\ 4.4\,Re^{-0.03} \text{ for } 0.2 < Re < 1\,, \\ 4.4\,Re^{-0.1} \text{ for } 1 < Re < 500\,, \\ 2.4 \text{ for } Re > 500\,. \end{cases} \tag{8}$$

Foscolo-Gibilaro replace 4.65 with 4.8=2(2.4) for reasons to be made clearer later.

There is a huge amount of fluid mechanics buried in the Richardson-Zaki correlation. This is hidden in the drag law for particles falling under gravity in steady flow. Let $F_d(\varepsilon)$ be the drag on a single particle in a freely falling suspension with a water fraction ε. When $\varepsilon=1$ we get a drag law for the free fall of a single sphere which is Stokes drag when V is small enough; for larger V the drag is given by

$$F_d(1) = \frac{\rho V^2}{2}\,\frac{\pi d^2}{4}\,C_D \tag{9}$$

where C_D is given by an empirical correlation, for example,

$$C_D = \left(0.63 + \frac{4.90}{\sqrt{re}}\right)^2$$

said to be due to Dallavalle. Foscolo and Gibilaro produce the formula

$$F_d(\varepsilon) = \varepsilon F_d(1) \tag{10}$$

from an argument which says that in a fluidized bed in steady flow, the total force F on a sphere is the sum

$$F(\varepsilon, Re) = F_d(\varepsilon) - F_p(\varepsilon) \tag{11}$$

where

$$F_p(\varepsilon) = \frac{\pi d^3}{6} (\rho_p - \rho_f)\ \ g \tag{12}$$

is the buoyant force using the effective density

$$\rho_f = \varepsilon\rho_p + \phi\rho_p \tag{13}$$

of the composite fluid. Since $\phi = 1 - \varepsilon$,

$$F_p(\varepsilon) = -\frac{\pi d^3}{\rho} (\rho_p - \rho_f)\ \ g\varepsilon = \varepsilon F_p(1) \tag{14}$$

In steady flow, F = 0 and

$$F_d(\varepsilon) = F_p(\varepsilon) = \varepsilon F_p(1) = \varepsilon F_d(1) \tag{15}$$

We never see steady flow in a fluidized bed, the particles always jiggle about; steady is in some statistical sense, whatever that may be. In any interpretation

$$u_p = 0 \text{ in steady flow .} \tag{16}$$

Equation (15) is all that is required to get the drag on a single particle in a fluidized suspension in steady flow. The hydrodynamic content is all buried in the drag correlation (9). We may write $F_d(\varepsilon) = \varepsilon F^1(V)$. To see how $F_d(\varepsilon)$ depends on the fluidizing velocity u_c, Foscolo and Gibilaro note that (9) implies that

$$F_d = \varepsilon \begin{cases} 3\pi\mu V & \text{(laminar)} \\ 0.055\pi\rho d^2 V^2 & \text{(turbulent)}. \end{cases}$$

They next note that the Richardson and Zaki correlation (6) and (8), with 4.8 replacing 4.65, implies that

$$F_d = \varepsilon^{-3.8} \begin{cases} 3\pi\mu d u_c & \text{(laminar)} \\ 0.055\pi\rho d^2 u_c^2 & \text{(turbulent)}. \end{cases} \tag{17}$$

This is good, we have $F_d(u_f, \varepsilon) = \varepsilon^{-3.8}F_d(u_f)$, independent of V for low and high Reynolds numbers. Now we look for an equivalent expression, valid for all Reynolds numbers in steady flow and

$$F_d(\varepsilon) = F_d(\varepsilon, u_f, V) = \varepsilon^{-3.8} g(u_f, V) \tag{18}$$

which will reduce to (17) at low and high Re. Clearly

$$g(u_c, v) = \varepsilon^{4.8} F_d(1) = \left(\frac{u_c}{V}\right)^{\frac{4.8}{n}} F_d(1).$$

Hence

$$F_d(\varepsilon, u_c, V) = \varepsilon^{-3.8}\left(\frac{u_c}{V}\right)^{\frac{4.8}{n}} F_d(1). \tag{19}$$

This is just another way of writing $F_d(\varepsilon) = \varepsilon F_d(1)$ when 4.65 is replaced with 4.8 which is useful in motivating the constitutive equation (21) below.

4. The first constitutive hypothesis giving the force per unit volume on the spheres

Foscolo and Gibilaro assume that in unsteady flow the force on a particle is given by the expression (19) with u_c replaced by the slip velocity

$$u_c - u_p = (1-\varepsilon)u_p + \varepsilon u_f - u_p = \varepsilon(u_p - u_f). \tag{20}$$

Then the unsteady drag force is

$$F_d(\varepsilon, u_c - u_p, V) = \varepsilon^{-3.8}\left(\frac{u_c - u_p}{V}\right)^{\frac{4.8}{n}} F_d(1). \tag{21}$$

In steady flow, u_p = 0, and (21) reduces to

$$F_d(\varepsilon) = \varepsilon F_d(1) \tag{22}$$

where balancing drag and buoyancy for a single sphere gives

$$F_d(1) = \frac{\pi d^3}{6}(\rho_p - \rho_f)g.$$

The total force on single particle in a fluidized suspension is given by

$$F = F_d - F_b = \frac{\pi d^3 g}{6}(\rho_p - \rho_f)\left\{\varepsilon - \left[\frac{u_c - u_p}{V}\right]^{\frac{4.8}{n}} \varepsilon^{-3.8}\right\}. \tag{23}$$

The force per unit volume due to all n spheres is

$$F = NF \tag{24}$$

where

$$N = \frac{\phi}{\pi d^3/6} = \frac{n}{\text{volume}}. \tag{25}$$

Hence, the total force on the particles per unit volume is

$$\mathbf{F} = \phi(\rho_p - \rho_f)g\left\{\varepsilon - \left[\frac{u_c - u_p}{V}\right]^{\frac{4.8}{h}} \varepsilon^{-3.8}\right\}. \tag{26}$$

In steady flow, u_p and F = 0.

5. The second constitutive hypothesis giving the particle phase pressure

The same force of the fluid on the particles acts at the boundary to keep the fluids from dispersing. However we need to multiply the force on a single particle by the number N_A per unit area

$$N_A = \frac{\phi}{\pi d^2/4}$$

Hence, the dynamic pressure is given by

$$\pi_p = N_A F = \frac{N_A}{N}\mathbf{F} = \frac{2}{3}d\mathbf{F}$$

The idea of making a constitutive equation for the pressure is more allied to gas dynamics where the pressure is a state variable than to incompressible fluid mechanics. In discussing forces which fluids exert on particles G.K. Batchelor [1] noted that in his list (2.3) of forces there is a "… mean force exerted on particles in this volume by the particles outside the volume." Further he notes that the nature of these two forces

> "… may be explained by reference to a hypothetical case in which the particles are electrically charged and exert repulsive electrostatic forces on each other. The range of action of these electrostatic forces is small by comparison with the dimensions of the dispersion, and so the mean resultant force exerted on the particles inside τ, that is, by stress, –S say, which is a function of the local particle concentration.
>
> "Electrostatic interparticle forces are conservative, and in that case one can interpret –S as the derivative of the mean potential energy per particle with respect to the volume of the mixture per particle. The contribution to the net force exerted on particles in our control volume by external particles is then
>
> $$-A\int_{x_1}^{x_2} \frac{\partial s}{\partial x}dx. \qquad (2.12)$$

A repulsive force between particles corresponds to a positive value of S (relative to zero when the particles are far apart), in which case S plays a dynamical role analogous to the pressure in a gas."

The equations of motion (1) and (2) are now reduced to

$$\frac{\partial\phi}{\partial t}+\frac{\partial}{\partial z}\phi u_p=0, \tag{27}$$

$$\rho_p\phi\left[\frac{\partial u_p}{\partial t}+u_p\frac{\partial u_p}{\partial z}\right]=\mathbf{F}-\frac{2}{3}d\frac{\partial\mathbf{F}}{\partial z} \tag{28}$$

where F is given by (26) and

$$\frac{\partial\mathbf{F}}{\partial z}=\frac{\partial\mathbf{F}}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial z}+\frac{\partial\mathbf{F}}{\partial u_p}\frac{\partial u_p}{\partial z} \tag{29}$$

and

$$\frac{\partial\mathbf{F}}{\partial\varepsilon}=\frac{\partial N}{\partial\varepsilon}F+N\frac{\partial F}{\partial\varepsilon}. \tag{30}$$

Equations (27) and (28) are two nonlinear equations in two unknowns, ε and u_p. These equations differ from the ones derived by Foscolo and Gibilaro [2] to which they reduce when the two additional terms

$$\frac{\partial\mathbf{F}}{\partial u_p}\frac{\partial u_p}{\partial z} \tag{31}$$

and

$$\frac{\partial N}{\partial\varepsilon}\ \mathrm{F} \tag{32}$$

are put to zero. The term (32) vanishes in the analysis of stability of uniform fluidization but (31) does not.

The term (31) also is absent from the list of forces which act in this problem developed by Batchelor [1]. Hence, we are obliged to consider the physical origin of such a term. We may regard the term (31) as arising from changes in the microstructure of the mixture. This has been well expressed in a recent paper by Ham and Homsy [3].

> "Analysis of the mean settling speed leaves unresolved the problem of microstructural evolution in suspensions. Such changes in the relative positions of particles are likely because each particle in a random suspension sees a slightly different local environment and is therefore expected to have a velocity which is, in general, different from that of any neighboring particle. The variations in particle velocities will lead to an adjustment of the particle distribution."

They note further that

> "…the microstructural dependence arises from the fact that the time between the velocities of the faster- and slower-setting particles, and the difference will be influenced by the relative position of the particles. The influence of ϕ comes about from the change in interparticle spacing with concentration of particles."

6. Stability of uniform fluidization

Equations (27) and (28) are satisfied by the steady state of uniform fluidization

$$u_p = 0, \varepsilon = \varepsilon_0 \tag{33}$$

where ε_0 is independent of z and t. Let ε and u_p now stand for small perturbations of ε_0 and 0. Then, we find that

$$\frac{\partial \varepsilon}{\partial t} = \phi \frac{\partial u_p}{\partial z}, \tag{34}$$

$$\phi \frac{\partial u_p}{\partial t} = -B(\phi_0 u_p + c_1 \varepsilon) + c_2^2 \frac{\partial \varepsilon}{\partial z} + \frac{2}{3}\phi_0 dB \frac{\partial u_p}{\partial z} \tag{35}$$

where

$$
\begin{aligned}
c_2^2 &= 3.2\phi_0 \tilde{g}\, d, \\
\tilde{g} &= \frac{g(\rho_p - \rho_f)}{\rho_p}, \\
c_1 &= nV\varepsilon_0^{n-1}\phi_0, \\
B &= \frac{4.8\tilde{g}}{nV\varepsilon_0^{n-1}}.
\end{aligned} \tag{36}
$$

After eliminating u_p between (34) and (35) we get

$$
\frac{\partial^2\varepsilon}{\partial t^2} - c_2^2\frac{\partial^2\varepsilon}{\partial z^2} - \frac{2}{3}dB\frac{\partial^2\varepsilon}{\partial t\partial z} + B\left(\frac{\partial\varepsilon}{\partial t} + c_1\frac{\partial\varepsilon}{\partial z}\right) = 0. \tag{37}
$$

This equation is in the form derived by Wallis [5] but the meanings of the coefficients are altogether different, as are the results of stability analysis using normal modes proportional to exp $\{-i\alpha\,(x - \omega t)\}$. We find that the uniform is unstable whenever

$$
c_1 + \frac{1}{3}\ dB > \sqrt{c_2^2 + \frac{1}{9}d^2\,B^2} \tag{38}
$$

This reduces to the well known criterion of Wallis when B = 0. In the present case, the inequality (38) is always satisfied and the uniform state is always unstable.

The unsteady drag law (21) used by Foscolo and Gibilaro is controversial. I have not seen a compelling argument for or against it. Perhaps we can finally judge (21) only after we know what it implies.

References

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Acknowledgements

This work was supported under grants from the Dept. of Energy, ARO, and NSF. I wish to thank Constantinos Sioutas for his help with some aspects of §6.