# HYPERBOLICITY, CHANGE OF TYPE, WAVE SPEEDS AND RELATED MATTERS

D. D. Joseph
Department of Aerospace Engineering and Mechanics
The University of Minnesota
Minneapolis, Minnesota 55455

Summary

In this paper I will review some consequences of instantaneous elasticity for the numerical analysis of flows of viscoelastic liquids. I will consider situations which are associated with hyperbolic waves of vorticity. The vorticity equation may change type in steady and unsteady flow. In the latter case we get instability to short waves, ill-posedness, and, in the former, transonic flow. The two types of change are related. The regularizing effect of an effective Newtonian viscosity and the physical origins of viscosity are briefly reviewed.

## 1. Introduction

Many of the topics treated in this paper have been treated in the papers by Joseph, Renardy and Saut [1985, hereafter called JRS] and by Joseph and Saut [1986, hereafter called JS]. Most of the references in this paper are to be found in JS. Only the references which are not in JS will be listed here.

The main impetus for recent work on topics of this review was the failure of numerical simulations for complex flows of viscoelastic fluids with relatively large relaxation times. These fluids are relatively elastic. The numerical problem was called the high Weissenberg, or high Deborah, number problem. Every two years for the last ten, some of us who are interested get together for discussions. In the Lake Morrey, Vermont, meeting in the summer of 1983 I suggested that the problem of numerical simulation might be associated with the mathematical problem of hyperbolicity and change of type of the governing equations. I also suggested that type dependent upwinding schemes like those used in transonic flow might be useful. At the time, it was thought that limit points might be responsible for the failure of fact that measured values of wave speeds are robust and not sensitive to various perturbations of measuring technique or gap size (see Joseph, Narain and Riccius, 1986).

It is well known that Newtonian fluids generally are asymptotic limits of slow, slowly varying flows. This was shown within the context of Coleman's theory of fading memory by Coleman and Noll and was called a retardation theorem by them. Actually one could argue that the memory never fades, while the fluid moves, because the viscosity represents the memory of elasticity long past. The idea of an effective viscosity uses an entirely different asymptotic limit in which different times of relaxation in fluids are compared with one another.

The most outstanding physical consequence of change of type in steady flow could be delayed die (or extrudate) swell (Joseph, Matta, Chen, 1987). When the extrusion speed is increased beyond a critical diameter dependent value, the swell doesn't occur at the exit of the pipe but is delayed, and the delay is an increasing function of the extrusion speed. Delayed die swell is like a hydraulic jump. We can define a viscoelastic Mach number as the ratio of the centerline velocity in the pipe to the speed of vorticity, or shear waves, into rest. In all eighteen polymer solutions so far tested, the critical extrusion velocity is greater than the wave speed; after the swell, the velocity is less than the wave speed. Hence, in the region of the delayed swell there are points at which the viscoelastic Mach number is one. The critical extrusion velocity is such that at the exit of the pipe the critical Mach number was not smaller than one, Mcrit $\geq 1$, with equality for large nozzles. Delayed die swell seems strongly associated with the propagation of hyperbolic waves of vorticity.

## 2. Models like Maxwell's model

We are going to assume that the part $\tau$ of the stress $T = -p1 + \tau$ in an incompressible fluid satisfies a constitutive equation like Maxwell's

$$\lambda\frac{\mathcal{D}\tau}{\mathcal{D}t} = 2\eta\, D[u] + \mathcal{L}[u, \tau]$$

where D[u] is the symmetric part of $\nabla u$, u is the velocity, $\lambda$ is the relaxation time, $\eta$ is the "elastic" viscosity,



simulations. It appears now that limit points are common in the numerical approximation but usually disappear after more careful analysis.

A finite difference numerical simulation of the flow of an upper convected Maxwell fluid through a planar 4:1 contraction using a type dependent upwinding scheme for the vorticity was recently reported by Song and Yoo [1987]. A similar method, with the addition of an artificial viscosity and using a Giesekus model, was employed by Choi, Song and Yoo [1987]. The introduction of these type dependent methods gives improved numerical results at higher values of Deborah number but they do not eliminate the problem.

The problem of numerical simulations is evidently associated with hyperbolicity, not so much for "transonic" steady problems as I suggested originally but with short wave length instabilities of the type associated with ill-posed problems. The first important results on ill-posed problems for Maxwell models can be found in a series of papers in the 1970s by Rutkevich; JRS showed that vorticity was the key variable and demonstrated that type changes could be expected in many, maybe nearly all, flows. The paper by Dupret, Marchal and Crochet [1985] is important in that it indicated that discretization errors could introduce short wave instabilities even for models, like the upper convected Maxwell model, which are always well posed for smooth solutions. Discretization for such problems allows one to step into forbidden and otherwise inaccessible regions of the added stress. A recent work by Marchal and Crochet [1987] seems to have partially solved the problem of "false" discretization induced instability by upwinding on streamlines. They introduced technical improvements and the addition of an artificial diffusivity which goes to zero with mesh refinement.

The physical consequences of hyperbolicity and wave propagation in fluids are only now coming to be understood. The speed of waves of vorticity into fluids at rest have recently been measured by Joseph, Riccius and Arney [1986], Riccius, Arney and Joseph [1987] and by Lee and Fuller [1987]. The speeds are slow, ranging from 10 to 2000 cm/sec. Such slow speeds are associated with "rubbery" elastic responses induced by altering the short range molecular order of large molecules. The small molecules give rise to very large wave speeds and short times of relaxation. The response of small molecules is glassy at very short times (say, less than 10-8 seconds). Glassy modes which have already relaxed give rise to an effective Newtonian viscosity. The idea of an effective viscosity and rigidity associated with large differences in the times of relaxation of molecules of different size and type appears to be required to explain the

$$\frac{\mathcal{D}\tau}{\mathcal{D}t} \overset{\text{def}}{=} (\frac{\partial}{\partial t} + u_0\nabla)\tau + \tau\Omega - \Omega\tau - a(D\tau + \tau D)$$

where a $(-1 \le a \le 1)$ is a real number, $\Omega[u]$ is the skew symmetric part of $\nabla u$ and $\ell[u, \tau]$ is of lower order; it does not depend on derivatives of u or $\tau$. Models like Maxwell's differ in lower order terms but have the same principal part. The Oldroyd-Maxwell models have $\ell = -\tau$; a = 1 is an upper-convected Maxwell model, a = -1 is a lower-convected Maxwell model and a = 0 is a corotational model. A model of Giesekus is associated with $\ell = -\tau - c_1\tau^2$, a = 1, where $c_1$ is constant. A model of Phan Thien-Tanner is associated with $\ell = -\tau - c_2\tau \operatorname{tr} \tau$ and a = 1, where $c_2$ is a constant.

## 3. Quasilinear systems of equations

The dynamical equations governing the motion of fluids like Maxwell's are quasilinear; they are nonlinear, but linear in derivatives. We can write the dynamical equations

$$G(Qt + u \cdot \nabla Q) + HQ_x + JQ_y = \ell[Q] \tag{3.1}$$

where G is not invertible (e.g., there is no $p_t$ in this system), Q is a system vector whose components are the velocity, stresses and pressure and G, H, J and $\ell$ depends on Q, but not on the derivatives of Q. In two dimensions we have velocity components u = (u, v) corresponding to x, y stress components

$$[\tau] = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix}$$

and

$$Q = [u, v, \sigma, \gamma, \tau, p] .$$

There are six quasilinear equations for the six scalar fields, $[u, v, \sigma, \gamma, \tau, p]$, linear in derivatives with lower order right-hand sides, $\ell_1$, $\ell_2$ and $\ell_3$.



$$\sigma_t + u\sigma_x + v\sigma_y + \tau(v_x - u_y) - a[2\sigma u_x + \tau(u_y + v_x)] - 2\mu u_x = \ell_1 ,$$

$$\tau_t + u\tau_x + v\tau_y + \frac{1}{2}(\sigma - \gamma)(u_y - v_x) - \frac{1}{2}a(\sigma + \gamma)(u_y + v_x) - \mu(u_y + v_x) = \ell_2 ,$$

$$\gamma_t + u\gamma_x + v\gamma_y + \tau(u_y - v_x) - a[2\gamma u_y + \tau(u_y + v_x)] - 2\mu v_y = \ell_3 , \tag{3.2}$$

$$\rho(u_t + uu_x + vu_y) + p_x - \sigma_x - \tau_y = 0 ,$$

$$\rho(v_t + uv_x + vv_y) + p_y - \tau_x - \gamma_y = 0 ,$$

$$u_x + v_y = 0 .$$

## 4. Stability

For the moment it is useful to think about how we might carry out an analyses of stability. First we suppose that $\hat{Q}$ is a solution of (3.1). It could even be an unsteady solution. Then we write

$$Q = \hat{Q} + q$$

and suppose that q is small. The linearized equations are

$$\hat{G}(q_t + \hat{u} \cdot \nabla q) + \hat{H}q_x + \hat{J}q_y = \hat{\ell}q \tag{4.1}$$

where $\hat{H}$, $\hat{J}$, $\hat{\ell}$ depend on $\hat{Q}$, derivatives of $\hat{Q}$ but not on q. We can imagine trying to solve (4.1) plus boundary conditions as an initial value problem for the stability of Q. If $\hat{Q}$ is steady, we could write $q = c^{\sigma t}\hat{q}(x)$ and determine stability from eigenvalues $\sigma$. To get the eigenvalues we would have to solve a complicated set of partial differential equations over the whole field of flow, satisfying boundary conditions.

## 5. Frozen coefficients on short waves

Now we shall treat the problem of stability for a special class of disturbances which lead to simple but deep results. We are going to consider short waves, tending to zero noting that $\hat{Q}$, hence $\hat{G}$, $\hat{H}$, $\hat{J}$ and $\hat{\ell}$ are nearly constant on any sufficiently small neighborhood $|x - x_0| < \epsilon$ of any point $x_0$.

The coefficients of (4.1) are constant on such a small neighborhood and we may try for a solution in terms of normal modes

$$q = a \exp(-i\omega t + i\alpha(x - x_0) + i\beta(y - y_0)) \quad (5.1)$$

where $a(x_0)$ is an amplitude, $\omega$ a frequency and x and $\beta$ are wave numbers. Of course, we cannot hope to satisfy boundary conditions with a solution of this form. It is a strange form for the solution because it applies at each and every point $x_0$, so we may find stability at some points and instability at others.

To set some notations, we define a wave vector.

$$\mathbf{k} \overset{\text{def}}{=} \mathbf{e}_x\alpha + \mathbf{e}_y\beta ,$$

$$|\mathbf{k}| = \sqrt{\alpha^2+\beta^2} . \quad (5.2)$$

Since $\exp(-i\omega t) = \exp(\omega_i t)\exp(-i\omega_r t)$ where $\omega = \omega_r + i\omega_i$ we may define a growth rate $\sigma = w_i$.

## 6. Instability to short waves

After putting the normal modes (5.1) into (4.1) we get

$$\hat{L}(\omega, \mathbf{k})\mathbf{a} = \hat{\ell}\mathbf{a} \quad (6.1)$$

where

$$\hat{L} = (-\omega + \hat{u} \cdot \mathbf{k})\hat{G} + \alpha\hat{H} + \beta\hat{J} .$$

We divide (6.1) by $|\mathbf{k}|$ and let $|\mathbf{k}| \to \infty$. Since $\hat{\ell}$ is independent of $\alpha$ and $\beta$, $\hat{\ell}\mathbf{a}/|\mathbf{k}| \to 0$ and

$$\hat{L}\left(\frac{\omega}{|\mathbf{k}|}, \frac{\mathbf{k}}{|\mathbf{k}|}\right)\mathbf{a} = 0 \quad (6.2)$$

Equation (6.2) represents six linear, homogeneous equations for the six unknown components of $\mathbf{a}$. Hence

$$\Delta \overset{\text{def}}{=} \det \hat{L} = -\rho c^2 + \hat{f} = 0 \quad (6.3)$$

where

$$c = (\omega - \mathbf{u} \cdot \mathbf{k})/|\mathbf{k}| \quad (6.4)$$



and

$$\hat{f} = \frac{\eta}{\lambda} - \frac{\hat{\gamma}}{2}(1 - a) + \frac{\hat{\sigma}}{2}(1 + a) . \quad (6.5)$$

The expression (6.5) for $\hat{f}$ has been simplified by choosing x so that $\beta = 0$, $\mathbf{k} = \alpha \mathbf{e}_x$. The growth rate $\sigma = \omega_i$ is given by

$$\text{imaginary } c = \frac{\sigma}{|\mathbf{k}|} = \pm \text{ imaginary} \sqrt{\frac{\hat{f}}{\rho}} .$$

It follows that

$$\sigma = 0 \text{ is } \hat{f} > 0$$

and there is a positive growth rate if $\hat{f} < 0$.

We may phrase the condition for stability to short waves in terms of the wave speed c, with stability only if $c^2$ is positive corresponding to real wave speeds.

The condition

$$0 > \hat{f} = \frac{\eta}{\lambda} - \frac{\hat{\gamma}}{2}(1 - a) + \frac{\hat{\sigma}}{2}(1 + a)$$

for instability to short waves is framed as a condition on the values of the normal stresses $\hat{\gamma}$ and $\hat{\sigma}$ in a coordinate system in which $\mathbf{k} = \alpha \mathbf{e}_x$. If the solution enters into this region of forbidden stress, a very ugly instability will ensue.

## 7. Catastrophic short wave instability and the loss of well-posedness

Suppose that $\hat{f} < 0$, then

$$\sigma = \pm |\mathbf{k}| \sqrt{|\hat{f}|\rho} . \quad (7.1)$$

This is a strange formula. When $\hat{f} = 0$, $\sigma = 0$ but when $f < 0$ and small, $\sigma \to \infty$ with $|\mathbf{k}|$. We can get stability to short waves at some points and instability at others, depending on the values of $\hat{\gamma}$ and $\hat{\sigma}$ through frozen coefficients. This kind of catastrophic short wave number instability is sometimes called Hadamard instability. Many well-known problems of mechanics and physics can undergo catastrophic instabilities to short waves. Some examples are the Kelvin-Helmholtz instability, the

Rayleigh-Taylor instability, the Taylor instability of a flat interface for oil displacement in a porous medium, magnetohydrodynamics with the Hall effect, two-fluid models of bicomponent media when viscosity is neglected and the flow of some models of viscoelastic liquids with instantaneous elasticity (see JS, 1986).

G. Birkhoff [1954] has called attention to the fact that the mathematical definition of ill-posed initial value problems requires the prior specification of set of functions. He argues convincingly that the class of functions possessing Fourier transforms is appropriate for this discussion because in this class it is easy to make a direct connection between Hadamard instabilities and ill-posed initial value problems. In this class he defines regular and irregular growth rates, "$\sigma(k)$ is regular if and only if it is bounded as $|k| = k \rightarrow \infty$. Otherwise it is irregular." Problems with irregular eigenvalues are ill-posed as initial value problems in the class of functions with Fourier transforms. There is no uniform continuity with respect to initial values: the short wave components of the disturbance will amplify with asymptotically unbounded growth rates. This kind of instability would not lead to bifurcation as in the case of regular eigenvalues. When the model giving rise to irregular eigenvalues is physically reasonable, the short wave instabilities are associated with fibril structures leading to fingering rather than bifurcation.

It is possible to have short wave instabilities with $k \rightarrow \infty$ without ill-posedness. This is the case when the growth rates are bounded with a maximum growth rate for the shortest waves; $k \rightarrow \infty$. An example of a regular short wave instability of a shear flow is superposed liquids with different viscosities when the surface is neglected (Hooper and Boyd, 1983).

Another connection between ill-posedness and irregular growth rates can be made in the class of functions having Fourier transforms. This connection states that problems with irregular growth rates have no solutions for any class of initial data more general than analytic. For example, the Cauchy problem on $\mathbb{R}^2$ which follows from freezing the coefficients has no solution with initial data in the $C_m$ class. This result is well known for the Laplace equation treated in the celebrated example by Hadamard. The solution of Laplace's equation is analytic, say in the half $(x, y)$ plane with $x > 0$, and it can be extended to $x < 0$ by reflections. Hence the initial data on $y = 0$ must also be analytic. This "nonexistence" result is valid generally. A proof of nonexistence will appear in a paper on short wave instability and ill-posed problems by me and J. C. Saut. A sketch of the proof is given below. Suppose that the coefficients in (4.1) are



constant (as will, in fact, be implied by frozen coefficients) and that q lies in a Fourier transform class:

$$p(\alpha, \beta, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\alpha x+\beta y)}\, q(x, y, t)\,dx\,dy \qquad (7.2)$$

is the transform of $q(x, y, t)$ and $p(\alpha, \beta, 0)$ is the transform of the Cauchy initial data $q(x, y, 0)$ for $q(x, y, t)$ and

$$q(x, y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta y)}\, p(\alpha, \beta, t)\,d\alpha\,d\beta \qquad (7.3)$$

The transform $p(\alpha, \beta, t)$ satisfies, the following ordinary differential equation:

$$\hat{G}(\frac{dp}{dt} + i\hat{u}\cdot kp) + i(\alpha\hat{H} + \beta\hat{J})p = 0 \qquad (7.4)$$

with a prescribed $p = p(\alpha, \beta, 0)$ at $t = 0$. The solution of this initial value problem is given by a vector valued function in $\mathbb{R}^6$

$$p(\alpha, \beta, t) = p(\alpha, \beta, 0)\exp(-i\omega t) \qquad (7.5)$$

where $\omega = \omega_r(\alpha, \beta) + i\omega_i(\alpha, \beta)$ is an eigenvalue of (6.1) and $\sigma(\alpha, \beta) \overset{def}{=} \omega i(\alpha, \beta)$ is the largest growth rate among the six eigenvalues, giving instability when positive. It follows now that

$$q(x, y, t) = \frac{1}{2\pi}\int_0^{\omega}\int_0^{\infty} e^{i(\alpha x+\beta y-\omega t)}\, p(\alpha, \beta, 0)\,d\alpha\,d\beta .$$

(7.6)

The crucial factor under the integral is

$$e^{\sigma(\alpha,\beta,)t}\, p(\alpha, \beta, 0) . \qquad (7.7)$$

If the growth rate $\sigma(\alpha, \beta)$ is irregular then

$$\lim_{k\sqrt{\alpha^2+\beta^2} \rightarrow\infty} \sigma(\alpha, \beta) \rightarrow \infty$$

and the integral defining $q(x, y, t)$ can exist only if $p(\alpha, \beta, 0)$ decays exponentially $\sigma(\alpha, \beta, t)$ as $k \to \infty$ with an exponent dominating $\sigma(\alpha, \beta, t)$.

Our investigation of the existence of solutions with irregular eigenvalues has now been reduced to the study of the asymptotic properties of the transform (7.2) of the initial data $q(x, y, 0)$. Suppose the second $x$ derivative of $q(x, y, 0)$ is discontinuous at $x = x^*$, that $q(x, y, 0)$ tends to zero at large $x$, and $y$ is as required for functions in the Fourier transform class. Then, after integrating by parts, we find

$$2\pi p(\alpha, \beta, 0) = -\int_{-\infty}^{\infty} dy e^{-i\beta y} \int_{-\infty}^{\infty} \frac{1}{i\alpha^3} e^{-i\alpha x} \frac{\partial^3 q(x,y,0)}{\partial x^3}$$

$$+ \frac{1}{i\alpha^3} \int_{-\infty}^{\infty} dy e^{-i\beta y} e^{-i\alpha x^*} [\frac{\partial^2 q}{\partial x^2}](y) \qquad (7.8)$$

where

$$[\frac{\partial^2 g(x^*,y)}{\partial x^2}](y) \qquad (7.9)$$

is the jump in $q(x, y)$ at $x = x^*$. It follows that, in general, $p(\alpha, \beta, 0)$ decays like $1/\alpha^3$ for large $\alpha$ when (7.9) holds. In this case (7.7) is unbounded as $\alpha \to \infty$ whenever $\sigma(\alpha, \beta) > 0$ is irregular.

The proof that (7.7) is unbounded when $q(x, y, 0)$ is $C^\infty(x, y)$ will be given in the forthcoming paper on short wave instabilities and ill-posed problems by Saut and me. Ill-posedness means nonexistence for all but analytic data.

## 8. Some further comments about frozen coefficients

Analyses of short waves on frozen coefficients has the following useful properties.

1. It leads us to linear equations. Richtmeyer and Morton [1967] note that "Indeed, it is in checking the 'local' stability of linearized equations obtained from truly nonlinear equations that the constant coefficient theory is mainly of use."



2. The short waves allow one to freeze the coefficients; the coefficients do not vary on a sufficiently small region, so the linearized problem has constant coefficients.

3. Since derivatives of the quasilinear system on short waves become unboundedly large, the lower order terms are increasingly unimportant and they may be neglected. This leads to a homogeneous system, to a set of linear homogeneous equations which can be solved only if a determinant of constants vanish. What could be easier?

4. The reduced homogeneous system is now in $\mathbb{R}^2$ and boundary conditions can be neglected. The short wave instabilities start as a local phenomenon.

The foregoing analysis of short waves on frozen coefficients was formal and not rigorous. The connection to ill-posedness can be framed as in §7, but more needs to be done. Kreiss [1962] gives an example of an equation where the problem with variable coefficients is properly posed, yet all the corresponding constant coefficient problems are improperly posed; on the other hand, he gives an example in which the constant coefficient problems are all properly posed, yet the variable coefficient problem is not. So Kreiss' examples show that, in general, local stability is neither necessary nor sufficient for the overall stability of the variable coefficient problem. However, as Strang [1966] points out, if a quasilinear system of first order equations is properly posed, then all frozen coefficient problems are also properly posed. So first order systems, like (4.1) must be properly posed locally, in the sense of frozen coefficients.

## 9. Regularization of ill-posed problems

The catastrophic loss of stability for short waves cannot be an acceptable mathematical description of any physical process. Some regularizing feature of physics, negligible in smoother motions, is brought into action by the rapid variations associated with exponentially growing short waves. A small viscosity or surface tension can be important in controlling short waves.

There is a temptation to discard constitutive equations which can lead to ill-posed initial value problems. However, constitutive equations which always have well-posed problems may be bad and those which may be ill-posed can be good. It is necessary to study how correct models which allow solutions that enter into domains of ill-posedness can be regularized. This type of study is a realization of the "axiom" of physics, which is "there is always a cut-off", or of medicine, "the bleeding always stops".

The addition of Newtonian viscosity to the constitutive model for fluids with instantaneous elasticity is the natural way to regularize ill-posed problems. Many models, like Oldroyd B, already have a Newtonian contribution, expressed by a retardation time. Physically we expect a Newtonian viscosity to arise from the decay of rapidly decaying modes associated with small molecules. All this puts forward the rheometrical problem of measurements of an "effective" Newtonian viscosity. This new problem needs a solution.

## 10. The vorticity is the key hyperbolic variable

For models like Maxwell's, it is possible to frame the discussion of hyperbolicity in terms of a second order PDE for the vorticity (see Equation (4.1) in JS).

In plane flow, there is one nonzero component of vorticity satisfying

$$\rho \frac{\partial^2 \zeta}{\partial t^2} + 2\rho(u \cdot \nabla)\frac{\partial \zeta}{\partial t} - A\frac{\partial^2 \zeta}{\partial x^2} - 2B\frac{\partial^2 \zeta}{\partial x \partial y} - C\frac{\partial^2 \zeta}{\partial y^2} + \ell = 0 . \qquad (10.1)$$

where $\ell$ is of lower order, and A, B, C are defined by

$$A = -\rho u^2 + \mu + \frac{1}{2}\sigma(1 + a) - \frac{1}{2}\gamma(1 - a) ,$$

$$B = \tau - \rho u \upsilon ,$$

$$C = -\rho \upsilon^2 + \mu - \frac{1}{2}\sigma(1 - a) + \frac{1}{2}\gamma(a + 1) .$$

Analysis of (10.1) using the method of short waves leads directly to the criterion of (6.3). The same criterion, positive wave speed $c^2 > 0$, is sufficient to guarantee that (10.1) is hyperbolic (see JS).

Two conclusions follow from the foregoing comparison:

1. The quasilinear system (3.2) is well-posed or ill-posed if the vorticity equation (10.1) is well-posed or ill-posed, respectively.



2. The quasilinear system (3.2) is well-posed if and only if the vorticity equation (10.1) is hyperbolic. It is useful here to remark that the property of well-posedness of an initial value problem is a more general one than hyperbolicity but in the present case there is a sense in which the two concepts coincide.

## 11. Hyperbolicity and change of type of the steady vorticity equation

Now we put the time derivatives to zero. This means that we have left behind the problem of ill-posed problems and short wave instabilities. The problem now is to find the regions of steady flow in which the vorticity equation gives rise to real characteristic directions. In general the analysis of the characteristics of (10.1) when the time derivatives vanish will lead to conditions for the emergence of transcritical flow, like transonic flow in aerodynamics, elliptic in some regions of flow and hyperbolic in others. Elementary analysis of the problem of characteristics in steady flow (see, for example, JRS and JS, Equation (5.7)) leads to the formula

$$\frac{dy}{dx} = \frac{B}{A} + \frac{\sqrt{B^2 - AC}}{A}$$

where A, B, C are defined under (10.1). Clearly there are real characteristics wherever the discriminant

$$B^2 - AC = -\mu^2 + \rho[\mu + a\sigma + a\gamma](u^2 + \upsilon^2) + \frac{1}{2}\rho(\gamma - \sigma)(u^2 - \upsilon^2)$$

$$+ \tau^2 + \frac{1}{4}\sigma^2(1 - a^2) + \frac{1}{4}\gamma^2(1 - a^2) - \mu a(\sigma + \gamma) - 2\rho\tau\upsilon > 0 , \qquad (10.2)$$

and the vorticity equation is elliptic wherever $B^2 - AC < 0$. The criterion (10.2) depends on inertia through the terms multiplying the density $\rho$ but the criterion $f < 0$ for ill-posedness is independent of $\rho$ (see 6.5). Usually regions of high speed steady flow will go hyperbolic when the velocities are large enough. However, it is possible for a steady flow of an inertia-less fluid with $\rho = 0$ to change type. The following result proved by JS relates the criterion for ill-posed problems, basically defined for evolution, to the criterion for change of type in steady flow. The quasilinear system (3.2) is well-posed if and only if the vorticity equation (10.1) is hyperbolic. If the vorticity equation of steady flow is hyperbolic when $\rho = 0$, then the unsteady vorticity equation is elliptic and the quasilinear system ill-posed. Conversely, if the vorticity of an inertia-less steady flow is elliptic and $A > 0$ the system (3.2) is well-posed.

It is easiest to examine the criterion just given in principal coordinates for the stress, $\tau = 0$. Then putting $\rho$ to zero we have a well-posed problem for $\sigma$ and $\gamma$ such that

$$A = \mu + \frac{1}{2}\sigma(1 + a) - \frac{1}{2}\gamma(1 - a) > 0$$

and

$$B^2 - AC = -\mu^2 + \frac{1}{4}\sigma^2(1 - a^2) + \frac{1}{4}\gamma^2(1 - a^2) - \mu a(\sigma + \gamma) < 0 \,.$$

For upper convected models, a = 1, this criterion reduces to

$$A = \mu + \sigma > 0 \,, \qquad B^2 - AC = -\mu(\mu + \sigma + \gamma) < 0 \,.$$

For lower convected models, a = −1, this criterion reduces to

$$A = \mu - \gamma > 0 \,, \qquad B^2 - AC = -\mu(\mu - \sigma - \gamma) < 0 \,.$$

These inequalities are always satisfied because of restrictions on the range of $\tau$ implied by the constitutive equations when expressed in integral form (see JRS, Equation (5.8) and (5.9)). Dupret and Marchal [1985] used the differential form of the upper and lower convected Maxwell models to show that if the criterion for well-posed problems is satisfied initially, it will not fail subsequently.

A large number of examples of problems which change type, using different constitutive models and different flows, were considered by JRS and JS. Some models are always evolutionary (well-posed) and do not change type in unsteady flow. The vorticity equation for steady flow of such models can and does change type. Other models can become ill-posed and undergo bad instability to short waves. Some flows of all these models, like simple shear or Poiseuille flow, are always well-posed while other flows, like plane extension or sink flow, can become ill-posed. Sink flows of upper convected and lower convected Maxwell models change type in steady flow, but cannot be ill-posed. On the other hand, sink flows of corotational Maxwell models change type in steady flow and are also ill-posed. Nearly every possibility is realized for some flow of some model.



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## References

1. G. Birkhoff, "Classification of partial differential equations," J. Soc. Indus. Appl. Math., 2(1), 57-67 (1954).

2. H. C. Choi, J. H. Song and J. Y. Yoo, "Numerical simulation of the planar contraction flow of a Giesekus fluid," Fifth International Symposium on Numerical Simulation of Non-Newtonian Flow, to appear in J. Non-Newtonian Fluid Mech.

3. B. Coleman and M. Noll, "An approximation theorem for functionals with applications in continuum mechanics," Arch. Rational Mech. Anal., 6, 355-370 (1960).

4. F. Dupret, J. M. Marchal, "Sur le signe des valeurs propres du tenseur des extra-constraints dans un ecoulement de fluide de Maxwell," J. Mec. Theor. Appl., 5, 403-427 (1986).

5. A. Hooper and W. Boyd, "Shear-flow instability at the interface between two viscous fluids," J. Fluid Mech., 128, 507-528 (1983).

6. D. D. Joseph, M. Renardy, J. C. Saut, "Hyperbolicity and change of type in the flow of viscoelastic fluids.," Arch. Rational Mech. Anal., 87, 213-251 (1985).

7. D. D. Joseph and J. C. Saut, "Change of type and loss of evolution in the flow of viscoelastic fluids," J. Non-Newtonian Fluid Mech., 20, 117-141 (1986).

8. D. D. Joseph and J. C. Saut, "Short wave instabilities and ill-posed initial value problems," Forthcoming.

9. D. D. Joseph, A. Narain and O. Riccius, "Shear-wave speeds and elastic moduli for different liquids, Part I. Theory," J. Fluid Mech., 171, 289-308 (1986).

10. D. D. Joseph, O. Riccius and M. Arney, "Shear-wave speeds and elastic moduli for different liquids, Part II. Experiments," J. Fluid Mech., 171, 309-338 (1986).

11. D. D. Joseph, J. Matta and K. Chen, "Delayed Die Swell," J. Non-Newtonian Fluid Mech., 24, 31-65 (1987).

12. H. O. Kreiss, Über die Stabilitätsdefinition für Differnzgleichungen die partielle Differentialgleichungen approximieren," Nordisk Tidskr Informations, Behandling, 2, 153 (1962).

13. J. S. Lee, and G. G. Fuller, "The spatial development of transient Couette and shear wave propagation in polymeric liquids by flow birefringence," to appear in J. Non-Newtonian Fluid Mech.

14. J. M. Marchal and M. J. Crochet, "A new mixed finite element for calculating viscoelastic flow," Fifth International Symposium on Numerical Simulation of Non-Newtonian Flow, to appear in J. Non-Newtonian Fluid Mech.

15. O. Riccius, D. D. Joseph and M. Arney, "Shear-wave speeds and elastic moduli for different liquids, Part 3. Experiments--update," Rheol. Acta, 26, 96-99 (1987).

16. R. Richtmeyer and K. W. Morton, "Different methods for initial value problems," John Wiley and Son, 1967.

17. J. H. Song and J. Y. Yoo, "Numerical simulation of viscoelastic flow through a sudden contraction using a type dependent difference method," J. Non-Newtonian Fluid Mech., 24, 221-243 (1987).

18. W. G. Strang, "Necessary and insufficient conditions for well-posed Cauchy problems," J. Differential Equations, 2, 107 (1966).