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APPLICATION OF BINARY SEQUENCES TO PROBLEMS OF CHAOS

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SUMMARY

Oil and water in equal proportions are set in motion between horizontal concentric cylinders when the inner one rotates. In a range of speeds where the water is Taylor unstable and the oil Taylor stable, we get Taylor cells. The main focus of this paper is the mathematical description of the apparently chaotic trajectory of a small oil bubble moving between an eddy pair in a single Taylor cell trapped between the oil bands of a banded Couette flow. We define a discrete autocorrelation sequence on a binary sequence associated with left and right transitions in the cell to show that the motion of the bubble is chaotic. A formula for a macroscopic Lyapunov exponent for chaos on binary sequences is derived and applied to the experiment and to the Lorenz equation to show how binary sequences can be used to discuss chaos in continuous systems. We use our results and recent results of Feeny and Moon (1989)[1] to argue that Lyapunov exponents for switching sequences are not convenient measures for distinguishing between chaos (short range predictability) and white noise (no predictability).

1. EXPERIMENTS

Experiments were carried out between two concentric cylinders with axis horizontal, perpendicular to gravity. The outer cylinder end plates are plexiglass, while the inner cylinder is aluminium. The inside diameter of the outer cylinder is 2·495 inches; the outside diameter is 2·986 inches. The inner cylinder has a diameter of 1·985 inches, and a length of 11·985 inches. The outer cylinder is fixed and the inner one rotates with angular velocity Ω.

Our Taylor apparatus uses two neoprene lip seals to prevent leakage. The shaft driving the inner cylinder is connected to a torque meter which has a provison for counting r.p.m. The torque meter is connected to a mechanical–digital converter which displays the value of the torque and rate of rotation. We used Mobil heavy duty oil with density $0\cdot97\ \mathrm{g\,cm^{-3}}$ and viscosity 0·95 poise, and tap water, both at the laboratory temperature of 25° C. The interfactial tension between Mobil heavy-duty motor oil and tap water is $30\cdot00\ \mathrm{dyne\,cm^{-1}}$. The major effect of the density difference occurs in slow or lubricated flow in which the oil floats up. When the angular velocity is smaller than 500 r.p.m., the two fluids are arranged in alternating bands. When the r.p.m. is greater than 10 the water bands are Taylor unstable whereas the oil bands are Taylor stable. The Taylor instability leads to the formation of two counter-rotating vortices in the water band.

2. CHAOTIC TRAJECTORIES OF OIL BUBBLES IN AN UNSTABLE WATER CELL

At sufficiently high values of the angular velocity prior to emulsification of motor oil, the secondary motion appears to be chaotic. In some situations we were able to get one small oil

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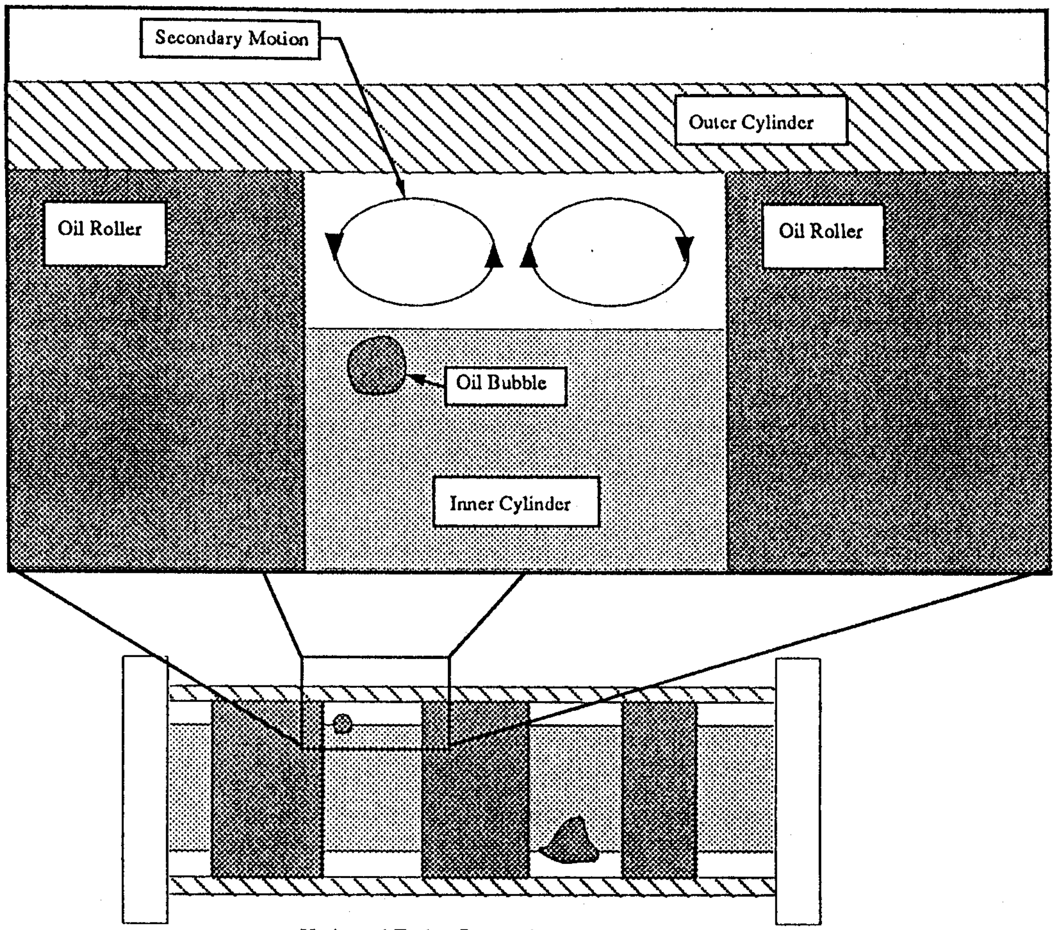

Figure 1. Horizontal Taylor–Couette apparatus

bubble into a Taylor cell. This oil bubble was carried around by water and dragged around in the secondary motion owing to Taylor instability. The accompanying videotape contains segments of a recording made of one such bubble. The small oil bubble on the left is the one for which the binary sequence is studied. Each time the oil drop went around it was either in the left eddy or in the right eddy. We monitored about 3000 terms in the sequence LRLL... and assigned the number -1 to the left and 1 to the right. It is difficult to get revealing still photographs of the motion of the small bubble in the leftmost water cell, but the video displays the motion quite well. A drawing of the apparatus including a cross-section of the water cell is presented in Figure 1.

2.1. Binary sequences

We are going to apply methods of estimation theory (see Singh and Joseph, 1989)[2] to characterize the chaos in the binary number sequence generated by the bubble in our experiments. Consider a sequence $u(n) = \pm 1$ of binary numbers. We assume that the sequence is ergodic so that time averages are the same as ensemble averages. In our experiment the

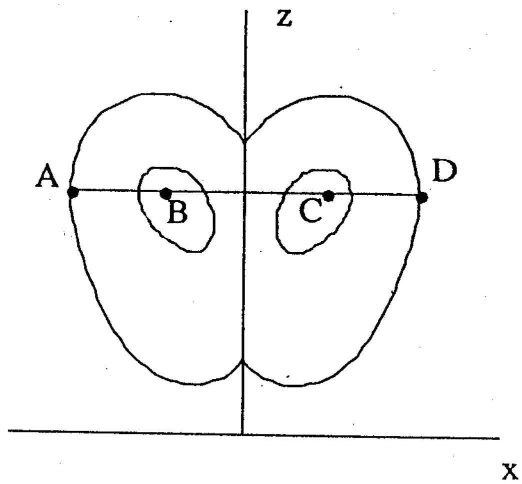

Figure 2. The projected trajectories of the Lorenz attractor remain inside the butterfly region and outside the ovals around the fixed points

average

$$E[u(n)] = \frac{1}{N}\sum_{n=1}^{N} u(n) \to 0$$

when N is large, left and right (or ± 1) are equally probable.

Singh and Joseph (1989)[2] showed how to generate a binary sequence for chaotic trajectories of the Lorenz system $[\dot{x}, \dot{y}, \dot{z}] = [\sigma(y-x),\ rx - y - xz,\ xy - bz]$ for $(\sigma, b, r) = (10, 8/3, 28)$. The binary sequence is generated by projecting the trajectories into the xz-plane, as shown in Figure 2, and monitoring the crossing points of trajectories on the segments AB and CD of the line AD. The crossing times are put into correspondence with the sequence n of integers, left crossings on AB are recorded as $u(n) = 1$, and the right crossings of CD as $u(n) = -1$. The time averages of these sequences vanish for large N, independent of the initial condition, so that left and right crossings are equally probable and we may assume that the sequences are ergodic.

2.2. *Autocorrelations*

An estimate of the autocorrelation function on an ergodic binary sequence can be obtained as follows:

$$r(n) = \frac{1}{N}\sum_{k=1}^{N} u(k+n)u(k), \quad n = 1, 2, \ldots, N \gg n \tag{1}$$

The value $r(1)$ represents the correlation between immediate neighbors $(1,2)$, $(2,3)$, $(3,4)$, etc. Value $r(2)$ gives the correlation between separated pairs $(1,3)$, $(2,4)$, etc. A chaotic response is one for which $r(1) \neq 0$ and $r(n) \to 0$ for large n.

For the oil bubble, the autocorrelation values $r(n)$ are not uniformly close to zero for large n, because of the relatively small length of the sequence, $N = 3000$ (Figure 3). We tried sequences of different length and found that the $r(n)$ approached zero uniformly for large n as the length of sequence was increased.

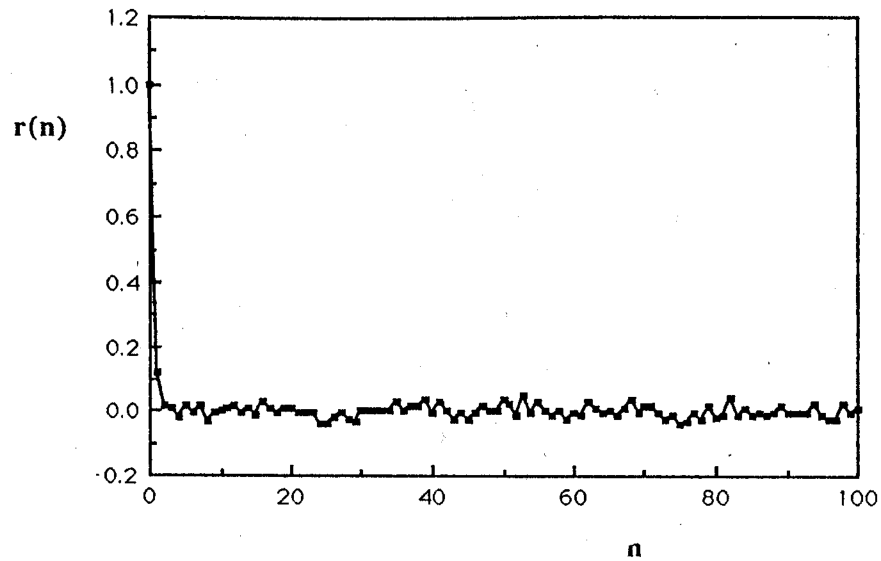

Figure 3. The autocorrelation sequence for the oil bubble, $N = 3000$

The Lorenz equations were integrated numerically using the NAG library. Subroutine DO2BBF was used for different tolerance levels in the range 10^{-4} to 10^{-10}. We projected into the xz-plane and formed a binary number symbol sequence with 76 000 entries. The autocorrelation function is shown in Figure 4. The tolerance level in the numerical scheme had absolutely no effect on the nature of the autocorrelation sequence, even though the sequences generated were quite different for different tolerance levels. For large $n, r(n)$ approached zero uniformly with the increase in length of the sequence, N.

In both cases the decay in the autocorrelation values is very rapid. For large n, autocorrelation values decrease monotonically with the length of the sequence. The decay of

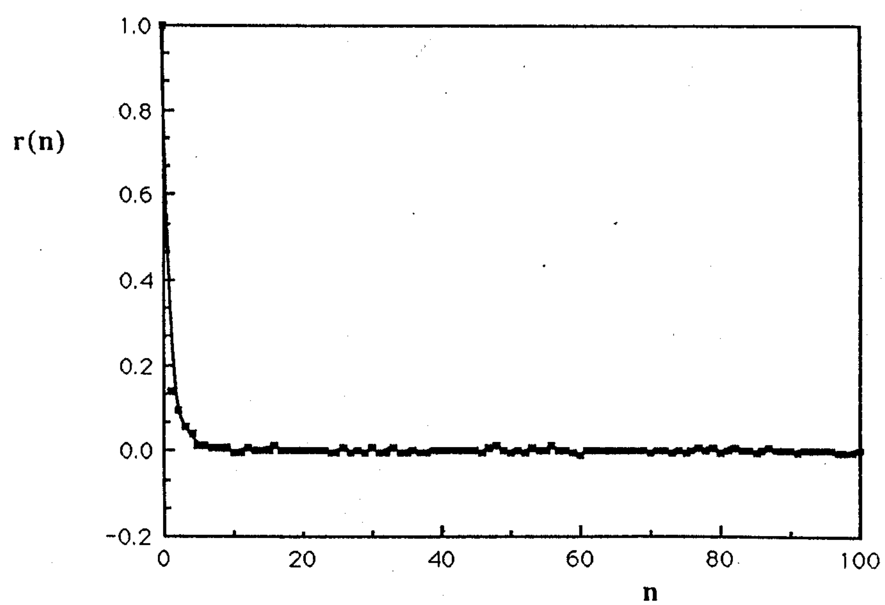

Figure 4. The autocorrelation sequence for the Lorenz attractor, $N = 76\,000$

autorcorrelation for the bubble is essentially complete after $n = 2$, a substantial correlation exists only for $r(1)$. The decay of correlation is slower for the Lorenz system with non-zero $r(n)$ for $n < 6$. We could say that the Lorenz system is less random.

2.3. *Lyapunov exponents*

Singh and Joseph (1989)[2] derived a macroscopic Lyapunov exponent for a binary sequence. Lyapunov exponents for continuous times are locally defined quantities which measure the tendency for chaotic trajectories to diverge exponentially for small time, on the average. One can define the first exponent by

$$\lambda = \frac{1}{t_{N+1} - t_1} \sum_{k=1}^{N} \log_2 \frac{d(t_{k+1})}{d_0(t_k)} \tag{2}$$

where $d_0(t_k)$ is the initial distance between two trajectories at time t_k and $d(t_{k+1})$ is the distance between these two trajectories at time $t_{k+1} > t_k$. In the continuous case $d_0(t_k)$ and $d(t_{k+1})$ are infinitesimal and $N \to \infty$.

The concept of distance is not natural to binary sequences. Two trajectories correspond to two strings of binary symbols. We replace the condition that the initial distances between trajectories is small with the condition that we shall only compare strings of symbols which start with the same symbol. We can compare the 'distance' between two strings of symbols which both start with $u = 1$ or both with -1, but not with starting values of $+1$ for one string and -1 for the other.

Another condition we need for comparing two strings of symbols is statistical independence. We want uncorrelated sequences so that theorems requiring ergocity, the use of 'time' averages, will be appropriate. This requirement is easy to fulfil for our binary sequence symbol string. We compare two strings $u(k)$, $k = 1, 2, \ldots$, with $u(k + M)$ where M is larger than the correlation time for the autocorrelation, $M > 2$ for the chaotic bubble and $M > 5$ for the Lorenz attractor.

We can replace (2) with

$$\lambda(t_{N+1} - t_1) = \sum_{k=1}^{N} \log_2 \frac{\bar{d}(k+1)}{\bar{d}_0(k)} \tag{3}$$

where $\bar{d}_0(k)$ is the average 'distance' between two statistically independent strings at the kth observation. If the two symbols at the $(k + 1)$th observation have the same sign, we say that the 'distance' is unchanged, on the average.

$$\bar{d}(k+1) = \bar{d}_0(k) \tag{4}$$

If the two symbols have different signs after one observation, then

$$\bar{d}(k+1) = c_1 \bar{d}_0(k) \tag{5}$$

where c_1 is the constant average change of distance. It follows from (4) and (5) that

$$\log_2 \frac{\bar{d}(k+1)}{\bar{d}_0(k)} = \begin{cases} 0 & \text{same sign} \\ \alpha & \text{sign change after iteration} \end{cases} \tag{6}$$

where $\alpha = \log_2 c_1$.

We now define the set S_1 of ergodic initial distances between strings of symbols

$$S_1 = \{k : u(k)u(k+M) = 1\} \tag{7}$$

The complementary set is

$$S_2 = \{k : u(k)u(k+M) = -1\} \tag{8}$$

Hence, we may write

$$\log_2 \frac{\bar{d}(k+1)}{\bar{d}_0(k)} = \frac{\alpha}{2}\{1 - u(k+1)u(k+1+M)\}$$

for all symbol sequences which have the same sign at time k for all $k \in S_1$. Hence,

$$\sum_{k=1}^{N} \log_2 \frac{\bar{d}(k+1)}{\bar{d}_0(k)} = \frac{\alpha}{2} \sum_{k \in S_1} \{1 - u(k+1)u(k+1+M)\} \tag{9}$$

The total number of k is N. Let N_1, N_2 be the number of k's in the sets S_1, S_2, and $N_1 + N_2 = N$. We have also that

$$Nr(M) = \sum_{k=1}^{N} u(k)u(k+M) = \sum_{k \in S_1} u(k)u(k+M)$$

$$+ \sum_{k \in S_2} u(k)u(k+M) = N_1 - N_2 = 0 \tag{10}$$

Since $r(M) = 0$ when M is larger than the correlation 'time'. Hence $N_1 = N_2 = N/2$.

We next define the macroscopic Lyapunov exponent as the average value

$$\lambda_{\mathrm{m}} = \frac{1}{N_1} \sum_{k \in S_1} \log_2 \frac{\bar{d}(k+1)}{\bar{d}_0(k)}$$

$$= \frac{\alpha}{N} \sum_{k \in S_1} \{(1 - u(k+1)(k+1+M)\}. \tag{11}$$

This is related to the average Lyapunov exponents by

$$\frac{\lambda(t_{N+1} - t_1)}{N} = \lambda_m \tag{12}$$

Singh and Joseph (1989)[2] showed that

$$\lambda_{\mathrm{m}} = \frac{\alpha}{2}[1 - r^2(1)] \tag{13}$$

2.4. Lyapunov exponents and white noise

Singh and Joseph (1989)[2] calculated the macroscopic Lyapunov exponent for the Lorenz system described in the section 'Binary sequences'. They calculated α as follows. The average distance between starting trajectories on the line AB (= CD) of Figure 2 is

$$\frac{|\mathrm{AB}|}{3} = \bar{d}_0(k)$$

The switching distance is $|\mathrm{AD}| - |\mathrm{AB}| = \bar{d}(k+1)$. Hence

$$\frac{\bar{d}(k+1)}{\bar{d}_0(k)} = 3\left[\frac{|\mathrm{AD}|}{|\mathrm{AB}|} - 1\right]$$

They found that $|\mathrm{AD}| = 4{\cdot}31\,|\mathrm{AB}|$. Then from (6) we calculate $\alpha = 3{\cdot}3$. The relation (12) between the average Lyapunov exponent λ and the macroscopic exponent λ_m may be simplified by putting $t_{N+1} - t_1 = NT$ where ΔT is the average period. Then

$$\bar{\lambda} = \frac{\lambda_m}{\Delta T} = \frac{\alpha}{2\Delta T}(1 - r(1)^2)$$

where $\Delta T = 0{\cdot}7519$ s. We get

$$\lambda_m = 1{\cdot}618 \text{ bits per period}$$

The largest Lyapunov exponent computed directly for the Lorenz attractor is

$$\lambda = 1{\cdot}30 \text{ bits per period}$$

Feeny and Moon (1989)[1] have studied a chaotic dry friction oscillator using the method of binary sequences of Singh and Joseph (1989).[2] They did an experiment with sliding friction in which an imposed change of the normal force caused the slider to stick. They also modeled their experiment with a second order forced ODE involving friction coefficient and normal load functions. They did Poincaré sections for the experiments with 2048 symbols and for the differential equation 10 000 symbols. The symbols form a string of binary numbers ± 1 corresponding to whether the motion is sticking or slipping at each pass through the Poincaré section. They measure distance on the Poincaré plot:

$$\bar{d}_0(k) = \tfrac{1}{3}, \quad \bar{d}(k+1) = 1$$

Hence, using (6), they get $\alpha = \log_2 3 = 1{\cdot}585$.

Feeny and Moon[1] studied the tent map and logistic map using the formula (13) with $\alpha = 1{\cdot}585$. They calculated $r(1)$ for $N = 10^5$ and $N = 2048$. The theoretical value of the largest Lyapunov exponent is $\lambda = 1$ for both the tent map and the logistic map. They compute

$$\lambda_m = \begin{cases} 0{\cdot}787515 \ (10^5 \text{ symbols}) \\ 0{\cdot}787705 \ (2048 \text{ symbols}) \end{cases} \text{tent map}$$

$$\lambda_m = \begin{cases} 0{\cdot}791578 \ (10^5 \text{ symbols}) \\ 0{\cdot}791116 \ (2048 \text{ symbols}) \end{cases} \text{logistic map}$$

A binary autocorrelation was obtained for their experiments and numerically from the differential equation for a symbol string with $N = 2048$. In both cases the autocorrelation $r(1)$ is very small, less than $\pm 0{\cdot}05$. They calculated

$$\lambda_m = \begin{cases} 0{\cdot}79055 \text{ experiment} \\ 0{\cdot}79219 \text{ numerical integration} \end{cases}$$

The calculation of the exponent for the Poincaré map from the equations of motion gives

$$\lambda = 0{\cdot}77$$

We draw the reader's attention to the fact that for all the calculations done by Feeny and Moon,[1] they obtained

$$\lambda_m = \frac{\alpha}{2}[1 - r(1)^2] = 0{\cdot}7925[1 - r(1)^2]$$

This shows that $r(1)^2$ is very small in the examples of the tent map, logistic map and experiments.

Short range predictability requires that $r(1), r(2), \ldots, r(M) \neq 0$ for small $M, r(n) \to 0$ for large n. For white noise, we have ($r(n) = 0$ for all n). The autocorrelation is good for distinguishing short range predictability and white noise. The macroscopic Lyapunov exponent is not useful for making this important distinction. In fact, the macroscopic Lyapunov exponent depends on distance through α, but λ_m/α is universal, does not depend on distance, and may be a more intrinsic measure of chaos. Certainly $r(1)$ contains much less information than the graph of $r(n)$.

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