

## CHANGE OF TYPE AND LOSS OF EVOLUTION OF THE WHITE–METZNER MODEL

CLAUDE VERDIER and DANIEL D. JOSEPH

*Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis,  
MN 55455 (U.S.A.)*

(Received September 25, 1988; accepted September 30, 1988)

### Summary

In this paper a mathematical study of the White–Metzner model is presented. This model gives rise to systems of first order nonlinear (not quasilinear) partial differential equations. The unsteady case is studied first to determine if the Cauchy problem is well posed, stable to short waves. Then we study two-dimensional steady flow by classifying the streamfunction equation for type. The roots of a polynomial have to be analyzed. We show how to use the roots to factorize an operator, we compute the roots and carry out the factorization globally, using a symbol manipulator, and locally, using perturbation analysis. The additional nonlinearities are found to add more hyperbolicity.

---

### 1. Introduction

There are by now many papers devoted to classifying the equations which govern the flow of viscoelastic fluids. This classification is important because ill-posed problems are catastrophically unstable to short waves; the finer the mesh the worse the result. On the other hand, steady problems which change type need not be unstable but the type change may signal a change in the physics of flow.

Almost everything is known about the Maxwell fluids. Rutkevich [1] found some conditions for well posedness for the upper, lower convected and corotational Maxwell models. Joseph et al. [2] found that the upper and lower convected models never become ill posed on smooth solutions. A similar result, using different methods was obtained by Dupret and Marchal [3,4]. Joseph and Saut [5] studied the Maxwell models which interpolate

between the upper and lower convected models for a loss of evolution (loss of well posedness). The interpolated models are known to some as Johnson–Segalman models. They showed that the interpolated models would go unstable in many common flows.

The interpolated models are quasilinear. Less is known about nonlinear models. The White–Metzner model is nonlinear, not quasilinear. It is obtained from the upper convected model by considering the relaxation time and viscosity ( $\lambda$  and  $\eta$ ) to depend on the second invariant of the rate of strain tensor.

Dupret and Marchal [4] studied the White–Metzner model and found some evolution criteria in abstract form. Here, another approach to the problem of the loss of evolution for the White–Metzner model is presented, similar to the one given in [5] and the results are compared. Two conditions for well posedness are found in the 3-dimensional case.

It is also of interest to classify the system of equations governing steady flow for type. This has to be done first by reducing the system of  $n$  nonlinear equations to a quasilinear system of  $n$  equations. This technique will be explained later. Usual methods are then used to find the type (hyperbolic or elliptic).

The roots of the system have been found to be also the roots of the vorticity equation for the interpolated models. This is not the case for the White–Metzner model, for there are terms that cannot be reduced to any component of the vorticity. In the 2-dimensional steady case, a streamfunction is more likely to be used, as it was done by Regirer and Rutkevich [6] and it satisfies a fourth order partial differential equation.

The type of the system is determined by studying the roots of a fourth order polynomial. Surprisingly, it is found that the dependence of  $\lambda$  and  $\eta$  on the second invariant gives rise to an extra part in the polynomial which is always purely hyperbolic (real roots). This is a very important result since it shows that the effect of hyperbolicity is even stronger than what we thought.

Using a symbol manipulator on a computer, we can determine in the most general case what the roots are and reconstitute the operator. This has been done but the formulae are very lengthy and will not be given here.

When the relaxation time and the viscosity do not depend very strongly on the second invariant ( $II_D$ ) of the rate of strain tensor, an expansion in powers of a small parameter can be done to determine how the roots behave.

Our study is helpful to understand what happens for some very special models. Gaidos and Darby [7,8] use a very specific formula for  $\lambda$  and  $\eta$  and they find a change of type in the developing flow at the entrance of a planar slit. Calderer et al. [9], although they use another nonlinear model (Bird–DeAguiar), find plane flows with 6 real roots so that the system of equations can even be purely hyperbolic.

## 2. Governing equations

We will start this analysis by writing the equations of motion for the White–Metzner model, the continuity equation, the momentum equation and the stress equation:

$$\operatorname{div} \mathbf{u} = 0, \quad (1)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \operatorname{div} \boldsymbol{\tau}, \quad (2)$$

$$\boldsymbol{\tau} + \lambda (II_D) \left( \frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\tau} - \nabla \mathbf{u} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \nabla \mathbf{u}^T \right) = 2\eta (II_D) \mathbf{D}[\mathbf{u}]. \quad (3)$$

One can see from (3) that the nonlinearity comes from the fact that  $\lambda$  and  $\eta$  depend on the second invariant of the stress tensor  $II_D$ , where  $II_D$  is defined by

$$II_D = \frac{1}{2} \operatorname{tr}(\mathbf{D}^2) = \frac{1}{2} \mathbf{D} : \mathbf{D} \geq 0 \quad (4)$$

## 3. Loss of evolution

To determine whether loss of evolution occurs, we perturb a basic solution designated with capital letters, with a perturbation, designated with primes; that is,

$$(\mathbf{u}, \boldsymbol{\tau}, p) = (\mathbf{U}, \mathbf{T}, P) + (\mathbf{u}', \boldsymbol{\tau}', p'). \quad (5)$$

We will now drop the primes on the perturbed quantities. Under linearization, we have

$$II_{D+d} = II_D + \mathbf{D} : \mathbf{d}, \quad (6)$$

where  $\mathbf{D} \stackrel{\text{def}}{=} \mathbf{D}[\mathbf{U}]$  and  $\mathbf{d} \stackrel{\text{def}}{=} \mathbf{D}[\mathbf{u}]$ .

We now use (5) and (6) in eqns. (1), (2) and (3) to obtain

$$\operatorname{div} \mathbf{u} = 0, \quad (7)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{u} \right) + \nabla p - \operatorname{div} \boldsymbol{\tau} = \text{l.o.t.}, \quad (8)$$

$$\begin{aligned} \lambda (II_D) \mathbf{L}_1(\boldsymbol{\tau}) + (\mathbf{D} : \mathbf{d}) \lambda' (II_D) \mathbf{L}_2(\mathbf{T}) - 2\eta (II_D) \mathbf{d} \\ - 2(\mathbf{D} : \mathbf{d}) \eta' (II_D) \mathbf{D} = \text{l.o.t.} \end{aligned} \quad (9)$$

$$\text{where } \mathbf{L}_1(\boldsymbol{\tau}) = \frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{U} \cdot \nabla \boldsymbol{\tau} - \nabla \mathbf{u} \cdot \mathbf{T} - \mathbf{T} \cdot \nabla \mathbf{u}^T$$

$$\text{and } \mathbf{L}_2(\mathbf{T}) = \frac{\partial \mathbf{T}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{T} - \nabla \mathbf{U} \cdot \mathbf{T} - \mathbf{T} \cdot \nabla \mathbf{U}^T$$

and l.o.t. are lower order terms without any derivative of  $\mathbf{u}$  or  $\boldsymbol{\tau}$ .

Following the method of [1] we now look for perturbations of the form

$$(\mathbf{u}, \tau, p) = (\omega_i, \sigma_{ij}, q) \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t). \quad (10)$$

We define

$$\mathbf{n} = \mathbf{k}/|\mathbf{k}| \quad (11)$$

and introduce the velocity of propagation of wave fronts for short waves  $C$ :

$$C = \frac{\omega}{|\mathbf{k}|} - \mathbf{U} \cdot \mathbf{n}. \quad (12)$$

We next let  $k = |\mathbf{k}| \rightarrow \infty$  since we are looking at short waves. This leads to the following system for the amplitudes:

$$\omega_i n_i = 0, \quad (13)$$

$$-\rho C \omega_i = -q n_i + \sigma_{ij} n_j, \quad (14)$$

$$\lambda(C\sigma_{ij} + T_{lj}\omega_l n_i + T_{il}\omega_j n_l) + \eta(\omega_l n_j + \omega_j n_l) + D_{kl}(\omega_k n_l + \omega_l n_k)V_{ij} = 0, \quad (15)$$

where we have defined the tensor  $V$  by

$$V = \mu' \lambda D + \frac{\lambda'}{2\lambda} T, \quad (16)$$

where

$$\mu = \frac{\eta(II_D)}{\lambda(II_D)} = \mu(II_D)$$

and

$$\mu' = \mu'(II_D), \quad \lambda' = \lambda'(II_D).$$

This gives a system of 10 equations with 10 unknowns. For a non-trivial solution, the determinant has to vanish:

Following [1], we can select a special system of coordinates such that

$$n_1 = 1, \quad n_2 = 0, \quad n_3 = 0 \quad \text{and} \quad T_{23} = 0.$$

The determinant  $\Delta$  of the  $10 \times 10$  system can be then factored out. We find that

$$\Delta = -(\lambda C)^4 (\rho \lambda C^2 - A) (\rho \lambda C^2 - A - 2D_{12}V_{12} - 2D_{13}V_{13}), \quad (17)$$

where  $A$  is defined by

$$A = \lambda T_{11} + \eta. \quad (18)$$

There are therefore two waves with velocities  $C_+$  and  $C_-$  given by:

$$C_+^2 = (\lambda T_{11} + \eta) / \rho \lambda, \quad (19)$$

$$C_-^2 = (\lambda T_{11} + \eta + 2D_{12}V_{12} + 2D_{13}V_{13}) / \rho \lambda. \quad (20)$$

The condition for well posedness is that the two squares of the propagation speeds have to be positive

$$\lambda T_{11} + \eta \geq 0, \quad (21)$$

$$\lambda T_{11} + \eta + 2D_{12}V_{12} + 2D_{13}V_{13} \geq 0. \quad (22)$$

In the 2D case we find that there is only one root:

$$C^2 = (\lambda T_{11} + \eta + 2D_{12}V_{12}) / \rho \lambda \geq 0. \quad (23)$$

Condition (21) is the same as the one obtained in [1] for an upper convected Maxwell model, whereas (22) is an additional condition which is needed for well posedness. It is not possible to conclude anything on the sign of  $D_{12}V_{12} + D_{13}V_{13}$  since it depends on the way that  $\lambda$  and  $\eta$  vary as functions of the second invariant  $II_D$ . In general we could write

$$\text{Min}(\lambda T_{11} + \eta, \lambda T_{11} + \eta + 2D_{12}V_{12} + 2D_{13}V_{13}) \geq 0. \quad (24)$$

It is of interest to carry out this kind of analysis for basic flows like shear flow or elongational flow, following the method introduced in [2] and [5].

### 3.1 Shear flow

We are looking at the 2D case of shear flow where we have

$$U = \kappa y, \quad V = 0, \quad T_{11} = 2\lambda\kappa T_{12}, \quad T_{22} = 0, \quad T_{12} = \eta\kappa \quad (25)$$

and

$$\lambda = \lambda(II_D) = \tilde{\lambda}(\kappa), \quad \eta = \eta(II_D) = \tilde{\eta}(\kappa).$$

After simplifications we can rewrite (23) as

$$\eta(1 + 2\lambda^2\kappa^2) + \frac{\kappa^2}{2}\eta' \geq 0. \quad (26)$$

The condition (26) for well posedness is satisfied always when the fluid is shear-thickening  $\eta' > 0$  or not too large. Obviously there are shear-thinning fluids for which shear flow is ill posed.

### 3.2 Elongational flow

The basic flow is

$$U = sx, \quad V = -sy, \quad T_{11} = \frac{2\eta s}{1 - 2\lambda s},$$

$$T_{12} = 0, \quad T_{22} = \frac{-2\eta s}{1 + 2\lambda s} \quad (27)$$

and

$$\lambda = \lambda(II_D) = \tilde{\lambda}(s), \quad \eta = \eta(II_D) = \tilde{\eta}(s).$$

The condition (23) becomes

$$1/(1 - 2\lambda s) \geq 0 \quad \text{or} \quad s \leq 1/2\lambda. \quad (28)$$

This condition is the same one as for elongational flow of an upper convected Maxwell model. There is ill posedness whenever (28) is not satisfied.

### 4. Analysis of the 2D steady case

In most problems we deal with systems that can be easily put into a quasilinear form convenient for the analysis of characteristic curves. Here we have a nonlinear system and a little more work needs to be done. We will give here as a reference the method of [7,8] and use it in a rather different way:

First we write (1), (2), and (3) as a system of six nonlinear equations:

$$\begin{aligned} \sigma + \lambda_{II_D}(u\sigma_x + v\sigma_y - 2\sigma u_x - 2\tau u_y) - 2\eta_{II_D}u_x &= 0, \\ \tau + \lambda_{II_D}(u\tau_x + v\tau_y - \gamma u_y - \sigma v_x) - \eta_{II_D}(u_y + v_x) &= 0, \\ \gamma + \lambda_{II_D}(u\gamma_x + v\gamma_y - 2\gamma v_y - 2\tau v_x) - 2\eta_{II_D}v_y &= 0, \\ \rho(uu_x + vu_y) + p_x - \sigma_x - \tau_y &= 0, \\ \rho(uv_x + vv_y) + p_y - \tau_x - \gamma_y &= 0, \\ \mathbf{u}_x + \mathbf{v}_y &= \mathbf{0}, \end{aligned} \quad (29)$$

where we wrote  $\lambda_{II_D}$  and  $\eta_{II_D}$  to keep in mind that  $\lambda$  and  $\eta$  are non constants.

Here we have

$$II_D = \frac{1}{2}\mathbf{D} : \mathbf{D} = \frac{1}{2}u_x^2 + \frac{1}{2}v_y^2 + \frac{1}{4}(u_x + v_y)^2. \quad (30)$$

The idea is to differentiate the system to reduce the nonlinear terms to quasilinear terms in differentiated variables. First we introduce the new variables, as in [7,8,10]:

$$\mathbf{h} = (u, v, p, \sigma, \tau, \gamma), \quad (31)$$

and also define

$$p_i = \partial h_i / \partial x, \quad (32)$$

$$q_i = \partial h_i / \partial y. \quad (33)$$

Of course, we have the compatibility relation

$$\partial p_i / \partial y = \partial q_i / \partial x. \quad (34)$$

Then the system can be rewritten in the form

$$\mathbf{F}(x, y, h_i, p_i, q_i) = \mathbf{0} \quad i = 1, \dots, 6. \quad (35)$$

We differentiate the system (35) with respect to  $x$  to get

$$\frac{\partial F_i}{\partial p_j} \frac{\partial p_j}{\partial x} + \frac{\partial F_i}{\partial q_j} \frac{\partial p_j}{\partial y} = -\frac{\partial F_i}{\partial x} - \frac{\partial F_i}{\partial h_j} p_j \quad (36)$$

after using (34). This gives rise to the first part of the system. We have only derivatives of the  $p_i$ 's. The lower order terms are on the right. This is the way we are going to emphasize the structure of the new system.

We now differentiate with respect to  $y$  to get

$$\frac{\partial F_i}{\partial p_j} \frac{\partial q_j}{\partial x} + \frac{\partial F_i}{\partial q_j} \frac{\partial q_j}{\partial y} = -\frac{\partial F_i}{\partial y} - \frac{\partial F_i}{\partial h_j} q_j. \quad (37)$$

This is the second part of the system involving the derivatives of the  $q_i$ 's only.

Finally we write mathematical identities, based on (34) to get the part of the system containing only derivatives of the  $h_i$ 's:

$$\frac{\partial F_i}{\partial p_j} \frac{\partial h_j}{\partial x} + \frac{\partial F_i}{\partial q_j} \frac{\partial h_j}{\partial y} = \frac{\partial F_i}{\partial p_j} p_j + \frac{\partial F_i}{\partial q_j} q_j. \quad (38)$$

We now have a system of 18 quasilinear equations for the variables ( $h_i, p_i, q_i$ ) and the structure is clear, since we can write

$$A_{ij} \frac{\partial w_j}{\partial x} + B_{ij} \frac{\partial w_j}{\partial y} = \text{l.o.t.}_i(w). \quad (39)$$

where  $w$  can be either  $h, p$  or  $q$  and where the lower order terms are different.  $A_{ij}$  and  $B_{ij}$  are defined by

$$A_{ij} = \partial F_i / \partial p_j, \quad (40)$$

$$B_{ij} = \partial F_i / \partial q_j. \quad (41)$$

It is not necessary to write this whole system since all the “blocks” have the exact same structure, except that the lower order terms are different. Therefore we only need to derive a system of 6 quasilinear equations by differentiating with respect to  $x$  for example. In general we can always find a quasilinear system of  $n$  equations equivalent to a non linear system of  $n$  equations.

Our system is ( $F$  actually does not depend on  $x$  and  $y$ )

$$\begin{aligned}
 h_4 + \lambda_{II_D}(h_1 p_4 + h_2 q_4 - 2h_4 p_1 - 2h_5 q_1) - 2\eta_{II_D} p_1 &= 0, \\
 h_5 + \lambda_{II_D}(h_1 p_5 + h_2 q_5 - h_6 q_1 - h_4 p_2) - \eta_{II_D}(q_1 + p_2) &= 0, \\
 h_6 + \lambda_{II_D}(h_1 p_6 + h_2 q_6 - 2h_6 q_2 - 2h_5 p_2) - 2\eta_{II_D} q_2 &= 0, \\
 \rho(h_1 p_1 + h_2 q_1) + p_3 - p_4 - q_5 &= 0, \\
 \rho(h_1 p_2 + h_2 q_2) + q_3 - p_5 - q_6 &= 0, \\
 p_1 + q_2 &= 0.
 \end{aligned} \tag{42}$$

We differentiate the 6 equations with respect to  $x$  to get, after using (34):

$$\mathbf{A} \frac{\partial \mathbf{p}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{p}}{\partial y} = \text{l.o.t.}, \tag{43}$$

where

$$\mathbf{A} = \begin{pmatrix} A_1 & A_2 & u & 0 & 0 & 0 \\ A_3 & A_4 & 0 & 0 & u & 0 \\ A_5 & A_6 & 0 & u & 0 & 0 \\ \rho u & 0 & -1 & 0 & 0 & 0 \\ 0 & \rho u & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

and

$$\mathbf{B} = \begin{pmatrix} B_1 & B_2 & v & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 & v & 0 \\ B_5 & B_6 & 0 & v & 0 & 0 \\ \rho v & 0 & -1 & 0 & 0 & 0 \\ 0 & \rho v & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$



The coefficients  $A_i$ 's and  $B_i$ 's are defined as follows:

$$\begin{aligned}
 A_1 &= -2 \frac{D_{11}V_{11}}{\lambda} - 2\sigma - 2\mu, & A_2 &= -2 \frac{D_{12}V_{11}}{\lambda}, \\
 A_3 &= -2 \frac{D_{11}V_{12}}{\lambda}, & A_4 &= -2 \frac{D_{12}V_{12}}{\lambda} - \sigma - \mu, \\
 A_5 &= -2 \frac{D_{11}V_{22}}{\lambda}, & A_6 &= -2 \frac{D_{12}V_{22}}{\lambda} - 2\tau, \\
 B_1 &= -2 \frac{D_{12}V_{11}}{\lambda} - 2\tau, & B_2 &= -2 \frac{D_{22}V_{11}}{\lambda}, \\
 B_3 &= -2 \frac{D_{12}V_{12}}{\lambda} - \gamma - \mu, & B_4 &= -2 \frac{D_{22}V_{12}}{\lambda}, \\
 B_5 &= -2 \frac{D_{12}V_{22}}{\lambda}, & B_6 &= -2 \frac{D_{22}V_{22}}{\lambda} - 2\gamma - 2\mu,
 \end{aligned} \tag{46}$$

where the components  $V_{ij}$  of the tensor  $V$  have been defined by (16).

Then we follow [2] and let  $\alpha = dy/dx$ ; the characteristic curves are determined by

$$\det(\alpha A - B) = 0. \tag{48}$$

After calculating the determinant we find, as in [7,8], that

$$\begin{aligned}
 (\alpha u - v)^2 \lambda^2 &(-A_1 \alpha^2 + B_1 \alpha - A_2 \alpha^3 + B_2 \alpha^2 + A_3 \alpha(1 - \alpha^2) + B_3(\alpha^2 - 1) \\
 &+ A_4 \alpha^2(1 - \alpha^2) + B_4 \alpha(\alpha^2 - 1) + A_5 \alpha^2 - B_5 \alpha + A_6 \alpha^3 - B_6 \alpha^2 \\
 &- \rho \lambda(1 + \alpha^2)(\alpha u - v)^2) = 0.
 \end{aligned} \tag{49}$$

Then we use (46) and (47). Surprisingly we find, after some calculations that (49) becomes

$$\begin{aligned}
 (\alpha u - v)^2 \lambda^2 &\{(1 + \alpha^2)((\sigma + \mu - \rho u^2)\alpha^2 - 2(\tau - \rho uv)\alpha + \gamma + \mu - \rho v^2) \\
 &+ \frac{2}{\lambda}(D_{12}\alpha^2 + (D_{11} - D_{22})\alpha - D_{12})(V_{12}\alpha^2 + (V_{11} - V_{22})\alpha - V_{12})\} = 0.
 \end{aligned} \tag{50}$$

First we note that the streamlines  $dy/dx = v/u$  are twice solutions of (50) so there are at least two real roots, but they are double. The system is not strictly hyperbolic in any case. It is also of interest to note the symmetric role of the tensors  $D$  and  $V$  in the second part of this formula.

The formula (50) shows that we add an extra part  $Q(\alpha)$ , to the usual polynomial  $P(\alpha)$ , where

$$P(\alpha) = (1 + \alpha^2)((\sigma + \mu - \rho u^2)\alpha^2 - 2(\tau - \rho uv)\alpha + \gamma + \mu - \rho v^2) \tag{51}$$

and

$$Q(\alpha) = \frac{2}{\lambda} (D_{12}\alpha^2 + (D_{11} - D_{22})\alpha - D_{12})(V_{12}\alpha^2 + (V_{11} - V_{22})\alpha - V_{12}). \quad (52)$$

The polynomial  $P(\alpha)$  is indeed the one that corresponds to the upper convected Maxwell model.

In addition to that, the extra part  $Q(\alpha)$  only has real roots since the discriminants of each quadratic from (52) are positive, i.e.

$$\Delta_1 = (D_{11} - D_{22})^2 + 4D_{12}^2 \geq 0, \quad (53)$$

$$\Delta_2 = (V_{11} - V_{22})^2 + 4V_{12}^2 \geq 0. \quad (54)$$

There are two limiting cases that can be considered.

- $\lambda$  and  $\eta$  do not depend very strongly on  $II_D$

In this case the coefficients  $V_{ij}$  are small and we can neglect the second part, the roots are the roots of (51),

$$\alpha = \pm i, \quad (55)$$

$$\alpha = \frac{\tau - \rho uv \pm \sqrt{(\tau - \rho uv)^2 - (\sigma + \mu - \rho u^2)(\gamma + \mu - \rho v^2)}}{\sigma + \mu - \rho u^2}. \quad (56)$$

Of course we have two imaginary roots, and the two other ones depend on the sign of the quantity  $\Delta = (\tau - \rho uv)^2 - (\sigma + \mu - \rho u^2)(\gamma + \mu - \rho v^2)$ . We may have a change of type.

- $\lambda$  and  $\eta$  depend very strongly on  $II_D$

Assuming we can neglect the first part  $P(\alpha)$ , then the roots are easy to find

$$\alpha = \frac{D_{22} - D_{11} \pm \sqrt{(D_{11} - D_{22})^2 + 4D_{12}^2}}{2D_{12}}, \quad (57)$$

$$\alpha = \frac{V_{22} - V_{11} \pm \sqrt{(V_{11} - V_{22})^2 + 4V_{12}^2}}{2V_{12}}. \quad (58)$$

These roots are always *real* and it does not even matter how  $\lambda$  and  $\eta$  depend on the second invariant  $II_D$ .

It is hard to say how the roots change by studying the whole polynomial. Since the streamlines are always doubly characteristic, we have to study the roots of a polynomial of order 4. There are some standard techniques to determine the roots of a quartic but all of them are very lengthy. The only thing that needs to be said is that each fourth order polynomial can be factored into a product of two quadratics. Therefore there are either

- Four real roots,
- Two real roots + a pair of complex conjugates,
- Two pairs of complex conjugates.

To these we append the usual two real roots corresponding to the streamlines.

In Section 6 we shall note that all the roots may be obtained in analytic form by using symbol manipulators on the computer. The expressions for the general case are very long and are not given here. We show how to use the formulae for the roots to factorize the fourth order operator on the streamfunction into quadratic operators. All this is carried out explicitly for small  $\lambda'$  and  $\eta'$  by perturbation in Section 7.

## 5. Vorticity equations and the streamfunction

The method of deriving the vorticity equation is well known. For the White-Metzner model, we find, for the components of the vorticity  $\zeta = \zeta_b e_b$ :

$$\rho \left( \frac{\partial^2 \zeta_b}{\partial t^2} + 2(\mathbf{u} \cdot \nabla) \frac{\partial \zeta_b}{\partial t} + (\mathbf{u} \cdot \nabla)^2 \zeta_b \right) - (\boldsymbol{\tau} : \nabla \nabla) \zeta_b - \mu (II_D) \nabla^2 \zeta_b - \epsilon_{bmi} V_{ij} D_{kl} \left( \frac{\partial^3 u_k}{\partial x_i \partial x_j \partial x_m} + \frac{\partial^3 u_l}{\partial x_k \partial x_j \partial x_m} \right) = \text{l.o.t.} \quad (59)$$

where  $V_{ij}$  are the components of the tensor  $V$  defined by (16). Note one more time the importance of the tensor  $V$ . The last terms in (59) are "spoilers" because they cannot be expressed in terms of the vorticity components.

In the 2D case we will introduce a streamfunction  $\psi$  defined by

$$u = \partial \psi / \partial y, \quad v = -\partial \psi / \partial x. \quad (60)$$

Inserting the streamfunction in (59), we determine a fourth order equation for  $\psi$  as in [6]. We find that

$$\left( \sigma + \mu - \rho u^2 + 2 \frac{D_{12} V_{12}}{\lambda} \right) \frac{\partial^4 \psi}{\partial x^4} + 2 \left( \tau - \rho uv + \frac{(D_{22} - D_{11}) V_{12}}{\lambda} + \frac{D_{12} (V_{22} - V_{11})}{\lambda} \right) \frac{\partial^4 \psi}{\partial x^3 \partial y}$$

$$\begin{aligned}
& + \left( \sigma + \mu - \rho u^2 + \gamma + \mu - \rho v^2 + 2 \frac{(D_{11} - D_{22})(V_{11} - V_{22})}{\lambda} \right. \\
& \left. - 4 \frac{D_{12}V_{12}}{\lambda} \right) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} \\
& + 2 \left( \tau - \rho uv + \frac{(D_{11} - D_{22})V_{12}}{\lambda} + \frac{D_{12}(V_{11} - V_{22})}{\lambda} \right) \frac{\partial^4 \psi}{\partial x \partial y^3} \\
& + \left( \gamma + \mu - \rho v^2 + 2 \frac{D_{12}V_{12}}{\lambda} \right) \frac{\partial^4 \psi}{\partial y^4} = \text{l.o.t.}, \tag{61}
\end{aligned}$$

where l.o.t. represents lower order terms including derivatives up to the third order of  $\psi$ .

We can get the quartic equation in the braces of (50) for the characteristic curves directly from (61) \*. We will now introduce different notations for the coefficients

$$\begin{aligned}
A &= \sigma + \mu - \rho u^2, & B &= \tau - \rho uv, & C &= \gamma + \mu - \rho v^2, \\
D &= \frac{D_{12}(V_{11} - V_{22})}{\lambda}, & E &= \frac{D_{12}V_{12}}{\lambda}, \\
F &= \frac{(D_{11} - D_{22})V_{12}}{\lambda}, & G &= \frac{(D_{11} - D_{22})(V_{11} - V_{22})}{\lambda},
\end{aligned} \tag{62}$$

and we have in addition to that:

$$DF = EG, \tag{63}$$

This allows us to rewrite (61) as

$$\begin{aligned}
& (A + 2E) \frac{\partial^4 \psi}{\partial x^4} + 2(B - D - F) \frac{\partial^4 \psi}{\partial x^3 \partial y} + (A + C + 2G - 4E) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} \\
& + 2(B + D + F) \frac{\partial^4 \psi}{\partial x \partial y^3} + (C + 2E) \frac{\partial^4 \psi}{\partial y^4} = \text{l.o.t.} \tag{64}
\end{aligned}$$

\* If a function  $\psi$  satisfies the equation

$$a_1 \frac{\partial^4 \psi}{\partial x^4} + a_2 \frac{\partial^4 \psi}{\partial x^3 \partial y} + a_3 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + a_4 \frac{\partial^4 \psi}{\partial x \partial y^3} + a_5 \frac{\partial^4 \psi}{\partial y^4} = \text{l.o.t.}$$

then the characteristic curves are given by:

$$a_1 \left( \frac{dy}{dx} \right)^4 - a_2 \left( \frac{dy}{dx} \right)^3 + a_3 \left( \frac{dy}{dx} \right)^2 - a_4 \frac{dy}{dx} + a_5 = 0$$

To see the structure of this operator, we shall rewrite it in terms which give rise to the polynomials  $P(\alpha)$  and  $Q(\alpha)$ . Thus

$$\left[ \left( A \frac{\partial^2}{\partial x^2} + 2B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2E \left( \frac{\partial^2}{\partial x^2} - \frac{F}{E} \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x^2} - \frac{D}{E} \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2} \right) \right] \psi = \text{l.o.t.} \quad (65)$$

We now let

$$\mathbf{L} = A \frac{\partial^2}{\partial x^2} + 2B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2}, \quad (66)$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ (Laplacian operator)}, \quad (67)$$

$$\mathbf{L}_1 = \frac{\partial^2}{\partial x^2} - \frac{F}{E} \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2}, \quad (68)$$

$$\mathbf{L}_2 = \frac{\partial^2}{\partial x^2} - \frac{D}{E} \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2}, \quad (69)$$

so that (65) becomes

$$(\mathbf{L}\Delta + 2E\mathbf{L}_1\mathbf{L}_2)\psi = \text{l.o.t.} \quad (70)$$

The first part of this operator  $\mathbf{L}\Delta$  is of mixed type and depends on the sign of  $B^2 - AC$ , and the second operator  $\mathbf{L}_1\mathbf{L}_2$  is the product of two hyperbolic operators therefore it is purely hyperbolic. This corresponds to what we found in the study of the polynomial.

This shows how strong the effect of hyperbolicity is in the flow of viscoelastic fluids. Quasilinear models like Maxwell's are already endowed with a fairly rich hyperbolic structure. The White-Metzner model could be considered to be one of the easiest nonlinear generalizations of Maxwell models, representative of an even larger class of nonlinear models. These nonlinear models can be seen to be even "more hyperbolic" than models like Maxwell's.

## 6. Computer obtained fourth order polynomial roots for operator factorization

There are standard techniques to determine the roots of a quartic. One is to factor the fourth order polynomial into two quadratics. Then it is easy to

solve for the roots. This can be done on a computer using routines involving symbol manipulators. The polynomial is now put into the following form:

$$(A + 2E)\alpha^4 - 2(B - D - F)\alpha^3 + \left(A + C + 2\frac{DF}{E} - 4E\right)\alpha^2 - 2(B + D + F)\alpha + (C + 2E) = 0. \quad (71)$$

We used the symbol manipulator MACSYMA available on a Vax machine (grant from the University of Minnesota ACSS) to determine the expression of the roots. We found the four roots that we can put together by pair of complex conjugates. Therefore we can reconstitute the two quadratics and also the two second order operators: indeed, if we assume that the roots are  $\alpha_1, \beta_1, \alpha_2, \beta_2$  (by pairs if they are complex), then the polynomial is proportional to  $(\alpha^2 - (\alpha_1 - \beta_1)\alpha + \alpha_1\beta_1)(\alpha^2 - (\alpha_2 + \beta_2)\alpha + \alpha_2\beta_2)$ , where the coefficients  $\alpha_1 + \alpha_2, \alpha_1\alpha_2, \alpha_2 + \beta_2, \alpha_2\beta_2$  are always real. The two second order operators are

$$\frac{\partial^2}{\partial x^2} + (\alpha_1 + \beta_1)\frac{\partial^2}{\partial x \partial y} + \alpha_1\beta_1\frac{\partial^2}{\partial y^2}$$

and

$$\frac{\partial^2}{\partial x^2} + (\alpha_2 + \beta_2)\frac{\partial^2}{\partial x \partial y} + \alpha_2\beta_2\frac{\partial^2}{\partial y^2}.$$

Unfortunately, the formulae giving the roots are very lengthy, but they are available on request.

## 7. Perturbation analysis of the roots

We are going to assume that for low shear rates  $\kappa, \eta'$  and  $\lambda'$  are small, so that  $\eta$  and  $\lambda$  can be written as

$$\eta = \epsilon\tilde{\eta} \quad \text{and} \quad \lambda = \epsilon\tilde{\lambda}, \quad (72)$$

where  $\epsilon$  is a small parameter and  $\tilde{\eta}$  and  $\tilde{\lambda}$  are  $O(1)$  for small  $\epsilon$ . We may define the left eigenvectors  $\mathbf{h}$  by

$$\mathbf{h}(\alpha\mathbf{A} - \mathbf{B}) = \mathbf{0}, \quad (73)$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are given by (44) and (45) and  $\alpha = dy/dx$  are the roots of our sixth order polynomial. We now expand in powers of  $\epsilon$ :

$$\alpha = \sum_{n=0}^{\infty} \alpha_n \epsilon^n, \quad (74)$$

$$\mathbf{h} = (u, v, \sigma, \gamma, \tau, p) = \sum_{n=0}^{\infty} \mathbf{h}_n \epsilon^n, \quad (75)$$

$$A = \sum_{n=0}^{\infty} A_n \epsilon^n, \quad (76)$$

$$B = \sum_{n=0}^{\infty} B_n \epsilon^n. \quad (77)$$

We know that the streamlines are always doubly characteristic so that we do not follow these roots.

At zero<sup>th</sup> order, the solutions of  $\mathbf{h}_0(\alpha_0 \mathbf{A}_0 - \mathbf{B}_0) = \mathbf{0}$  are given in [5], and the left eigenvectors have also been computed.

$$\alpha_0 = \pm i$$

and the corresponding eigenvectors are

$$\mathbf{h} = \left( -\alpha_0, 2, \alpha_0, -\alpha_0 u_0 + v_0, u_0 + \alpha_0 v_0, -\rho \alpha_0 (-\alpha_0 u_0 + v_0)^2 + 2\alpha_0(\gamma_0 - \sigma_0) + 2\tau_0 \right). \quad (78)$$

$$\alpha_0 \text{ is a root of } A_0 \alpha_0^2 + 2B_0 \alpha_0 + C_0 = 0,$$

where  $A_0, B_0, C_0$  are defined to be the coefficients of  $A, B, C$  (see (62)) when  $\epsilon = 0$ . The left eigenvectors can be reduced to

$$\mathbf{h}_0 = \left( -\alpha_0, 1 - \alpha_0^2, \alpha_0, -\alpha_0 u_0 + v_0, \alpha_0(-\alpha_0 u_0 + v_0), \alpha_0(\gamma_0 - \sigma_0) \right). \quad (79)$$

To determine the values of the perturbed roots  $\alpha_1$ , we are going to follow [11]. We define an adjoint problem

$$(\alpha_0 \mathbf{A}_0 - \mathbf{B}_0) \mathbf{h}_0^* = \mathbf{0}. \quad (80)$$

Following [5], we obtain the general formula valid for all the different roots

$$\mathbf{h}_0^* = \left( -\alpha_0 u_0 + v_0, \alpha_0(-\alpha_0 u_0 + v_0), 2\tau_0 - 2\alpha_0(\sigma_0 + \mu_0), 2\alpha_0(\gamma_0 + \mu_0 - \tau_0 \alpha_0), \gamma_0 + \mu_0 - \alpha_0^2(\sigma_0 + \mu_0), -\alpha_0 \rho(-\alpha_0 u_0 + v_0)^2 + \alpha_0^3(\sigma_0 + \mu_0) - 2\tau_0 \alpha_0^2 + \alpha_0(\gamma_0 + \mu_0) \right). \quad (81)$$

If  $\alpha_0$  is root of  $A_0 \alpha_0^2 + 2B_0 \alpha_0 + C_0 = 0$  then the sixth component vanishes. At first order, we have

$$\mathbf{h}_1(\alpha_0 \mathbf{A}_0 - \mathbf{B}_0) + \mathbf{h}_0(\alpha_0 \mathbf{A}_1 + \alpha_1 \mathbf{A}_0 - \mathbf{B}_1) = \mathbf{0}. \quad (82)$$

After multiplying by  $\mathbf{h}_0^*$  on the right and using (78) we obtain

$$\alpha_0 \langle \mathbf{h}_0 \mathbf{A}_1 \cdot \mathbf{h}_0^* \rangle + \alpha_1 \langle \mathbf{h}_0 \mathbf{A}_0 \cdot \mathbf{h}_0^* \rangle - \langle \mathbf{h}_0 \mathbf{B}_1 \cdot \mathbf{h}_0^* \rangle = 0. \quad (83)$$

Hence

$$\alpha_1 = \frac{\langle \mathbf{h}_0 (\mathbf{B}_1 - \alpha_0 \mathbf{A}_1) \cdot \mathbf{h}_0^* \rangle}{\langle \mathbf{h}_0 \mathbf{A}_0 \cdot \mathbf{h}_0^* \rangle}, \quad (84)$$

where

$$\langle \mathbf{a} \cdot \mathbf{b} \rangle \stackrel{\text{def}}{=} \sum_{i=1}^6 a_i b_i. \quad (85)$$

We have everything to determine  $\alpha_1$  from (76), (77) and (79) where  $\mathbf{A}_0$ ,  $\mathbf{A}_1$ , and  $\mathbf{B}_1$  are prescribed.

The calculations that need to be carried out are easy but rather lengthy so we will not go through them here. Another way to find  $\alpha_1$  is to expand the terms in (50) in powers of  $\epsilon$ . We find that

$$\begin{aligned} \alpha_1 = & - \{ (A_1 + 2E_1)\alpha_0^4 + 2(B_1 - D_1 - F_1)\alpha_0^3 + (A_1 + C_1 + 2G_1 - 4E_1)\alpha_0^2 \\ & + 2(B_1 + D_1 + F_1)\alpha_0 + C_1 + 2E_1 \} \\ & \times \{ 4A_0\alpha_0^3 + 6B_0\alpha_0^2 + 2(A_0 + C_0)\alpha_0 + 2B_0 \}^{-1}. \end{aligned} \quad (86)$$

This formula for  $\alpha_1$  has been checked against (84). The interest of (86) is that it is easier to handle for the analysis of the roots, where  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ , defined in (62) have also been expanded in powers of  $\epsilon$ .

We can now look at the behavior of the roots  $\alpha = \alpha_0 + \epsilon\alpha_1 + \dots$  of the quartic for the streamfunction. We name the four roots  $\alpha_0$ ,  $\beta_0$ ,  $\gamma_0$ ,  $\delta_0$  and their “ $\epsilon$ -parts”  $\alpha_1$ ,  $\beta_1$ ,  $\gamma_1$ ,  $\delta_1$ .

In the new notation, the roots  $\alpha_0 = \pm i$  become  $\alpha_0 = i$ ,  $\beta_0 = -i$  and

$$\alpha_1 = \frac{4E_1 - G_1 + 2(D_1 + F_1)i}{2B_0 + (A_0 - C_0)i} \quad \text{and} \quad \beta_1 = \frac{4E_1 - G_1 - 2(D_1 + F_1)i}{2B_0 - (A_0 - C_0)i}. \quad (87)$$

It is easy to see that the two values of  $\alpha_1$  are complex conjugates ( $\bar{\beta}_1 = \alpha_1$ ). In the new notation, the roots of  $A_0\alpha_0^2 + 2B_0\alpha_0 + C_0 = 0$  become  $\gamma_0$  and  $\delta_0$ . If  $B_0^2 - A_0C_0 \geq 0$  then the two values of  $\alpha_1$  (we call them  $\gamma_1$  and  $\delta_1$ ) are real since everything is real in (84). If  $B_0^2 - A_0C_0 \leq 0$  then  $\bar{\delta}_0 = \gamma_0$  and the two values of  $\alpha_1$  are complex conjugates,  $\bar{\delta}_1 = \gamma_1$ .

We may conclude that if  $\lambda'$  and  $\eta'$  are small, then the two imaginary roots become a pair of complex conjugates, whereas the two other roots stay real if  $B_0^2 - A_0C_0 \geq 0$  or complex conjugates if  $B_0^2 - A_0C_0 \leq 0$ .

Once we have determined  $\alpha_1$ ,  $\beta_1$ ,  $\gamma_1$ ,  $\delta_1$  we can reconstitute the operator: it is easy to get the two factors of the operator  $\mathbf{L} = \mathbf{L}^* \mathbf{L}^{**}$  of fourth order, we find that

$$\mathbf{L}^* = \frac{\partial^2}{\partial x^2} - 2\epsilon \operatorname{Re}(\alpha_1) \frac{\partial^2}{\partial x \partial y} + (1 - 2\epsilon \operatorname{Im}(\alpha_1)) \frac{\partial^2}{\partial y^2} + \mathcal{O}(\epsilon^2), \quad (88)$$



$$\begin{aligned}
 \mathbf{L}^{**} = & (A_0 + \epsilon(A_1 + 2E_1)) \frac{\partial^2}{\partial x^2} \\
 & - \left( 2B_0 + \epsilon \left( \frac{2B_0}{A_0} (A_1 + 2E_1) - A_0(\gamma_1 + \delta_1) \right) \right) \frac{\partial^2}{\partial x \partial y} \\
 & + \left( C_0 + \epsilon \left( \frac{C_0}{A_0} (A_1 + 2E_1) + A_0(\gamma_1 \delta_0 + \gamma_0 \delta_1) \right) \right) \frac{\partial^2}{\partial y^2} + O(\epsilon^2). \tag{89}
 \end{aligned}$$

Note that if  $B_0^2 - A_0 C_0 \leq 0$ , then  $\gamma_1 + \delta_1 = 2 \operatorname{Re}(\gamma_1)$  and  $\gamma_1 \delta_0 + \gamma_0 \delta_1 = 2 \operatorname{Re}(\gamma_0 \bar{\gamma}_1)$

We can evaluate the discriminants of the two operators  $\mathbf{L}^*$  and  $\mathbf{L}^{**}$ . Thus

$$\Delta^* = -1 + 2\epsilon \operatorname{Im}(\alpha_1) + O(\epsilon^2), \tag{90}$$

$$\Delta^{**} = \Delta_0 + \epsilon \theta_0 + O(\epsilon^2), \tag{91}$$

where  $\Delta_0$  and  $\theta_0$  are defined by

$$\Delta_0 = B_0^2 - A_0 C_0, \tag{92}$$

$$\theta_0 = \left( \frac{2B_0}{A_0} - 2C_0 \right) (A_1 + 2E_1) - A_0(\gamma_1 + \delta_1) - A_0^2(\gamma_1 \delta_0 + \gamma_0 \delta_1). \tag{93}$$

We plotted the linear relationship between  $\Delta^*$ ,  $\Delta^{**}$  and  $\epsilon$  in Figs. 1, 2 and 3.

Figure 1 shows that for small  $\epsilon$ , the discriminant  $\Delta^*$  remains negative, as long as  $\epsilon < |\epsilon_0|$ . The roots  $\pm i$  perturb to complex roots, the underlying operator  $\mathbf{L}^*$  remains elliptic.

Figure 2 shows that in a neighborhood of  $\epsilon = 0$  the discriminant is positive but if  $\epsilon > |\epsilon_1|$ , we can get complex roots.

Figure 3 shows that the operator remains elliptic in the neighborhood of  $\epsilon = 0$ .

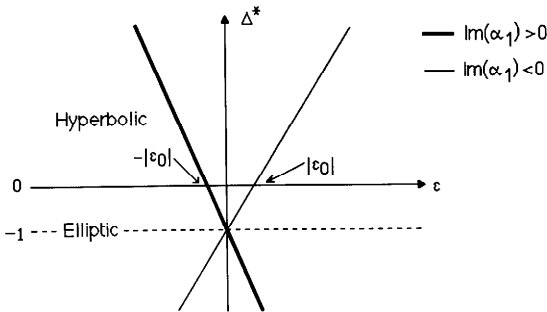


Fig. 1.  $\Delta^* = \Delta^*(\epsilon)$  where  $\epsilon_0 = 1/2 \operatorname{Im}(\alpha_1)$ .

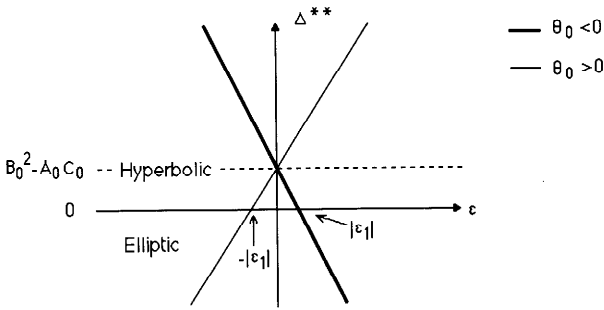


Fig. 2.  $\Delta^{**} = \Delta^{**}(\epsilon)$  where  $\epsilon_1 = -\Delta_0/\theta_0$  and  $B_0^2 - A_0 C_0 > 0$ .

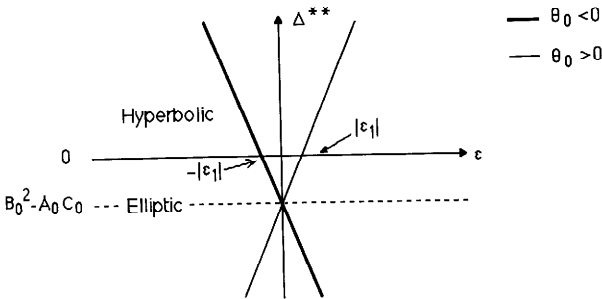


Fig. 3.  $\Delta^{**} = \Delta^{**}(\epsilon)$  where  $\epsilon_1 = -\Delta_0/\theta_0$  and  $B_0^2 - A_0 C_0 < 0$ .

We close with comments about the results of [7,8] and [9]. Gaidos and Darby find a change of type in the flow of a White–Metzner fluid through a slit. They have four real roots and 2 complex conjugates. This may be due to a weak dependence on the second invariant of the strain tensor. Perhaps there are six real roots when  $\lambda'$  and  $\eta'$  are larger. Calderer et al. [9] use a Bird–DeAguiar model, which is a nonlinear model derived from molecular theory. They found six real roots for steady plane flow in some cases. Although this result is for a different model, it suggests strongly that, like the White–Metzner model, other nonlinear models may also become purely hyperbolic in steady flows.

### Acknowledgments

This work was sponsored by the Army Research Office, Mathematics and the National Science Foundation, Fluid Mechanics.

## References

- 1 I.M. Rutkevich, Some general properties of the equations of viscoelastic incompressible fluid dynamics, *J. Appl. Math. Mech. (PMM)*, 33(1) (1969) 42–51.
- 2 D.D. Joseph, M. Renardy and J.C. Saut, Hyperbolicity and change of type in the flow of viscoelastic fluids, *Arch. Ration. Mech. Anal.*, 87(3) (1985) 213–251.
- 3 F. Dupret and J.M. Marchal, Loss of evolution in the flow of viscoelastic fluids, *J. Non Newtonian Fluid Mech.*, 20 (1986) 143–171.
- 4 F. Dupret and J.M. Marchal, Sur le signe des valeurs propres du tenseur des extra-contraintes dans un écoulement de fluide de Maxwell, *J. Méc. Théor. Appl.*, 5(3) (1986) 403–427.
- 5 D.D. Joseph and J.C. Saut, Change of type and loss of evolution in the flow of viscoelastic fluids, *J. Non Newtonian Fluid Mech.*, 20 (1986) 117–141.
- 6 S.A. Regirer and I.M. Rutkevich, Certain singularities of the hydrodynamic equations of non-Newtonian media, *J. Appl. Math. Mech. (PMM)*, 32(5) (1968) 942–945.
- 7 T. Nikoleris and R. Darby, Numerical simulation and change in type in the developing flow of a nonlinear viscoelastic fluid, *J. Non Newtonian Fluid Mech.*, 31 (1989) 193–207.
- 8 R. Gaidos, Ph.D. Thesis, Texas A & M University, TX, 1987.
- 9 M.C. Calderer and L.P. Cook, An analysis of the Bird–DeAguiar model for polymer melts, *J. Non Newtonian Fluid Mech.*, 31 (1989) 209–225.
- 10 W.E. Williams, *Partial Differential equations*, Clarendon Press, Oxford, 1980.
- 11 G. Iooss and D.D. Joseph, *Elementary Stability and bifurcation Theory*. Springer-Verlag, New York, NY, 1980.