ONE-DIMENSIONAL, PARTICLE BED MODELS OF FLUIDIZED SUSPENSIONS*

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Abstract. One-dimensional unsteady models of a fluidized suspension based on modeling the forces that the fluid exerts on the particles are considered. Four different theories are discussed. The first, by Foscolo and Gibilaro [1984, 1987], gives a criterion for the loss of stability of uniform fluidization. A second theory by Joseph [1989] which appears to carry the Foscolo-Gibilaro theory to a logical conclusion with the addition of a term proportional to the particle velocity gradient, leads always to instability. A third theory by G.K. Batchelor [1988] is formally similar to the one by Foscolo-Gibilaro, but is more generally derived. A fourth theory which takes into account the finite size of particles and can be used in any of the other three theories is derived here. We show that the finite size of particles is a regularizer of the short wave instability of uniform fluidization which occurs when the particle phase pressure is neglected. We introduce the problem of losing range. If the fluids and solids fractions are both intitially in the interval (0, 1), will they stay on that interval as they evolve? An answer is given.

1. Fluidized Beds. A particle is fluidized when it is lifted against gravity by the drag of upward moving fluid. The particle is in equilibrium under weight and drag. An assemblage of particles in a container which is not fluidized rests on the bottom of the container. Below this speed, fluid passing up through such a bed will see the bed as a porous media. There is a critical speed above which the particles are fluidized. The bed expands to maintain a balance between drag and weight when the flow rate is increased. A statistically homogeneous fluidized bed with constant flow throughput is called a state of uniform fluidization. Such states are notoriously difficult to achieve. It appears to be true that gas fluidized beds of light particles can be stable above minimum fluidization. When the flow throughput is large the gas collects into large gas bubbles which rise through the bed. This is a failure of fluidization since the individual particles are basically insufficiently fluidized to promote efficient heat and mass transfer. The transition to bubbling is said to be a transition from particulate to aggregate fluidization, the particles aggregate, with gas in clear regions. It is not clear that uniform fluidization is the same as particulate fluidization; for example, waves may appear destroying uniformity without marked aggregation of particles. The concept of stability itself is not clear since stability is defined only in a statistical sense. The particles are probably always shaking about.

Stability analysis for bubbling beds is a matter of great interest for the technology of beds used in catalytic cracking and coal combustion. It also plays a certain role in the theory of multiphase flow as a test problem.

*This research was supported under grants from the US Army Research Office, Mathematics, the Department of Energy and the National Science Foundation.

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2. Two-fluid Equations. We can form continuum equations for two-fluids, even when one of the two constituents is solid, by ensemble averaging (Drew, 1983). Joseph and Lundgren [1989] have derived the following set of ensemble averaged equations for incompressible fluid-particle suspensions.

$$\frac{\partial \varepsilon}{\partial t} + \operatorname{div} \varepsilon \mathbf{u}_f = 0 \ , \tag{1}$$

$$\frac{\partial \phi}{\partial t} + \operatorname{div} \phi \mathbf{u}_p = 0 \ , \tag{2}$$

$$\rho_f \varepsilon \left(\frac{\partial \mathbf{u}_f}{\partial t} + \mathbf{u}_f . \nabla \mathbf{u}_f \right) + \rho_f \operatorname{div} \langle H(\mathbf{V} - \mathbf{u}_f)(\mathbf{V} - \mathbf{u}_f) \rangle$$

$$= -\nabla(p_f \varepsilon) + \mu \nabla^2 \mathbf{u}_c - \langle \delta_\Sigma(\mathbf{x}) \mathbf{t} \rangle + \rho_f \varepsilon \mathbf{b}_f \ , \tag{3}$$

$$\rho_p \phi \left(\frac{\partial \mathbf{u}_p}{\partial t} + \mathbf{u}_p . \nabla \mathbf{u}_p \right) + \rho_p \operatorname{div} \langle (1 - H)(\mathbf{V} - \mathbf{u}_p)(\mathbf{V} - \mathbf{u}_p) \rangle$$

$$= -\nabla(p_p \phi) + \langle \delta_\Sigma(\mathbf{x}) \mathbf{t} \rangle + \rho_p \phi \mathbf{b}_p \ , \tag{4}$$

where

$H(\mathbf{x})$ is an indicator function, 0 if $\mathbf{x}$ is in the fluid, 1 otherwise
$\langle . \rangle(\mathbf{x}, t)$ ensemble average
$\varepsilon = \langle H \rangle$ fluid fraction
$\phi = \langle 1 - H \rangle$ solid fraction
$\mathbf{u}_f = \langle H \mathbf{V} \rangle$ average fluid velocity, where $\mathbf{V}$ is the true velocity
$\mathbf{u}_p = \langle (1 - H) \mathbf{V} \rangle$ average particles velocity
$(1 - \phi) p_f = \langle H p \rangle$ fluid phase pressure, where p is the mean normal stress
$\phi p_p = \langle (1 - H) p \rangle$ particle phase pressure
$\mathbf{u}_c = \varepsilon \mathbf{u}_f + \phi \mathbf{u}_p$
τ is the extra stress $\mathbf{T} = -p\mathbf{l} + \tau$
$\mathbf{b}_f$ and $\mathbf{b}_p$ ensemble averaged body forces in the fluid and solid
δ_Σ is a Dirac delta function across the solid-fluid interface
$\mathbf{t} = \mathbf{n}.\mathbf{T}$ is the traction vector on the solid-fluid interface.

If we add equations (1) and (2) we get

$$\operatorname{div} \mathbf{u}_c = 0 \ . \tag{5}$$

The boundary conditions between the fluid and the particle involves the traction vector term in (3) and (4) and it is probably best not to combine the two equations.

The existence of two fluid equations even when one of the fluids is solid is perfectly justified by ensemble averaging. These equations, like other two fluid models, are not closed and methods of closure, or constitutive models for the interaction terms, are required to put the equations into a form suitable for applications. Moreover, since averaging over repeated identical trials is not a realizable proposition, the ensemble average variables are conceptually abstract and their relation to more physically intuitive variables, like the ones which arise from spatial averaging, is uncertain.

Equations (1) through (5) are appropriate for fluidized suspensions with $\mathbf{b}_f = \mathbf{b}_p = \mathbf{g}$, gravity. Particle bed models decouple the fluids and solids equations and work with the solids equations alone.

3. On Losing Range. One of us (DDJ) was worried for a week in June 1989 about the possibility of losing range. The range $0 < \varepsilon(\mathbf{x}, t) < 1$ must be preserved by dynamics; if $\varepsilon(\mathbf{x}, 0)$ is between 0 and 1 is it possible for ε to go negative or grow larger than one? This should not happen; moreover, the protection of the range should not depend on $\mathbf{u}_f$ and $\mathbf{u}_p$ because these fields can be changed at will by changing the constitutive equations or initial conditions. The problem is this: given equations (1) and (2) and sufficiently smooth field $\mathbf{u}_f$ and $\mathbf{u}_p$ and $0 < \varepsilon(\mathbf{x}, 0) < 1$, what are the conditions such that $0 < \varepsilon(\mathbf{x}, t) < 1$ for all t. This problem was given to Sir James Lighthill, who gave the following solution (in one space dimension). We write

$$\frac{\partial \varepsilon}{\partial t} + u_f \frac{\partial \varepsilon}{\partial x} + \varepsilon \frac{\partial u_f}{\partial x} = 0 \tag{6}$$

and deduce that along a curve

$$\frac{dx}{dt} = u_f \tag{7}$$

we have

$$\frac{d\varepsilon}{dt} = -\varepsilon \frac{\partial u_f}{\partial x} \ . \tag{8}$$

Therefore, if $\varepsilon = \varepsilon_0$ on this curve at $t = 0$, then we have

$$\varepsilon = \varepsilon_0 \exp\left(-\int_0^t \frac{\partial u_f}{\partial x}\, dt \right) \tag{9}$$

which is always positive since $\varepsilon_0 > 0$. Similarly we have

$$\frac{d(1-\varepsilon)}{dt} = -(1-\varepsilon)\frac{\partial u_p}{\partial x} \ . \tag{10}$$

Then, if $\varepsilon = \varepsilon_0$ on this curve at $t = 0$, we have

$$1 - \varepsilon = (1 - \varepsilon_0) \exp\left(-\int_0^t \frac{\partial u_p}{\partial x}\, dt \right) \tag{11}$$

which is always positive.

This proof works for the three dimensional case with $\dfrac{\partial u}{\partial x}$ under the integral replaced by $\operatorname{div} \mathbf{u}$. It also works for compressible constituents, not discussed here.

It is clear that we can protect the range if and only if both (1) and (2) are satisfied. In particle bed models only (2) is satisfied, so we are in danger of losing range.

4. Particle Bed Models. If we knew the force on each and every particle we could in principle track their motion. The ultimate in particle bed calculations would be a molecular dynamic simulation. For this to work we would need to know the force that the fluid exerts on the particles. There is no perfect way to do this without doing the fluid dynamics. In fact, exact numerical solutions correlating certain small motions of particles with forces generated by the flow of a Navier-Stokes fluid is a viable proposition (for example, see Singh, Caussignac, Fortes, Joseph and Lundgren [1989]) with a great future. Unfortunately it is not possible to know perfectly how the fluid forces will effect a particle without actually doing the fluid mechanics. The hope behind the particle bed models discussed below is that our experience and undertanding of fluid-particle interactions will allow us to guess correctly what form these interactions ought to take, at least in an average sense. It is by no means certain that this hope can be realized.

One-dimensional particle bed models have been given by Foscolo and Gibilaro [1984, 1987] and G.K. Batchelor [1988]. These theories will be reviewed below. They use mass conservation of the solid in the ensemble averaged form (2)

$$\frac{\partial \phi}{\partial t} + \operatorname{div} \phi \mathbf{u}_p = 0 \ .$$

This equation does not acknowledged either structure or size effects of particles on the continuity of flow. Approaches, like the one given below, based on geometry rather than ensemble averaging may be preferred.

5. Mass Balance Equations for Balls of Radius R. We are going to derive a one-dimensional mass balance for spherical particles of uniform radius R. Consider a plane at $z = Z$, perpendicular to gravity, in figure 1. Let us consider the area $A = L^2$ of a square in this plane with $L \gg R$, so many spheres intersect the plane at Z. Let x be the distance from the plane $z = Z$. All spheres whose centers are at $|x| \leq R$ pass through the plane Z. Spheres with $|x| \geq R$ do not touch Z. The area of the hole cut out by the sphere at $|x| \leq R$ is $\pi(R^2 - x^2)$.

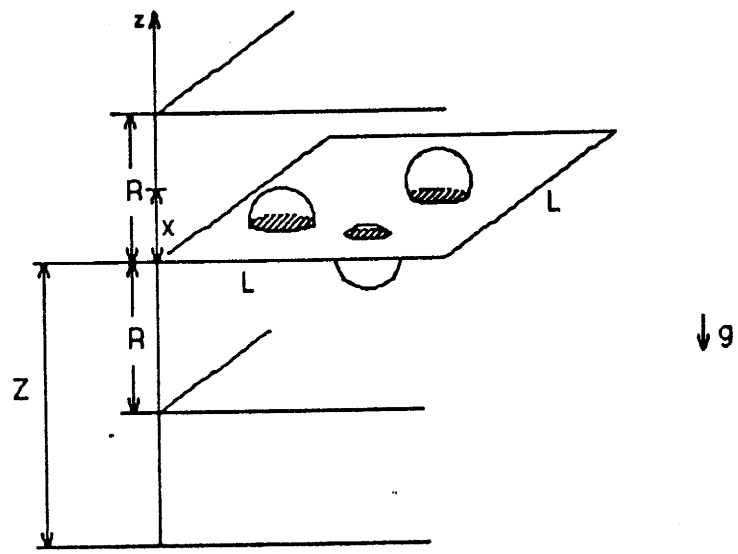

FIGURE 1. Cross-section of fluidized spheres of radius R in the plane at $z = Z$.

Now we define a wafer of influence for the plane at z. Its area is A and thickness is $2R, -R \leq x \leq R$. The total volume of the waver is $2RA$.

The next quantity to be defined is the number density per unit area

$$N(z+x,t) = \frac{\text{number of spheres whose centers are at } z+x}{A}$$

(12)
$$= \frac{\text{number of holes in the plane } Z \text{ of area } \pi(R^2 - x^2)}{A} .$$

Let V_s be the volume of spheres in the wafer of influence and $\phi = \dfrac{V_s}{2RA}$ be the solid fraction. Then

$$dV_s = N(z+x,t)A\{\pi(R^2 - x^2)\}dx$$

is the element of solids volume swept out by number of holes of area $\pi(R^2 - x^2)$ times the volume of one of these holes as x moves through dx. Then

(13)
$$V_s = \int_{-R}^{R} N(z+x,t)A\pi(R^2 - x^2)dx$$

and

(14)
$$\phi = \frac{1}{2R} \int_{-R}^{R} N(z+x,t)\pi(R^2 - x^2)dx .$$

If $N = N_0$ is constant, then

(15)
$$\phi_0 = \frac{\left(\frac{4}{3}\,\pi R^3\right)(N_0 A)}{2RA} = \frac{2}{3}\,\pi R^2\, N_0$$

is the solid fraction for a uniform distribution of spheres.

Now we write out a mass balance. Let $u_p(z+x,t)$ be the velocity of a sphere whose center is at $z+x$. Then

$$A\; N(z+x,t)u_p(z+x,t)\pi(R^2-x^2)$$

is the flux of area through the plane at z of spheres centered at $z+x$ and the increase of concentration at z is balanced by the fluxes of all areas of spheres intersecting z; that is

(16)
$$\frac{\partial}{\partial t}\int_{-R}^{R} N(z+x,t)(R^2-x^2)dx + \frac{\partial}{\partial z}\int_{-R}^{R} N(z+x,t)u_p(z+x,t)(R^2-x^2)dx = 0\,.$$

6. The Particle Bed Model of G.K. Batchelor. Batchelor [1988] established the form of the momentum equation for one-dimensional unsteady mean motion of solid particles in a fluidized bed or sedimenting dispersion from physical arguments. He works with area averaged quantities which because of statistical homogeneity can be identified with ensemble averages. He asserts a definite point of view preferring to establish equations carefully, with plausible physical reasoning and a minimum of hypotheses concerning the relation between mean quantities. He avoids the introduction of any parameters that do not have a clear physical meaning and are not calculable or measurable, at least in principle. He obtains the following differential equations for the area averaged mean quantities balancing momentum
(17)

$$mn(1+\theta)\left(\frac{\partial V}{\partial t}+V.\nabla V\right) = -\frac{\partial(mn\langle v^2\rangle)}{\partial t} + n\{F_h(V,\phi)-F_h(U,\phi)\} - \frac{D}{B}\frac{\partial n}{\partial x} + \frac{\partial\left(\phi\rho_f\eta'\frac{\partial V}{\partial x}\right)}{\partial x}$$

where

$m =$ mass of particle,

n local number density, $mn = \rho_p \phi$,

$\theta = \dfrac{\rho_f}{\rho_p} C$ virtual mass term, where $C = C(\phi)$ depends on ϕ. But here we take C to be a constant for convenience which makes the term dependent on the derivative of C to drop out.

V mean particle velocity. The axis of reference is such that the mean of material volume across a horizontal plane is zero.

When $V \neq 0$; the axis of reference moves with a constant velocity,

v velocity fluctuation with a zero horizontal average, $\langle v \rangle = 0$.

Horizontal averages are assumed to be same as ensemble averages,

$F_h(V, \phi)$ mean force exerted by the fluid on a particle whose mean velocity is V in a homogeneous dispersion of concentration ϕ,

(18) $F_h(U, \phi) = -m\tilde{g}$

where $\tilde{g} = g\dfrac{\rho_p - \rho_f}{\rho_p}$ is the reduced buoyancy and $V = U(\phi)$ is the mean particle velocity in a uniform bed in which the particles are in equilibrium under weight and drag

B bulk mobility. This is the ratio of the small change of velocity produced by a small change of force,

D local hydrodynamic diffusivity. $\dfrac{D}{B}$ is a diffusivity coefficient,

$\phi \rho_f \eta'$ a viscosity coefficient.

He considers first the approximate form of the momentum equation when the departure from homogeneity is small and the spatial gradients $\dfrac{\partial \phi}{\partial x}$ and $\dfrac{\partial V}{\partial x}$ are small in some sense. Withoug going very deeply into these approximations we list them below: First

$$F_h(V, \phi) - F_h(U, \phi) = \gamma \frac{V - U}{U} F_h(U) = -\gamma m \tilde{g} \frac{V - U}{U} \tag{19}$$

where $F_h(V) = F_h(U) + \dfrac{\partial F_h}{\partial V}(V - U)$ defines γ, believed to be slowly varying in V and ϕ, and (18) is used to eliminate $F_h(U)$. Second

$$\langle v^2 \rangle = H(\phi) U^2 - \eta''(\phi) \frac{\partial V}{\partial x} - \eta'''(\phi) U \frac{\partial \phi}{\partial x} \tag{20}$$

where following kinetic theory $\eta'''(\ll \eta'' > 0)$ is discarded, $H(0) = 0$ and $H(\phi_0) = 0$ where ϕ_0 is for the close packing because fluctuations must vanish in these two limits. After inserting these approximations, Batchelor obtains

(21)

$$\phi(1+\theta)\left(\frac{\partial V}{\partial t} + V.\nabla V\right) = -\frac{d(\phi H U^2)}{d\phi}\,\frac{\partial \phi}{\partial x} - \frac{\gamma \tilde{g}}{U}\left(\phi(V - U) + D\frac{\partial \phi}{\partial x}\right) + \frac{\partial\left(\phi \eta \dfrac{\partial V}{\partial x}\right)}{\partial x}$$

where $\rho_p \eta = \rho_p \ \eta'' + \rho_f \ \eta'$ can be called the particle viscosity and $\rho_p \ \eta''$ is an eddy viscosity.

To compare Batchelor's theory with that of Foscolo and Gibilaro it is convenient to make a Galilean transformation to a laboratory fixed frame from the zero material flux axis. Thus

$$x = x_0 + U_0 t, \quad V = U_0 + u_p$$

where $U_0 = U(\phi_0)$ is independent of x and t and $U_0 = u_c$ is the composite or the fluidizing velocity. Then we have equation (2) and

$$(22) \quad \phi(1+\theta)\left(\frac{\partial u_p}{\partial t} + u_p.\nabla u_p\right) = -Q\frac{\partial \phi}{\partial x} - \frac{\gamma \tilde{g}}{U}\ \phi(V - U - u_p) + \frac{\partial(\left(\phi\eta\frac{\partial u_p}{\partial x}\right)}{\partial x}$$

where

$$Q = \frac{d(\phi H U^2)}{d\phi} + \frac{\gamma\tilde{g}}{U}\ D.$$

In the next section we will present some results which suggest that there ought to be a term proportional to $\dfrac{\partial u_p}{\partial x}$ on the left of (22).

The total coefficient of $-\dfrac{\partial \phi}{\partial x}$ in (22) can be interpreted as a bulk modulus of elasticity of the configuration of particles. This term could also be identified as arising from a particle phase pressure. Batchelor regards the contribution of the part proportional to D to be the more important of the two. He thinks of this as new contribution representing the diffusion of particles against a gradient. We will see later how these gradient terms regulate the short wave instabilities which arise when the derivative terms on the left of (22) are put to zero.

7. The Particle Bed Model of Foscolo-Gibilaro. Foscolo and Gibilaro [1984] start with coupled one-dimensional equations for the particles and fluid phase. The particle phase equations are

$$(23) \qquad \frac{\partial \phi}{\partial t} + \frac{\partial \phi u_p}{\partial z} = 0,$$

$$(24) \qquad \phi\rho_p\left[\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial z}\right] = -\phi\rho_p g + \mathfrak{F} - \frac{\partial p_p}{\partial z}\ .$$

where $\mathfrak{F}$ is the interaction force, the force that the fluid exerts on the particle, and p_p is the particle phase pressure. The fluid equations are of the same form except that the subscript p is replaced by f, ϕ is replaced by ε and $\mathfrak{F}$ by minus $\mathfrak{F}$.

Foscolo and Gibilaro modeled the interaction force $\mathfrak{F}$ and the particle phase pressure in a manner that decouples the equations for the fluid and solid phases. This gives rise to a system of equations for the particles only, called the particle bed model.

It is convenient to introduce a dynamic pressure π_p into (24) by writing

$$p_p == P + \pi_p$$

$$\phi p_p g + \frac{\partial P}{\partial z} = 0.$$

Then (24) reduces to

$$\phi\rho_p\left[\frac{\partial u_p}{\partial t}+u_p\frac{\partial u_p}{\partial z}\right]=\mathcal{F}-\frac{\partial \pi_p}{\partial z}\,. \tag{25}$$

To get their equations they first derived an interesting expression $F_d(1)$ for the drag force exerted by the fluid on a single particle in a uniform fluidized suspension. This expression relies strongly on the well-known correlation of Richardson and Zaki for fluidized and sedimenting beds of monosized spherical particles

$$u_c = V\varepsilon^n \tag{26}$$

where

$$u_c = u_p\phi + u_f\varepsilon \tag{27}$$

is the composite velocity, the volume flux divided by total area and u_c is independent of z, $\frac{\partial u_c}{\partial z}=0$. Of course $V=u_f$ when $\varepsilon=1$,the steady terminal velocity of a freely falling single sphere in a sea of fluid. The exponent n depends on the Reynolds number $Re=\frac{dV}{\nu}$ where d is the diameter

$$n=\begin{cases} 4.65 \text{ for } Re<0.2, \\ 4.4\ Re^{-0.03} \text{ for } 0.2<Re<1, \\ 4.4\ Re^{-0.1} \text{ for } 1<Re<500, \\ 2.4 \text{ for } Re>500. \end{cases} \tag{28}$$

Foscolo–Gibilaro replace 4.65 with 4.8=2(2.4) for reasons to be made clearer later.

There is a huge amount of fluid mechanics buried in the Richardson–Zaki correlation. This is hidden in the drag law for particles falling under gravity in steady flow. Let $F_d(\varepsilon)$ be the drag on a single particle in a freely falling suspension with a fluid fraction ε. When $\varepsilon=1$ we get a drag law for the free fall of a single sphere which is Stokes drag when V is small enough; for larger V the drag is given by

$$F_d(1)=\frac{\rho V^2}{2}\frac{\pi d^2}{4}C_D \tag{29}$$

where C_D is given by an empirical correlation. Foscolo and Gibilaro produce the formula

$$F_d(\varepsilon)=\varepsilon F_d(1) \tag{30}$$

from an argument which says that in a fluidized bed in a steady flow, the total force F on a sphere is the sum

$$F(\varepsilon, Re)=F_d(\varepsilon)-F_p(\varepsilon)$$

where

$$F_p(\varepsilon) = \frac{\pi d^3}{6}(\rho_p - \rho_c)g$$

is the buoyant force using the effective density

$$\rho_c = \varepsilon\rho_p + \phi\rho_p$$

of the composite fluid. Since $\phi = 1 - \varepsilon$,

$$F_p(\varepsilon) = \frac{\pi d^3}{\rho}(\rho_p - \rho_f)g\varepsilon = \varepsilon F_p(1)$$

in steady flow, $F = 0$ and

(31) $$F_d(\varepsilon) = F_p(\varepsilon) = \varepsilon F_p(1) = \varepsilon F_d(1)\ .$$

We never see steady flow in a fluidized bed, the particles always jiggle about; steady is in some statistical sense, whatever that may be. In any interpretation

$$u_p = 0 \quad \text{in steady flow.}$$

Equation (31) is all that is required to get the drag on a single particle in a fluidized suspension in steady flow. The hydrodynamic content is all buried in the drag correlation (29). We may write $F_d(\varepsilon) = \varepsilon F_d(1)$. To see how $F_d(\varepsilon)$ depends on the fluidizing velocity u_c, Foscolo and Gibilaro note that (29) implies that

$$F_d = \varepsilon \begin{cases} 3\pi\mu V & \text{(laminar)} \\ 0.055\pi\rho d^2 V^2 & \text{(turbulent).} \end{cases}$$

They next note that in the Richardson and Zaki correlation (26) and (28), with 4.8 replacing 4.65, implies that

(32) $$F_d = \varepsilon^{-3.8} \begin{cases} 3\pi\mu d u_c & \text{(laminar)} \\ 0.055\pi\rho d^2 u_c^2 & \text{(turbulent).} \end{cases}$$

This is good, we have $F_d(u_f, \varepsilon) = \varepsilon^{-3.8} F_d(u_f)$, independent of V for low and high Reynolds numbers. Now we look for an equivalent expression, valid for all Reynolds numbers in steady flow and

$$F_d(\varepsilon) = F_d(\varepsilon, u_f, V) = \varepsilon^{-3.8} g(u_f, V)$$

which will reduce to (32) at low and high Re. Clearly

$$g(u_c, V) = \varepsilon^{4.8} F_d(1) = \left(\frac{u_c}{V}\right)^{\frac{4.8}{n}} F_d(1).$$

Hence

$$F_d(\varepsilon, u_c, V) = \varepsilon^{-3.8} \left(\frac{u_c}{V}\right)^{\frac{4.8}{n}} F_d(1). \tag{33}$$

This is just another way of writing $F_d(\varepsilon) = \varepsilon F_d(1)$ when 4.65 is replaced with 4.8 which is useful in motivating the constitutive equation (34) below.

Foscolo and Gibilaro assume that in unsteady flow the force on a particle is given by the expression (33) with u_c replaced by the slip velocity

$$u_c - u_p = (1-\varepsilon)u_p + \varepsilon u_f - u_p = \varepsilon(u_p - u_f).$$

Then the unsteady drag force is

$$F_d(\varepsilon, u_c - u_p, V) = \varepsilon^{-3.8} \left(\frac{u_c - u_p}{V}\right)^{\frac{4.8}{n}} F_d(1). \tag{34}$$

In steady flow, $u_p = 0$, and (34) reduces to

$$F_d(\varepsilon) = \varepsilon F_d(1)$$

where balancing drag and buoyancy for a single sphere gives

$$F_d(1) = \frac{\pi d^3}{6} (\rho_p - \rho_f) g.$$

The total force on single particle in a fluidized suspension is given by

$$F = F_d - F_b = \frac{\pi d^3 g}{6}(\rho_p - \rho_f)\left\{\varepsilon - \left[\frac{u_c - u_p}{V}\right]^{\frac{4.8}{n}} \varepsilon^{-3.8}\right\} .$$

The force per unit volume due to all n spheres is

$$\mathcal{F} = NF$$

where

$$N = \frac{\phi}{\pi d^3/6} = \frac{n}{\text{volume}} .$$

Hence, the total force on the particles per unit volume is

$$\mathcal{F} = \phi(\rho_p - \rho_f) g \left\{\varepsilon - \left[\frac{u_c - u_p}{V}\right]^{\frac{4.8}{n}} \varepsilon^{-3.8}\right\}. \tag{35}$$

In steady flow, u_p and $F = 0$.

To compare their theory, Foscolo and Gibilaro need to model the particle phase pressure and they do so, but their argument is unclear. Their final expression appears to leave out terms that ought to be included. This issue was addressed in a recent paper by Joseph [1989] which is discussed below.

The same force of the fluid on the particles acts at the boundary to keep the particles from dispersing. However, we need to multiply the force on a single particle by the number N_A per unit area

$$N_A = \frac{\phi}{\pi d^2/4} \, .$$

Hence, the dynamic pressure is given by

$$\pi_p = N_A F = \frac{N_A}{N}\mathcal{F} = \frac{2}{3}d\mathcal{F} \, . \tag{36}$$

The idea of making a constitutive equation for the pressure is more allied to gas dynamics where the pressure is a state variable than to incompressible fluid mechanics. In discussing forces which fluids exert on particles G.K. Batchelor noted that in his list (2.3) of forces there is a "...mean force exerted on particles in this volume by the particles outside the volume." Further he notes that the nature of these two forces

> "...may be explained by reference to a hypothetical case in which the particles are electrically charged and exert repulsive electrostatic forces on each other. The range of action of these electrostatic forces is small by comparison with the dimensions of the dispersion, and so the mean resultant force exerted on the particles inside τ, that is, by stress, -S say, which is a function of the local particle concentration.
>
> "Electostatic interparticle forces are conservative, and in that case one can interpret -S as the derivative of the mean potential energy per particle with respect to the volume of the mixture per particle. The contribution to the net force exerted on particles in our control volume by external particles is then
>
> $$-A\int_{x_1}^{x_2}\frac{\partial S}{\partial x}dx.$$
>
> A repulsive force between particles corresponds to a positive value of S (relative to zero when the particles are far apart), in which case S plays a dynamical role analogous to the pressure in a gas."

The equations of motion (23) and (24) are now reduced to

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\phi u_p = 0, \tag{37}$$

$$\rho_p \phi\left[\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial z}\right] = \mathcal{F} - \frac{2}{3}d\frac{\partial \mathcal{F}}{\partial z} \tag{38}$$

where $\mathcal{F}$ is given by (35) and

$$\frac{\partial \mathcal{F}}{\partial z} = \frac{\partial \mathcal{F}}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial z} + \frac{\partial \mathcal{F}}{\partial u_p}\frac{\partial u_p}{\partial z} \tag{39}$$

and

$$\frac{\partial \mathcal{F}}{\partial \varepsilon} = \frac{\partial N}{\partial \varepsilon} F + N \frac{\partial F}{\partial \varepsilon} \,. \tag{40}$$

Equations (37) and (38) are two nonlinear equations in two unknowns, ε and u_p. These equations differ from the ones derived by Foscolo and Gibilaro [1984] to which they reduce when the two additional terms

$$\frac{\partial \mathcal{F}}{\partial u_p} \frac{\partial u_p}{\partial z} \tag{41}$$

and

$$\frac{\partial N}{\partial \varepsilon} \mathcal{F} \tag{42}$$

are put to zero. The term (42) vanishes in the analysis of stability of uniform fluidization but (41) does not.

The term (41) also is absent from the list of forces which act in this problem developed by Batchelor. Hence, we are obliged to consider the physical origin of such term. We may regard the term (41) as arising from changes in the microstructure of the mixture. This has been well expressed in a recent paper by Ham and Homsy [1988].

> "Analysis of the mean settling speed leaves unresolved the problem of microstructural evolution in suspensions. Such changes in the relative positions of particles are likely because each particle in a random suspension sees a slightly different local environment and is therefore expected to have a velocity which is, in general different from that of any neighboring particle. The variations in particle velocities will lead to an adjustment of the particle distribution."
>
> They note further that
>
> "...the microstructural dependence arises from the fact that the time between the velocities of the faster-and slower-setting particles, and the difference will be influenced by the relative position of the particles. The influence of ϕ comes about from the change in interparticle spacing with concentration of particles."

8. Classification of Type and Hadamard Instability.

The theory of classification of type of a second order partial differential equation

$$\widehat{A} \frac{\partial^2 \phi}{\partial t^2} + \widehat{B} \frac{\partial^2 \phi}{\partial t \partial z} + \widehat{C} \frac{\partial^2 \phi}{\partial z^2} + \text{ lower order terms } = 0 \tag{43}$$

is well known. Everything depends on the discriminant

$$D = \widehat{B}^2 - 4\widehat{A}\widehat{C}.$$

Equation (43) is parabolic, elliptic or hyperbolic depending on whether the discriminant $D = 0,\ D < 0$ or $D > 0$, respectively. Its characteristics are given by

$$\frac{dz}{dt} = \frac{\widehat{B} \pm \sqrt{\widehat{B}^2 - 4\widehat{A}\widehat{C}}}{2\ \widehat{A}} .$$

Hadamard instability is an explosive instability to short waves, the growth rates of unstable disturbances tend to infinity with α, that is the wave length $\frac{2\pi}{\alpha} - > 0$. Problems which are Hadamard unstable are ill-posed as initial value problems. In the analysis of short waves lower order terms are unimportant because the highest order derivatives dominate and the coefficients of these derivatives can not vary much in the length of short wave. This gives rise to the second order equation (43) with no lower order terms and constant coefficients. To show that the initial value problem is Hadamard unstable when it is elliptic we use normal modes

$$\phi(z,t) = P\ e^{i\alpha(z-\omega t)}$$

in equation (43) and obtain

$$\widehat{A}(\alpha\omega)^2 - \widehat{B}\alpha^2\omega + \widehat{C}\alpha^2 = 0.$$

Hence

$$\alpha\omega = \alpha \left[\frac{\widehat{B} \pm \sqrt{\widehat{B}^2 - 4\widehat{A}\widehat{C}}}{2\ \widehat{A}} \right] .$$

Then the growth rate is

$$\sigma = Im[\alpha\omega] = \alpha Im \left[\frac{\widehat{B} \pm \sqrt{\widehat{B}^2 - 4\widehat{A}\widehat{C}}}{2\ \widehat{A}} \right] = \pm\frac{\alpha}{2\ \widehat{A}}\ Im[\sqrt{D}]$$

where $Im[.]$ stands for the imaginary part. Clearly, if $D > 0$ then $\sigma = 0$ so the problem is not Hadamard unstable. but if $D < 0$ then $\sigma - > \pm\infty$ as $\alpha - > \infty$, i.e. if a problem is elliptic it is also Hadamard unstable in the sense of an initial value problem.

9. Stability of Uniform Fluidization. In a state of uniform fluidization $\phi = \phi_0$ and $U = U(\phi_0) = U_0$ are constant and $u_p = 0$. Equations (2) and (22) are satisfied. Since $F = F_0 = 0$ for uniform fluidization and π_p is constant, (23) and (25) are satisfied. The mass balance equation (16) for balls of radius R is also satisfied. The state of uniform fluidization satisfies all the required equations.

Now we linearize the equations around the state of uniform fluidization. Let ϕ be the perturbation of ϕ_0, $N(z + x, t)$ be the perturbation of the uniform number density N_0 and u_p be the perturbation of u_p from zero. The mass balance equation (2) becomes

$$\frac{\partial \phi}{\partial t} + \phi_0 \frac{\partial u_p}{\partial z} = 0 . \tag{44}$$

The mass balance equation (16) becomes

$$\text{(45)} \quad \frac{\partial}{\partial t} \int_{-R}^{R} N(z+x,t)\ (R^2-x^2)dx + N_0\ \frac{\partial}{\partial z} \int_{-R}^{R} u_p(z+x,t)(R^2-x^2)dx = 0\ .$$

The momentum equation (22) becomes

$$\text{(46)} \qquad \phi_0(1+\theta)\frac{\partial u_p}{\partial t} = -Q_0\frac{\partial \phi}{\partial z} + \frac{\gamma \tilde{g}}{U_0}\phi 0\left(\frac{\partial U}{\partial \phi_0}\ \phi + u_p\right) + \frac{\partial\left(\phi_0\eta_0\dfrac{\partial u_p}{\partial z}\right)}{\partial z}\ .$$

The linearization of (38) becomes

$$\text{(47)} \qquad \phi_0\frac{\partial u_p}{\partial t} = -B(\phi_0 u_p - C_1\phi) - C_2^2\frac{\partial \phi}{\partial z} + \frac{2}{3}\phi_0 d\ B\frac{\partial u_p}{\partial z}$$

where $B = \dfrac{4.8\tilde{g}\ (1-\phi_0)}{nu_c}$, $C_1 = \dfrac{n\ u_c\phi_0}{1-\phi_0}$ and $C_2^2 = 3.2\phi_0\tilde{g}d$.
After eliminating u_p between (44) and (47) we get the following second order equation

$$\text{(48)} \qquad \frac{\partial^2\phi}{\partial t^2} - \frac{2}{3}\ d\ B\frac{\partial^2\phi}{\partial t\partial z} - C_2^2\frac{\partial^2\phi}{\partial z^2} + B\left(\frac{\partial\phi}{\partial t} + C_1\frac{\partial\phi}{\partial z}\right) = 0\ .$$

Similarly, (45) and (47) give the following second order equation

$$\text{(49)} \qquad \begin{aligned} &\frac{\partial^2 u_p}{\partial t^2} - \frac{2}{3}\ d\ B\frac{\partial^2 u_p}{\partial t\partial z} - C_2^2\frac{\partial^2}{\partial z^2}\int_{-R}^{R} u_p(z+x,t)(R^2-x^2)dx \\ &\qquad + B\left(\frac{\partial u_p}{\partial t} + C_1\frac{\partial}{\partial z}\int_{-R}^{R} u_p(z+x,t)\ (R^2-x^2)dx\right) = 0\ , \end{aligned}$$

and (44) and (46) give

$$\text{(50)} \qquad (1+\theta)\frac{\partial^2 u_p}{\partial t^2} = Q_0\frac{\partial^2 u_p}{\partial z^2} + \frac{\gamma\tilde{g}}{U_0}\left(-\phi_0\frac{\partial U}{\partial \phi_0}\frac{\partial u_p}{\partial z} + \frac{\partial u_p}{\partial t}\right) + \eta_0\frac{\partial^3 u_p}{\partial t\partial z^2}\ .$$

For (48) D (see section 8) is

$$D = \left(\frac{2}{3}\ d\ B\right)^2 + 4C_2^2\ ,$$

and the characteristics are given by

$$\frac{dz}{dt} = -\frac{1}{3}\ d\ B \pm \sqrt{\left(\frac{1}{3}\ d\ B\right)^2 + C_2^2}\ .$$

This shows that if B and C_2 are not both zero (there is a particle phase pressure) then the governing second order equation is hyperbolic and has two real characteristics. This means the initial value problem for this equation is well posed and will not give rise to Hadamard instability. In equation (49) the second order term with an integral can be considered to be of a lower order and hence the expression for D becomes

$$D = \left(\frac{2}{3} d\ B\right)^2 ,$$

and the characteristics are given by

$$\frac{dz}{dt} = 0,\ -\frac{2}{3}\ d\ B\ .$$

If the third order term in (50) is dropped then D and the characteristics are given by

$$D = \sqrt{4\ Q_0\ (1 + \theta)}\ ,$$

$$\frac{dz}{dt} = \pm\sqrt{\frac{Q_0}{1+\theta}}\ .$$

We next look for normal mode solutions of (48), (49) and (50)

$$\phi(z,t) = P\ e^{i\alpha(z-\omega t)}, \quad u_p(z,t) = Q\ e^{i\alpha(z-\omega t)}$$

where $i = \sqrt{-1}$, P and Q are constants, α is the wave number and $\alpha\omega$ is the angular frequency. By putting the above solutions in (48), (49) and (50) we get following dispersion relations

(51) $$(\alpha\omega)^2 + \frac{2}{3} d\ B\alpha^2\omega - C_2^2\alpha^2 + B(i\alpha\omega - C_1\ i\alpha) = 0\ ,$$

(52) $$(\alpha\omega)^2 + \frac{2}{3} d\ B\alpha^2\omega - C_2^2\alpha^2\Theta(\alpha) + B(i\alpha\omega - C_1\ i\alpha\Theta(\alpha)) = 0\ ,$$

(53) $$(1+\theta)(\alpha\omega)^2 = Q_0\alpha^2 + \frac{\gamma\tilde{g}}{U_0}\left(\phi_0\frac{\partial U}{\partial\phi_0}\quad i\alpha + i\alpha\omega\right) - \eta_0\ \alpha^3\omega i,$$

where $\Theta(\alpha) = 3\left[\frac{\sin\ \alpha R}{(\alpha R)^3} - \frac{\cos\ \alpha R}{\alpha R)^2}\right]$ (see figure 2).

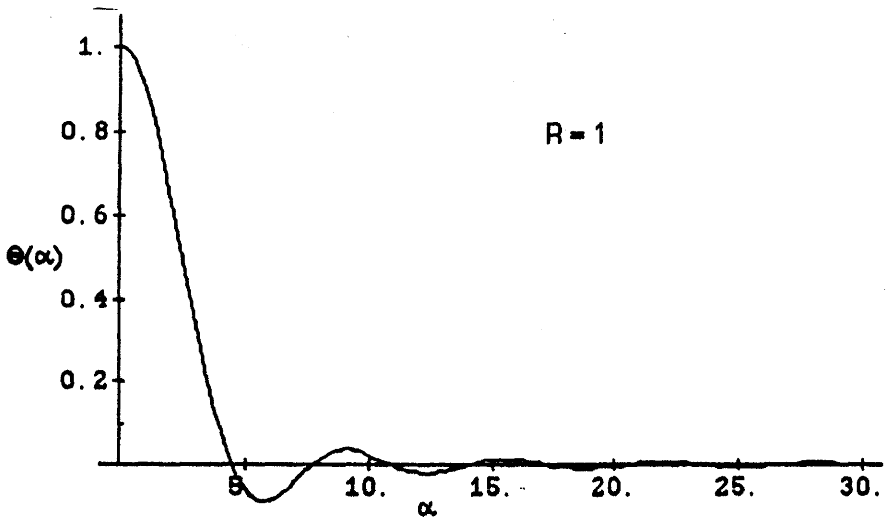

FIGURE 2. For $R = 1$, $\Theta(\alpha)$ is plotted as a function of α.

From (51) we find that the system is unstable if

$$C_1 + \frac{1}{3} d\ B > \sqrt{C_2^2 + \left(\frac{1}{3}\ d\ B\right)^2}\ . \tag{54}$$

This reduces to the well known criterion of Wallis when $\dfrac{\partial^2 \phi}{\partial t \partial z}$ term in (48) is dropped. From (52) the bed is unstable if

$$\Theta(\alpha) C_1 + \frac{1}{3}\ d\ B > \sqrt{\Theta(\alpha) C_2^2 + \left(\frac{1}{3}\ d\ B\right)^2}\ . \tag{55}$$

This also reduces to a form very similar to the well known criterion of Wallis when $\dfrac{\partial^2 \phi}{\partial t \partial z}$ term in (49) is dropped. In the present case, the inequalities (54) and (55) are always satisfied and so the uniform state is always unstable. From equation (53) the bed is unstable if

$$\frac{\dfrac{\gamma \tilde{g}}{U_0} \phi_0\ \dfrac{\partial U}{\partial \phi_0}}{\dfrac{\gamma \tilde{g}}{U_0} + \eta_0\ \alpha^2} > \sqrt{\frac{Q_0}{1 + \theta}}\ . \tag{56}$$

Conditions (55) and (56) are different because of their dependence on the wave number. The wave number dependence of (55) makes the modes with $\Theta(\alpha) = 0$ neutrally stable. Relation (56) shows that Batchelor's equations are stable to short waves and there exists a lower bound on α above which all modes are stable.

Viscosity stabilizes short waves. We want readers to notice that the criterion for stability in the absence of finite size of particles or viscosity effects is independent of the wave number (cf. Jones and Prosperetti [1985], Prosperetti and Jones [1987] and Prosperetti and Satrape [1989]).

Now consider the case when in (52) terms coming from the particle phase pressure are dropped. We solve for $\alpha\omega$

$$\alpha\omega = \frac{1}{2}\left(-Bi \pm \sqrt{-B^2 + 4\ B\ C_1\ i\alpha\ \theta}\right)$$

$$= \frac{B\ i}{2}\left(-1 \pm \sqrt{1 - \frac{4\ C_1\ i\alpha\ \Theta}{B}}\right)$$

(57)
$$= \frac{B\ i}{2}\left(-1 \pm \sqrt{1 - \Sigma\ i}\right)$$

where $\Sigma = \dfrac{4\ C_1\ \alpha\Theta}{B}$.

For $|\Sigma| < 1$ we can take Taylor series expansion

$$\alpha\omega = \frac{B\ i}{2}\ \left(-1 \pm \left(1 - \frac{\Sigma i}{2} + \frac{\Sigma^2}{8} + \ldots\right)\right) .$$

The growth rate

$$\sigma = Im[\alpha\omega] = \frac{B}{2}\ \left(-1 \pm \left(1 + \frac{\Sigma^2}{8} + \ldots\right)\right)$$

where $Im[\alpha\omega]$ is the imaginary part of $\alpha\omega$. The above expression for the growth rate and (57) implies

(1) Instability is weak when Σ is small.

(2) Σ is a bounded function of α with $\Sigma- > 0$ as $\alpha- > 0$ or $\alpha- > \infty$ (see figure 3). Thus both short and long waves are neutrally stable.

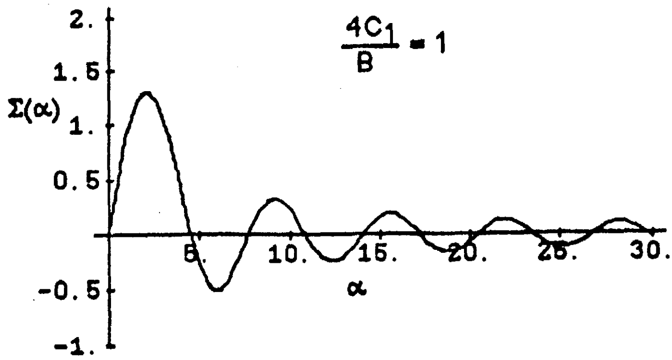

FIGURE 3. For $\frac{4\,C_1}{B} = 1$ and $R = 1$, $\Sigma(\alpha)$ is plotted as a function of α.

(3) Σ is maximum for $\alpha_{\max} \approx \frac{2.1}{R}$ and $\Sigma_{\max} \approx \frac{4\,C_1(1.3)}{B\,R}$ (see figure 3). Note that both $\alpha_{\max}$ and $\Sigma_{\max}$ increase as R is decreased. Hence the problem becomes ill-posed as $\alpha -> \infty$ for $R = 0$ (because $\Sigma_{\max} -> \infty$). To see that the problem is Hadamard unstable we put $R = 0$ in equation (57) and find that

(58)
$$\alpha\omega = \frac{B\,i}{2}\left(-1 \pm \sqrt{1 - \frac{4\,C_1\,\alpha}{B}i}\right)$$

where we have used $\lim_{R->0} \Theta = 1$. Therefore $\sigma = Im[\alpha\omega] = c\sqrt{\alpha}$ for some constant c, independent of α, for large α. The growth rate is unbounded for large α. Hence, the uniform fluidization is Hadamard unstable. Consider again equation (49) which reduces to the following form in the present case

(59)
$$\frac{\partial^2\phi}{\partial t^2} + B\left(\frac{\partial\phi}{\partial t} + C_1\frac{\partial\phi}{\partial z}\right) = 0.$$

This equation is parabolic because $D = 0$. So Hadamard instability here does not arise because the characteristics are imaginary. This kind of Hadamard instability is similar to that of the backward heat equation, here with the roles of time and space interchanged. One can see this by looking for spatially growing modes

(60)
$$\phi(z,t) = P\,e^{i\alpha t}e^{\sigma z}$$

where σ is complex and α is real. After combining (59) and (60), we get

$$\sigma = \frac{\alpha^2}{C_1} - \frac{Bi\alpha}{C_1}\,.$$

This is the kind of dispersion one get for the Hadamard instability of the backward heat equation.

(4) If we put the values of C_1 and B in the expression for Σ we get

$$\Sigma_{\max} \approx 6.9\frac{C_1^2}{C_2^2} \, .$$

So if $C_2 > \sqrt{6.9}\, C_1$ then Σ is smaller than one. In this case the instability is expected to be weak. This may be compared with the criterion $C_2 > C_1$ for stability derived in the (1984) paper of Foscolo and Gibilaro.

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