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Short Communication

REMARKS ON INERTIAL RADII, PERSISTENT NORMAL STRESSES, SECONDARY MOTIONS, AND NON-ELASTIC EXTENSIONAL VISCOSITIES

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Summary

In this note I discuss some consequences of the balance of inertia and normal stresses in nearly steady slow motions. I argue that the fluid's elasticity cannot be determined from its extensional viscosity. A formula is given for the extensional viscosity at quadratic order in the stretch rate and show that one and the same intensity factor for the extensional viscosity arises universally as the intensity factor for the non-Newtonian contribution to the vorticity. It is argued that the appearance of intense vortices and relatively large extensional viscosities are linked in nearly steady slow flow.

1. Second order fluids and Reiner–Rivlin fluids

The stress $\boldsymbol{T}$ in an incompressible simple fluid can be expressed as

$$\boldsymbol{T} = -p\boldsymbol{1} + \boldsymbol{S}, \tag{1}$$

where the extra stress is determined as a functional of the history of the relative Cauchy strain. The isotropic part of the stress has no constitutive equation; it is determined by dynamics as one of the unknown fields governed by the equations of motion. Most liquids which are isotropic in the rest state fit in this framework. Many models for the constitutive equation relating to $\boldsymbol{S}$ to deformation have been proposed. Whatever may be the choice for the functional relating stress to relative strain, it may be approximated by expressions of the form

$$\boldsymbol{S}^{(n)} \stackrel{\text{def}}{=} \boldsymbol{S}_1[\boldsymbol{U}] + \boldsymbol{S}_2[\boldsymbol{U}] + \cdots + \boldsymbol{S}_n[\boldsymbol{U}], \tag{2}$$

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with an error of $O(\|U\|^{n+1})$, where $U(x, t)$ is the velocity of a nearly steady slow motion

$$U(x, t) = \varepsilon v(x, \tau, \varepsilon), \qquad \tau = \varepsilon t, \tag{3}$$

and ε is a small parameter measuring slowness usually associated with prescribed data.

$$\| v(x, \tau, 0) \| < \infty \tag{4}$$

and t is a slow time, O(1) variations in time interval $\Delta\tau = \varepsilon\,\Delta t$ are slow in the time interval Δt; that is $\partial/\partial t = \varepsilon\,\partial/\partial\tau$. Moreover,

$$S^{(n)} = \varepsilon S_1[v] + \varepsilon^2 S_2[v] + \ldots \varepsilon^n S_n[v]. \tag{5}$$

For slow, steady motions, S_n is homogeneous of degree n in U. Nearly steady slow motions are equivalent to those which Coleman and Noll [1] call retarded; they are discussed in greater detail in references [7] and [10].

The expansions (2) were introduced by Coleman and Noll [1] and they were justified in the context of their theory of fading memory. These expansions are robust in the sense that they carry over to more general theories [10], which contain most of the constitutive equations presently used by rheologists. The form which such expansions must take is strongly suggested in the (1955) theory of Rivlin and Ericksen [4]. We are interested in the stress $S^{(2)}$ of grade two where

$$S_1 = \mu A_1[U],$$
$$S_2 = \alpha_1 A_2[U] + \alpha_2 A_1^2[U], \tag{6}$$

μ is the zero shear viscosity, α_1 and α_2 are constants, which I call the quadratic constants of $S^{(2)}$,

$$A_1[U] = \nabla U + \nabla U^{\mathrm{T}},$$

and

$$A_2[U] = \frac{\partial A_1}{\partial t} + (U \cdot \nabla) A_1 + A_1 \nabla U + \nabla U^{\mathrm{T}} A_1.$$

The stress $S^{(2)}$ has been widely used in rheological studies because it is one of the simpler nonlinear constitutive equations and, in fact, is the form which is taken by every constitutive equation when the motion is nearly steady and slow.

The stress $S = f(A_1)$ is a nonlinear viscous fluid may be represented, without loss of generality, in the form

$$S = \phi_1(II, III) A_1 + \phi_2(II, III) A_1^2, \tag{7}$$

where ϕ_1 and ϕ_2 are functions of the second and third invariants of A_1. The expression (7) is called a Reiner–Rivlin fluid. It is not presently used in rheology because it has a zero first normal stress in shear flow. On the other hand, every constitutive equation can be represented by eqn. (7) in steady motions which are purely extensional since the history of such motions can be expressed in terms of $\boldsymbol{A}_1$ alone. $\boldsymbol{S}^{(2)}$ is also of the form (7) because in pure extension $\boldsymbol{A}_2 = \boldsymbol{A}_1^2$.

It is not uncommon for rheologists to say that the extensional viscosity is associated with the elasticity, or the memory of a fluid. This type of thinking is flawed because we have just seen that the extensional viscosity is described uniquely, rigorously, and completely by an equation (7) for a purely viscous fluid without memory. Since $\boldsymbol{A}_1$ is the same for all past time in steady extension, the fluid has nothing different than its present state to remember. The elasticity of the fluid cannot be detected. To illustrate this with an elastic fluid, we may think of microstructural models of elastic dumbbells or long flexible molecules in which high extensional viscosities are associated with stretching of the model "molecules" in the flow, an elastic response in the sense that we have stored energy which is not manifest (because it remains constant) during a steady extensional flow. On the other hand, enormous extensional viscosities can be achieved in fluids with long, slender, inextensible and completely non-elastic particles. It is certainly possible to obtain the extensional functions of any simple fluid by devising an appropriate purely viscous fluid.

2. Quadratic constants

Tables of $3\alpha_1 + 2\alpha_2$, obtained from rod climbing measurements, can be found in the paper of Joseph and Beavers [5]. It is possible to determine the value of the ratio of normal stress differences $-N_2/N_1$ by back extrapolation toward $\kappa = 0$, even when N_1 and N_2 separately cannot be measured for small κ. In this case

$$-N_2/N_1 = 1 + \alpha_2/2\alpha_1$$

with an error proportional to κ^2. Therefore the measurement of the climbing constant and the normal stress ratio determine the separate values of α_1 and α_2. Keentok et al. [6] have measured limiting values of $-N_2/N_1$ for some fluids.

The zero shear viscosity μ and the quadratic constants α_1 and α_2 are related to moments of kernels appearing in the integral expansions of the stress (see, for example, Joseph and Beavers [7]) and to limiting values of the first and second normal stress. The zero shear viscosity μ is given by

$$\mu = \int_0^\infty G(s)\,\mathrm{d}s = \lim_{\kappa \to 0} T_{12}(\kappa)/\kappa, \tag{8}$$

the first quadratic constant is given by

$$\alpha_1 = -\int_0^\infty sG(s)\,\mathrm{d}s = -\lim_{\kappa\to 0} N_1(\kappa^2)/2\kappa^2 \tag{9}$$

and the second quadratic constant is given by [7]

$$\alpha_2 = \int_0^\infty\int_0^\infty \gamma(s_1, s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 = \lim_{\kappa\to 0}\frac{N_1(\kappa^2)+N_2(\kappa^2)}{\kappa^2} = \phi_2(0, 0), \tag{10}$$

where $\phi_2(II, III)$ appears in (7). These relations imply that μ is positive, α_1 is negative and the sign of α_2 is indeterminate. In polymeric liquids so far studied, N_2 is small and negative and α_2 is positive.

The parameter α_1 is associated with the extra tension $N_1 = T_{11} - T_{22}$ along streamlines in shear flow. At the same time $-\alpha_1/\mu$ may be thought to be a kind of relaxation time for the fluid, associated with the memory or elasticity. It could be said that N_1 is associated with elasticity. On the other hand, α_2 is present already in the Reiner–Rivlin fluid, which is instantaneously viscous and need not be associated with memory. We are going to show that any fluid with a large positive first normal stress difference and small α_2 will give rise to a *negative* extensional viscosity in slow steady flow. This type of elasticity aids rather than resists stretching.

3. Balance of inertia and normal stress. Something for nothing

Two features of flow are going to be discussed in the range of validity of second order theory which are not represented in the rheological literature. The first is the characteristic signature of dynamics of flow of second order fluids, the balance of inertia and normal stress at the quadratic level. The second is the persistence of normal stress in regions of vanishing dimension which we call "*something for nothing*".

Let us consider flows which perturb rest with steady flow proportional to a small parameter by writing

$$\boldsymbol{U} = \varepsilon \boldsymbol{U}_1 + \varepsilon_2 \boldsymbol{U} + \text{higher order terms}. \tag{11}$$

The perturbation equations for slow steady flow are well known. At first order, Stokes flow of Newtonian fluid is obtained. In the limit $\epsilon \to 0$, there is no difference between Newtonian and non-Newtonian fluids. At second order we get

$$\mu \nabla^2 \boldsymbol{U}_2 - \nabla p_2 = \rho \boldsymbol{U}_1\ \nabla \boldsymbol{U}_1 - \operatorname{div}\left(\alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{A}^2\right), \tag{12}$$

where $\boldsymbol{A}[\boldsymbol{U}_1] \overset{\text{def}}{=} \boldsymbol{A}_1$ is linear in $\boldsymbol{U}_1$ and $\boldsymbol{B}[\boldsymbol{U}_1] \overset{\text{def}}{=} \boldsymbol{A}_2$ is bilinear in $\boldsymbol{U}_1$. The inhomogeneous terms are the force of inertia

$$\rho \boldsymbol{U}_1\ \nabla \boldsymbol{U}_1$$

and the divergence of the normal stress

$$\operatorname{div}\left(\alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{A}^2\right).$$

The same eqn. (12), but with $\alpha_1 = 0$, governs the flow of the Reiner–Rivlin fluid (7). Suppose now that $\boldsymbol{U}_1$ scales with U and that $\boldsymbol{U}_1$ has an order one variation in a distance L. Then inertia scales like $\rho U^2/L$ and normal stresses like $\alpha U^2/L^3$ where α is representative of an appropriate linear combination of α_1 and α_2. The two effects are equally important at distances of the order

$$L = \sqrt{\alpha/\rho}\,.$$

In many problems there is a critical distance, of order L, with inertia dominant for larger distances and normal stresses dominant for smaller distances. We could call the critical distance an inertial radius. This type of balance appears in many problems; for example, in the flow outside a rotating sphere or between a cone and plate [8], outside a rotating wavy rod [9], or in the flow associated with rod climbing [5], [7], among others. There is an inertial radius for secondary motions which in rotating flows takes form as a dividing surface for eddies with a different direction of circulation. There is also an inertial radius for the pressure. Suppose that $U = r\omega(r)$ where $\omega(r)$ is the angular velocity of rotation in the first order Newtonian problem. Inertia

$$\rho r^2 \omega^2$$

goes to zero quadratically with r, but the normal stresses

$$\alpha \frac{U^2}{r^2} \sim \alpha \omega^2$$

do not go to zero, no matter how small is the distance r. The same type of phenomena occurs in pressure holes in which normal stresses persist (give something) even when the hole diameter d tends to zero (for nothing); $U = \kappa d$ and normal stresses scale with $\alpha U^2/d^2 = \alpha\kappa^2$, independent of d.

The astonishing persistence of normal stresses for small rods and holes appears from experiments to be a valid physical description though the evidence at the smallest dimensions is obscured by other small dynamical effects like surface tension, which become relatively large in this limit.

The concept of an inertial radius or critical distance dividing fields into regions dominated by inertia and regions dominated by normal stress requires a competition between normal stresses and inertia. Sometimes people refer to this as a competition between elasticity and inertia, but since α_2 is involved, this choice of words may be misleading. Examples of competing effects are abundant [7–9,11]. In steady flows around rotating bodies, it appears that inertia drives eddies with one sense, and normal stresses drive them in the opposite sense. The inertia of fluid will cause it to sink near a rotating rod and normal stresses will make it rise. The effect of normal stresses is to make the extrudate from a pipe swell, while inertia makes it contact. Inertia produces an error of one sign at pressure hole and the normal stresses produce an error of the other sign.

It is not obligatory that normal stresses and inertia must compete. Counter examples can be found. For example, when a rod is rotated in a two-layer system, with the viscoelastic liquid above, this liquid will climb down the rod. Analysis shows that the effect of gravity on the rotating heavy fluid below is to depress the free surface into a "bathtub vortex" with greater depression near the rod. The normal stresses push the viscoelastic fluid up into the air and down into the heavy fluid. Hence, inertia and normal stresses are in conflict at the air–liquid surface and in concert at the two-fluid interface. A casual inspection of the equations of motion might suggest that normal stresses and inertia do not need to act in opposition in general, in spite of so many examples in which they are opposed. Perhaps it is possible to find a general condition in which opposing effects are more than a rule of thumb.

4. Extensional viscosity and secondary motions

We can show that fluids which have large extensional viscosities at low stretching are likely to give rise to intense secondary motions; both are proportional to $\alpha_1 + \alpha_2$.

At first order we find that the vorticity $\zeta_1 = \text{curl}\ \boldsymbol{u}_1$ is a harmonic vector

$\nabla^2 \zeta_1 = 0$.

At second order we get

$$\rho \frac{\partial \zeta_2}{\partial t} + \rho\ \text{curl}(\zeta_1 \wedge \boldsymbol{u}_1) = \mu \nabla^2 \zeta_2 + (\alpha_1 + \alpha_2)\ \text{curl div}\ \boldsymbol{A}^2, \qquad (13)$$

where $\boldsymbol{A} = \boldsymbol{A}[\boldsymbol{u}_1]$ and the Giesekus [8] relation div $\boldsymbol{B} = \text{div}\ \boldsymbol{A}^2 + \text{grad}\ \phi$ has been used. Equation (13) shows that no couples arise from normal stresses at second order when

$$\alpha_1 + \alpha_2 = \lim_{\kappa \to 0} \left(\tfrac{1}{2} N_1 + N_2\right)/\kappa^2 \qquad (14)$$

vanishes. In polymeric liquids so far studied N_2 is small and negative. For these liquids $\alpha_1 + \alpha_2$ is positive.

In the case in which inertia and normal stresses compete, we may conclude that the inertial radius for secondary flows is given by

$L = O((\alpha_1 + \alpha_2)/\rho)^{1/2}$

At distances greater than the inertial radius for secondary motions, inertia will drive secondary motions; at lesser distances secondary motions are driven by normal stresses.

The motion known as simple extension in the direction $\boldsymbol{e}_1$ is given by

$$[\boldsymbol{A}_1] = \begin{pmatrix} 2\dot{s} & 0 & 0 \\ 0 & -\dot{s} & 0 \\ 0 & 0 & -\dot{s} \end{pmatrix},$$

$$[\boldsymbol{A}_1^2] = \begin{pmatrix} 4\dot{s}^2 & 0 & 0 \\ 0 & \dot{s}^2 & 0 \\ 0 & 0 & \dot{s}^2 \end{pmatrix},$$

$$\boldsymbol{A}_2 = \boldsymbol{A}_1^2,$$

where $\dot{s}$ is the rate of stretching, $\mathrm{d}U_1/\mathrm{d}x_1 = \dot{s}$. The stress differences in extension

$$T_{11} - T_{22} = T_{11} - T_{33} = S_{11} - S_{22} = S_{11} - S_{33}$$

are evaluated on $\boldsymbol{S}^{(2)} = \mu \boldsymbol{A}_1 + (\alpha_1 + \alpha_2)\boldsymbol{A}_1^2$ as

$$T_{11} - T_{22} = 3\left[\dot{s}\mu + (\alpha_1 + \alpha_2)\dot{s}^2\right] = 3\dot{s}\left[\mu + (\alpha_1 + \alpha_2)\dot{s}\right], \tag{15}$$

where $\mu + (\alpha_1 + \alpha_2)\dot{s}$ is the extensional viscosity at low stretching. This relation, expressed in terms of normal stress differences as in (14), was first given by Zahorski [12] and later in the form (15) in [5]. Equation (15) is the only model independent formula for the extensional viscosity known to the author. It applies to all steady extensional motions (small $\dot{s}$ with errors $O(\dot{s}^3)$).

Second order fluids with $\alpha_1 = 0$ are second order Reiner–Rivlin fluids. For these the extensional viscosity is larger than for any second order fluid with the same α_2, because α_1 is negative. The fact that the intensity factor $\alpha_1 + \alpha_2$ appears both in (15) and (13) leads to a theorem:

The extensional viscosity is greater and the vortical motions are more intense in the slow steady motions of a Reiner–Rivlin fluid ($\alpha_1 = 0$) and than in any elastic fluid ($\alpha_1 < 0$) with the same α_2.

Many intense secondary flows, like those with strong vortices in the flow through contractions, should then arise from Reiner–Rivlin fluids. The larger the value of α_2, the more intense the motion.

These observations about the relation of extensional viscosity to secondary motions seem to offer a theoretical foundation for recent numerical results presented by Debbaut and Crochet [2] and by Debbaut et al. [3]. They show that "...by including the dependence of the viscosity function upon the third as well as the second invariant in generalized Newtonian and viscoelastic fluids, it is possible to isolate the effects of extensional viscosity and first normal stress difference in axisymmetric flows. It is shown that a high Trouton's ratio leads to vortex enhancement in an abrupt 4:1 circular contraction and to drag increase for the flow round a sphere moving along the axis of a circular tube. For low values of elasticity, it is shown that the first normal stress difference has the opposite effect". We may interpret their observation as related to the fact that for a fixed value of α_2 the intensity of vortex activity and the Trouton viscosity are greatest when $|\alpha_1|$ is smallest.

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