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NOTE

A Note on the Net Force and Moment on a Drop Due to Surface Forces

It is shown that the net force and moment on a smooth drop or bubble due to surface forces are zero. The net force and moment due to the jump in traction are also zero. © 1993 Academic Press, Inc.

INTRODUCTION

When two immiscible fluids lie in contact, forces arise in the interface because of the differing attraction between like and unlike molecules. It is an important fact that the *net* force $\boldsymbol{F}$ and moment $\boldsymbol{M}$ (about a point $\boldsymbol{x}_0$) exerted by these forces on a smooth drop (bubble) of one fluid in another both vanish. That is,

$$\boldsymbol{F} = 0 \quad [1]$$

and

$$\boldsymbol{M} = 0. \quad [2]$$

It is clear intuitively that [1] and [2] must hold. Otherwise, an air bubble could not exist in translational and rotational equilibrium in the absence of external forces.

The purpose of this note is to provide a proof of this fundamental result and to discuss some of its consequences. Because it is no more difficult, the proof will be given for the general case in which the force ξ per unit length, exerted by the surface forces across a line element in the surface $\Sigma(t)$ of the drop, has a *shear*, as well as a tension component. We will see, however, that ξ cannot have a component *normal* to Σ, if angular momentum is to be conserved.

A proof of [1] (though not of [2], [21], or [22]) was given by Prosperetti and Jones for the special case in which ξ has only a tension component (see (1), Appendix B). Though they claim the result only for *constant* tension, their proof is valid for non-constant tension as well.

EFFECTIVE SURFACE FORCE AND MOMENT DENSITIES

To compute $\boldsymbol{F}$ and $\boldsymbol{M}$, the effective local force $\boldsymbol{f}$ and moment $\boldsymbol{m}$ per unit area, exerted by the surface forces on the bulk fluid adjacent to Σ, must be integrated over Σ. That is,

$$\boldsymbol{F} = \oint_{\Sigma} \boldsymbol{f} dA \quad [3]$$

and

$$\boldsymbol{M} = \oint_{\Sigma} \boldsymbol{m} dA. \quad [4]$$

The expressions for $\boldsymbol{f}$ and $\boldsymbol{m}$ in terms of ξ, the force per unit length acting *between adjacent portions of* Σ, will be derived from the conservation laws for the linear and angular momentum of an arbitrary small material volume $V(t)$ intersecting Σ,

$$\frac{d}{dt}\int_V \rho \boldsymbol{u} dV = \int_V \rho \boldsymbol{g} dV + \oint_{\partial V} \boldsymbol{t} dA + \oint_{\partial V \cap \Sigma} \xi ds \quad [5]$$

and

$$\frac{d}{dt}\int_V \boldsymbol{r} \times \rho \boldsymbol{u} dV = \int_V \boldsymbol{r} \times \rho \boldsymbol{g} dV + \oint_{\partial V} \boldsymbol{r} \times \boldsymbol{t} dA + \oint_{\partial V \cap \Sigma} \boldsymbol{r} \times \xi ds, \quad [6]$$

using standard arguments. Here $\boldsymbol{t}$ is the traction force and $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$.

It follows from [5], using the Cauchy stress theorem and its surface analog, that

$$\boldsymbol{t} = \mathbf{T}^T \hat{\boldsymbol{n}} \quad [7]$$

and

$$\xi = \mathcal{T}^T \hat{\tau}, \quad [8]$$

where $\hat{\boldsymbol{n}}$ is the unit outward normal to the surface element across which $\boldsymbol{t}$ acts, $\hat{\tau}$ is the unit outward normal—in Σ—to the line element across which ξ acts, $\mathbf{T}$ is the Cauchy stress tensor, and $\mathcal{T}$ is the surface stress tensor. Note that [8] defines $\mathcal{T}^T$ only for vectors $\hat{\tau}$ *tangent* to Σ. The value of $\mathcal{T}^T \hat{\boldsymbol{n}}_\Sigma$, where $\hat{\boldsymbol{n}}_\Sigma$ is the unit normal to Σ pointing from side 1 to side 2, is undetermined. For later convenience, we define

$$\mathcal{T}^T \hat{\boldsymbol{n}}_\Sigma = \mathbf{0}. \quad [9]$$

Introducing [7] and [8] into [5], applying the generalized transport theorem, divergence theorem, and surface divergence theorem (see (2), Appendix A), and making use of [9] and the continuity of $\boldsymbol{u}$ across Σ, we obtain the Cauchy equation of motion

$$\rho \frac{d\boldsymbol{u}}{dt} = \rho \boldsymbol{g} + \operatorname{div} \mathbf{T}^T \quad [10]$$

and the interface condition

$$\operatorname{div}_{\text{II}} \mathcal{T}^T - [\![\mathbf{T}^T]\!] \hat{\boldsymbol{n}}_\Sigma = \mathbf{0} \quad \text{on } \Sigma, \quad [11]$$

where $\operatorname{div}_{\text{II}}$ is the surface-divergence operator on Σ and $[\![\cdot]\!] \overset{def}{\equiv} (\cdot)_1 - (\cdot)_2$ is the jump in $(\cdot)$ across Σ.

Operating similarly on [6] and making use of [10] and [11], we find that

$$\mathbf{T}^T = \mathbf{T}, \quad [12]$$

$$\mathcal{T}^T = \mathcal{T}, \quad [13]$$

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and

$$\hat{\boldsymbol{n}}_\Sigma \cdot \mathcal{T}^T \hat{\tau} = \hat{\boldsymbol{n}}_\Sigma \cdot \boldsymbol{\xi} = \mathbf{0}. \quad [14]$$

Thus, $\boldsymbol{\xi}$ can have no component normal to Σ, as claimed earlier.

The expressions for $\boldsymbol{f}$ and $\boldsymbol{m}$ can now be determined using the conservation laws for the linear and angular momentum of the portion V_1 of V on side 1 of Σ (which in the present setting is a *material* volume),

$$\frac{d}{dt}\int_{V_1} \rho \boldsymbol{u} dV = \int_{V_1} \rho \boldsymbol{g} dV + \int_{\partial V_1 \setminus \Sigma} \mathbf{T}\hat{\boldsymbol{n}} dA + \int_{\partial V_1 \cap \Sigma} \mathbf{T}_2 \hat{\boldsymbol{n}}_\Sigma dA + \int_{\partial V_1 \cap \Sigma} \boldsymbol{f} dA \quad [15]$$

and

$$\frac{d}{dt}\int_{V_1} \boldsymbol{r} \times \rho \boldsymbol{u} dV = \int_{V_1} \boldsymbol{r} \times \rho \boldsymbol{g} dV + \int_{\partial V_1 \setminus \Sigma} \boldsymbol{r} \times \mathbf{T}\hat{\boldsymbol{n}} dA + \int_{\partial V_1 \cap \Sigma} \boldsymbol{r} \times \mathbf{T}_2 \hat{\boldsymbol{n}}_\Sigma dA + \int_{\partial V_1 \cap \Sigma} \boldsymbol{m} dA. \quad [16]$$

Applying the transport and divergence theorems to [15] and making use of [10] and [11], we find that

$$\boldsymbol{f} = \text{div}_{\text{II}} \mathcal{T}. \quad [17]$$

Operating similarly on [16] and making use of [12], we find that

$$\boldsymbol{m} = \boldsymbol{r} \times \text{div}_{\text{II}} \mathcal{T}. \quad [18]$$

If $\boldsymbol{\xi} = \sigma \hat{\tau}$ is a pure *tension* with (possibly varying) magnitude $\sigma > 0$, then, in light of [8] and [9],

$$\mathcal{T} = \sigma \mathbf{1}_{\text{II}},$$

where $\mathbf{1}_{\text{II}} = \mathbf{1} - \hat{\boldsymbol{n}}_\Sigma \otimes \hat{\boldsymbol{n}}_\Sigma$ is the two-dimensional identity tensor on Σ. In this case, we obtain

$$\boldsymbol{f} = \text{div}_{\text{II}}(\sigma \mathbf{1}_{\text{II}}) = \mathbf{grad}_{\text{II}} \sigma + 2H\sigma \hat{\boldsymbol{n}}_\Sigma, \quad [19]$$

where $\mathbf{grad}_{\text{II}}$ is the surface-gradient operator on Σ and $H = -\text{div}_{\text{II}} \hat{\boldsymbol{n}}_\Sigma / 2$ is the mean curvature of Σ. When σ is constant, this reduces to the usual Laplace formula.

COMPUTATION OF THE NET FORCE AND MOMENT

Let Σ be divided into two subsurfaces Σ_1 and Σ_2 by a closed curve in Σ and let $\hat{\tau}_1$ and $\hat{\tau}_2$ be the unit outward normals—in Σ—to $\partial\Sigma_1 = \partial\Sigma_2$ for Σ_1 and Σ_2, respectively. Clearly,

$$\hat{\tau}_2 = -\hat{\tau}_1. \quad [20]$$

Introducing [17] into [3], expressing the right-hand side as a sum of integrals over Σ_1 and Σ_2, and using the surface divergence theorem together with [9] and [20], we obtain

$$\begin{aligned} \boldsymbol{F} &= \int_{\Sigma_1} \text{div}_{\text{II}} \mathcal{T} dA + \int_{\Sigma_2} \text{div}_{\text{II}} \mathcal{T} dA \\ &= \int_{\partial\Sigma_1} \mathcal{T}\hat{\tau}_1 ds + \int_{\partial\Sigma_2} \mathcal{T}\hat{\tau}_2 ds \\ &= \mathbf{0}, \end{aligned}$$

proving [1]. Similarly, introducing [18] into [4] and using [12], [13], and [14], we obtain

$$\begin{aligned} \boldsymbol{M} &= \int_{\Sigma_1} \boldsymbol{r} \times \text{div}_{\text{II}} \mathcal{T} dA + \int_{\Sigma_2} \boldsymbol{r} \times \text{div}_{\text{II}} \mathcal{T} dA \\ &= \int_{\partial\Sigma_1} \boldsymbol{r} \times \mathcal{T}\hat{\tau}_1 ds + \int_{\partial\Sigma_2} \boldsymbol{r} \times \mathcal{T}\hat{\tau}_2 ds \\ &= \mathbf{0}, \end{aligned}$$

and [2] is proved.

A consequence of [1] is that the net force exerted on the drop by the jump in traction also vanishes. That is,

$$\int_\Sigma [\![\mathbf{T}]\!] \hat{\boldsymbol{n}}_\Sigma dA = \mathbf{0}. \quad [21]$$

This can be seen by integrating [11] over Σ and applying [17], [3], and [1]. Similarly we find, as a consequence of [2] and [11], that

$$\int_\Sigma \boldsymbol{r} \times [\![\mathbf{T}]\!] \hat{\boldsymbol{n}}_\Sigma dA = \mathbf{0}; \quad [22]$$

that is, the net moment exerted on the drop by the jump in traction vanishes.

DISCUSSION

The essence of the proof of [1] and [2] is that $\boldsymbol{f}$ and $\boldsymbol{m}$ are *exact differentials* and therefore have zero integral over any closed surface Σ, since $\partial\Sigma = \varnothing$ for such a surface. It is instructive to examine [1] in two dimensions. In this case, $\boldsymbol{\xi}$ is necessarily a pure tension. Introducing [19] into [3] and noting that $\mathbf{grad}_{\text{II}}\sigma = (d\sigma/ds)\hat{\tau}$ and $2H\hat{\boldsymbol{n}}_\Sigma = d\hat{\tau}/ds$, where s is the arclength along Σ (which is now a *curve* in the plane), we obtain

$$\boldsymbol{F} = \oint_\Sigma \left(\frac{d\sigma}{ds}\hat{\tau} + \sigma \frac{d\hat{\tau}}{ds} \right) ds = \oint_\Sigma \frac{d}{ds}(\sigma\hat{\tau}) ds = \mathbf{0},$$

since the initial and terminal points of the curve Σ coincide. Note that $\hat{\tau}$ is now the unit tangent vector to the curve Σ. When σ is not constant, the integrals of the two terms in the integrand do not (necessarily) vanish separately—only in combination.

Of course, $\boldsymbol{f}$ and $\boldsymbol{m}$ are exact differentials because they were originally obtained from [5] and [6] by localization, using the surface divergence theorem. Thus, in a sense, the proof is redundant. First it uses the surface divergence theorem to reduce the original integrals over $\partial\Sigma$ to integrals over Σ in order to obtain $\boldsymbol{f}$ and $\boldsymbol{m}$ as exact differentials. Then it uses the surface divergence theorem *backward* to integrate these exact differentials over Σ and obtain integrals over $\partial\Sigma$ again.

Thus, it might seem that [1] and [2] could be deduced *directly* by examination of the surface terms in [5] and [6]. The difficulty with this approach is that [5] and [6] apply only to material volumes V for which $\partial V \cap \Sigma$ is a *curve* in ∂V. If V is the interior of the drop, then $\partial V \cap \Sigma$ is *all* of ∂V, since $\partial V = \Sigma$ in this case.

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